KAMAK 2019

Penzion Korýtko, Filipova Huť

September 22 – 27

Charles University, Prague

Organizers:

Pavel Dvořák
Robert Šámal

Brochure of open problems,
Prague, 2019.

Contents

Program

8:00 breakfast

9:00 morning session I

10:30 break

11:00 morning session II

12:30 lunch

15:00 afternoon session I

16:30 break

17:00 afternoon session II

18:30 progress reports

19:00 dinner

Wednesday is planned to be free to make an excursion in the neighborhood with everyone who would like to come.

OPEN PROBLEMS

Problem 1. *Hat chromatic number (suggested by Václav Blažej)*

Source: Proposed by Bosek, Dudek, Farnik, Grytczuk, and Mazur in Hat Chromatic Number of Graphs 2019.

We are given a graph G and number of colors k. Vertices of G are occupied by players (called *Bears*) which can see all their neighbors but not themselves. Each of them will be given a *colored hat* and their task is to come up with a collective *strategy* such that at least one of them will guess his hat correctly. The guess may depend on hat colors of Bear's neighbors. The coloring of G is chosen by an evil adversary (called *Demon*) who knows Bears' strategy. If the Bears' strategy guarantees that at least one always guesses correctly they win.

Definitions.
• *The* hat chromatic number $\mu(G)$ *is the maximum number of colors k such that there is a winning strategy for Bears.*

Example:
• $\mu(K_2) = 2$. Note that each Bear recognizes 2 situations based on its neighbor's hat colors. If $k = 3$ then the adversary can choose a color which each bear never uses. Bears' winning strategy for $k = 2$ is for the first one to guess the same color as his neighbor has, and the second one will guess the other color.
• $\mu(K_n) \geq n$. The winning strategy is not hard and is left as an excercise.
• $\mu(K_n) \leq n$. The upper bound uses probabilistic method – excercise.

Question:

• Conjecture (max degree): $\mu(G) \leq \Delta + 1$
• Conjecture (max clique): $\chi(G) \leq \mu(G) + 1$
• Conjecture (Hadwiger number): $\mu(G) \leq h(G)$
• Conjecture (weak coloring number): $\mu(G) \leq f(col(G))$ for a function f
• Conjecture (strong coloring number) : $\mu(G) \leq col(G)$
• Conjecture (weak planar): $\mu(G) \leq C$ for planar graph for a constant C
• Conjecture (strong planar): $\mu(G) \leq 4$ for planar graphs

Related results:
• Theorem (trees): $\mu(T) \leq 2$
• Theorem (planar): $\mu(G) \leq 6$ for planar graphs G with girth at least 14
• Theorem (max degree): $\mu(G) \leq e(\Delta + 1)$
• Lemma (max clique): $\omega(G) \leq \mu(G)$
• Theorem (paths): $\mu(P_n) = 2$ for $n \geq 2$

References: An extended list of results, conjectures, and sources is available online at `http://users.fit.cvut.cz/~blazeva1/res/bears/`

Problem 2. *Centered coloring of grids (suggested by Michał Dębski)*

Source: Proposed by Felsner, Micek and Schröder in 2019 year.

Definitions.
• *A vertex coloring* ψ *of a graph* G *is* p-centered *if for every connected subgraph* H *of* G *either* ψ *uses more than* p *colors on* H *or there is a color that appears exactly once on* H.
• *A graph* G *is a* grid *if it is a subgraph of a Cartesian product of two paths, i.e. vertices of* G *are pairs of integers and vertices* (x, y) *and* (x', y') *form an edge iff* $|x - x'| + |y - y'| = 1$.

Example:
1-centered coloring of G is a proper vertex coloring of G.

Question: Determine the minimum number $f(p)$ such that every grid has a p-centered coloring that uses at most $f(p)$ colors.

Related results:
• The answer is at most linear in p – from a general result for bounded degree graphs it follows that $f(p) \leq 16384p$.
• A trivial lower bound is that $f(p) \geq p$.

References:

M. Dębski, S. Felsner, P. Micek and F. Schröder, Improved bounds for centered colorings, arxiv:1907.04586, 2019.

M. Pilipczuk and S. Siebertz, Polynomial bounds for centered colorings on proper minor-closed graph classes. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '19, pages 1501–1520, 2019.

Problem 3. *Large monochromatic rectangle for function of short randomized protocol (suggested by Pavel Dvořák)*

Source: Proposed by Arkadev Chattopadhyay et al. [1]

Definitions.
Let f be a function $\{0,1\}^n \times \{0,1\}^n \to \{0,1\}$.
• *Cost of randomized protocol computing f is a number of bits, which Alice and Bob sends to each other to compute $f(x,y)$ with high probability when Alice gets x, Bob gets y and they can use a shared random bits.*
• *A matrix M_f is a boolean matrix of size $2^n \times 2^n$ and $M_f(x,y) = f(x,y)$.*
• Rectangle *R is a submatrix of M_f. The rectangle R is* monochromatic *if all entries of R are the same (0 or 1). The size of rectangle $|R|$ is the number of entries in R.*

Question: Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ have a randomized protocol of cost c. Is it true that M_f contains a monochromatic rectangle R of size $|R| \geq 2^{-O(c)} 2^{2n}$?

Related results:
• The question relates to the separation of communication classes P^{NP} and BPP.

- P^{NP} – class of functions computable by deterministic protocol of length polylog(n) with access to the NP oracle.
- BPP – class of functions computable by randomized protocol of length polylog(n).

References:

[1] Arkadev Chattopadhyay, Shachar Lovett and Marc Vinyals. Equality Alone Does not Simulate Randomness. 34th Computational Complexity Conference (CCC 2019).

Problem 4. *Linear space hypothesis for 2SAT$_3$ (suggested by Miloš Chromý):*

Source: Proposed by Tomoyuki Yamakami 2016.

Hypothesis: For any choice of $\varepsilon \in (0, 1)$ and any polylogarithmic function l, no deterministic Turing machine solves 2SAT$_3$ parameterized by number of variables n simultaneously in polynomial time using $n^{\varepsilon} l(|x|)$ space, where x refers to an input instance to 2SAT$_3$.

Definitions.
• *2SAT$_3$ is an instance of SAT, where each clause of an input CNF has at most two literals and each variable has at most three occurences.*
• *n is a paremetr for a problem P if for every input sequence x of a problem P, n can be computed in log-space and there exists a polynomial function p such that $n \leq p(|x|)$, where $|x|$ is a number of bits to represent an instance x.*

Related results:
• Undirected graph reachibility can be done in sublinear time .
• Polytime algortithm for a directed graph reachibilty using "sublinear" space $n^{1-c/\sqrt{logn}} l(m+n)$ where l is polylog function and m number of edges and n number of vertices. This algorithm can be used for implication graph of a 2SAT$_3$ instance.

Question: Can we get polytime algorithm with better space complexity?

References:

Yamakami, Tomoyuki, The 2CNF Boolean Formula Satisfiability Problem and the Linear Space Hypothesis. 2017. https://arxiv.org/abs/1709.10453.

Greg Barnes, Jonathan F. Buss, Walter L. Ruzzo, and Baruch Schieber. 1998. A Sublinear Space, Polynomial Time Algorithm for Directed s-t Connectivity. SIAM J. Comput. 27, 5 (October 1998), 1273-1282.

Problem 5. *Partitioning posets into chains and antichains (suggested by Vít Jelínek)*

Source: Proposed by Brandstädt and Kratsch (for permutations) in 1986.

Definitions.
• *A* poset *is a pair* $P = (X, \preccurlyeq)$ *where* X *is a set and* $\preccurlyeq$ *is a partial order on* X. *A* chain *in a poset is a set* $C \subseteq X$ *whose every two elements are comparable by* $\preccurlyeq$, *while an* antichain *is a set* $A \subseteq X$ *whose every two distinct elements are incomparable.*
• *A* permutation of size n *is a sequence* $\pi = \pi_1, \pi_2, \ldots, \pi_n$, *in which every number from the set* $\{1, \ldots, n\}$ *appears exactly once. To such a permutation, we may associate a partial order* $\preccurlyeq$ *by putting* $\pi_i \preccurlyeq \pi_j$ *if an only if* $i \leq j$ *and* $\pi_i \leq \pi_j$. *Notice that the chains of the resulting poset correspond to increasing subsequences of* π, *while the antichains correspond to decreasing subsequences.*
• *To* partition a poset $P = (X, \preccurlyeq)$ into k chains and ℓ antichains *means to find* k *chains* $C_1, \ldots, C_k$ *and* ℓ *antichains* $A_1, \ldots, A_\ell$ *in* P *such that*

$$X = C_1 \cup \cdots \cup C_k \cup A_1 \cup \cdots \cup A_\ell$$

(we may assume without loss of generality that $C_1, \ldots, C_k, A_1, \ldots, A_\ell$ *are all disjoint).*

Question: Is there an algorithm which, for a given poset P and a given integer k determines whether P can be partitioned into 1 chain and k antichains? And dually, is there an algorithm which determines whether P can be partitioned into 1 antichain and k chains? The problem is open even for permutation posets (for which the two questions become equivalent by symmetry).

Related results:

There is a simple polynomial algorithm which, for a given poset P, finds the smallest k such that P can be partitioned into k chains;

this k is also equal to the size of the largest antichain in P, by Dilworth's theorem. There is also an (even simpler) algorithm that finds an optimal partition of P into antichains. Moreover, for any two fixed constants k and ℓ, there is an algorithm which for a given poset P determines whether P can be partitioned into k chains and ℓ antichains, in time polynomial in $|P|$; this follows from a more general result of Kézdy, Snevily and Wang.

On the other hand, for k and ℓ part of the input, the previous problem is NP-complete, as shown by Wagner. Fomin et al. later showed that for a given P, the smallest value of $k + \ell$ for which such a partition exists can be approximated in polynomial time within a factor of $1 + 1/\sqrt{2} \approx 1.71$. It is not known whether this approximation ratio is optimal.

References:

A. Brandstädt, D. Kratsch: On partitions of permutations into increasing and decreasing subsequences, Journal of Information Processing and Cybernetics 22 (1986), 263–273.

F. V. Fomin, D. Kratsch. J.-C. Novelli: Approximating minimum cocolorings, Information Processing Letters 84 (2002), 285–290.

A. E. Kézdy, H. S. Snevily, C. Wang: Partitioning permutations into increasing and decreasing subsequences, Journal of Combinatorial Theory, Series A 73(2) (1996), 353–359.

K. Wagner: Monotonic coverings of finite sets, Journal of Information Processing and Cybernetics 20 (1984), 633–639.

Problem 6. *Dilworth number (suggested by Tereza Klimošová)*

Source: Proposed by Marthe Bonamy at Dagstuhl Workshop in June 2019.

Definitions.
• *For two vertices x and y, we say that x* dominates *y if $N[x] \supseteq N(y)$.*
• *Dilworth number $dilw(G)$ is the size of a largest set of pairwise incomparable vertices.*

Question: If $dilw(G)$ is bounded

- is coloring in P?
- is clique-width bounded?

Random facts about Dilworth numbers
• Graphs with Dilworth number at most four are perfect [1]. (Best possible—C_5 has Dilworth number five and is not perfect.)
• $dilw(G) = dilw(\overline{G})$
• Deciding whether a graph has Dilworth number k can be done in $O(k^2n^2)$ time [2].
• Some characterisation of graphs with Dilworth number k is given in [3].

References:

[1] Payan, Charles. "Perfectness and Dilworth number." Discrete Mathematics 44.2 (1983): 229-230.

[2] Felsner, Stefan, Vijay Raghavan, and Jeremy P. Spinrad. "Recognition algorithms for orders of small width and graphs of small Dilworth number." Technical report (1999).

[3] Calamoneri, Tiziana, and Rossella Petreschi. "On pairwise compatibility graphs having Dilworth number k." Theoretical Computer Science 547 (2014): 82-89.

Problem 7. *Size Ramsey number of a path (suggested by Tereza Klimošová)*

Source: Deepak Bal and Louis DeBiasio in [1].

Definitions.
• *Given a graph H, let $\hat{R}_r(H)$ be the minimum m such that there exists a graph G with m edges such that in every r-coloring of G, there is a monochromatic copy of H.*
• *We call $\hat{R}_2(H)$ the* size-Ramsey number *of H.*

Known
• $74n \geq \hat{R}_2(P_n) \geq (3.75 - o(1))n$ [1] and [2].

Question: What is the largest monochromatic path one can find in an arbitrary 2-coloring of a d-regular graph on n vertices?

Note that the upper bound on the size Ramsey number gives an upper bound for this question. Thus, the question is for which values of d one can do better.

References:

[1] Deepak Bal, Louis DeBiasio: New lower bounds on the size-Ramsey number of a path, arXiv:1909.06354.

[2] A. Dudek, P. Prałat. On some multicolor Ramsey properties of random graphs. SIAM Journal on Discrete Mathematics 31, no. 3 (2017): 2079–2092.

Problem 8. *Small elements in the colouring poset (suggested by Robert Lukoťka)*

Definitions.

• *A colour is an element from* $\{0, 1, 2\}$*. An* n*-colour-tuple is an element from* $\{0, 1, 2\}^n$*. We denote* i*-th colour,* $i \leq 0 < n$ *(indexing from zero), of an* n*-colour-tuple* t *as* t_i*. An* n*-colour-set is a set of* n*-colour-tuples.*

• *An* n*-colour-tuple satisfies the parity lemma if each colour is used the same number of times mod* 2*.*

• *An* n*-tuple-set* S *is proper if every element of* S *satisfies the parity lemma and if every element* t *of* S *allows Kempe switches, that is for each* $0 \leq i < n$*, and for each set* C *of two colours that includes* t_i*, there exists* $j \neq i$*,* $0 \leq i < n$*, such that* $t_j \in C$ *and if we change the colours of* t_i *and* t_j *to the other colour from* C*, the resulting* n*-colouring-tuple is in* S*.*

• *We can identify some* n*-colouring sets into equivalence classes, one can consider swapping of the colours irrelevant, and one can consider the order in the tuples irrelevant (that is to change the order of all tuples at once).*

• *A cubic* n*-network* $N = (G, o)$*, where* G *is a graph that has of* n *vertices of degree* 1 *(terminals) and vertices of degree* 3*, and* o *is an ordering of its terminals. We say that* N *admits a colouring tuple* t *if there exists a colouring such that the edge incident with* i*-th terminal. We say that* N *represents a colouring set if it admits exactly the tuples of the set.*

• *The set of all* n*-colouring-sets creates a c-colouring-poset.*

Every n-network represents some proper n-tuple-set. However, it is not known if every proper n-tuple-set is represented by some n-network. For $n < 6$ this statement holds. For $n = 6$ we do not have a representant of several n-colour-sets. Then case when $n = 6$ might be the only interesting case as no snark without non-trivial ≤ 6-cuts is known. As we are not able to find the 6-networks representing certain proper 6-tuple-sets, maybe one can somewhat restrict the notion of properness. This is, however, probably too much to ask I propose a partial problem to work on.

Problem: Pick a property that is satisfied only of n-colouring-sets that are very close to $\emptyset$ in the poset and try to characterize the networks representing this n-colouring set.

An interesting property is e.g. that the n-colouring-set is "uniquely-proper", that is we modify the definition of properness to "for every t and i there exists exactly one j". It seems intuitive, that in this setting, small edge-cuts should appear.

References:

J. Karabáš, E. Máčajová, R. Nedela: 6-decomposition of snarks, European Journal of Combinatorics 34 (2013), 111 – 122.

Problem 9. *Unbalanced flow values on cycles (suggested by Robert Lukoťka)*

Definitions.
• *A* $(\mathbb{Z}_2 \times \mathbb{Z}_2)$*-flow on a graph* G *is a mapping* $\varphi : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ *such that for every* $v \in V(G)$ *the sum of* φ *on all edges incident with* v *is zero.*
• *The support of* φ $(S(\varphi))$ *is the set of edges of* G *that have non-zero value in* φ*.*

Problem: Let G be a cubic graph and let C by a cycle of G. Let φ be a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow on G. Let φ' be a flow on G such that $S(\varphi) - E(C) = S(\varphi') - E(C)$. The aim is to choose φ' so that $|S(\varphi')|$ is as big as possible. The aim is to find a good bound on $|S(\varphi')|$ with respect to $E(C)$.

There are several interesting variants of this problem we can work on: planar, weighted, non-cubic etc.

Related results:
• One can easy show that there exists φ' such that $|S(\varphi')| \leq 1/4 \cdot |E(C)|$.
• If one can find a flow where all four flow values are not present approximately equally, one can modify this flow to contain few zero values (and thus, the flow has large support)
• Fan showed, that there exists φ' such that $|S(\varphi')| < 1/4 \cdot |E(C)|$.

References:
G. Fan, Integer 4-Flows and Cycle Covers, Combinatorica 37 (2017), 1097–1112.

Problem 10. *(List) 3-coloring of $2P_4$-free graphs (suggested by Jana Novotná)*

Definitions.
• *A graph is P_t-free if it does not contain a path on t vertices as an induced subgraph.*

Question: What is the time complexity of (list) 3-coloring of $2P_4$-free graphs? Can you find a polynomial algorithm?

Related results: • For every $k \geq 3$, if H is not a linear forest (disjoint union of paths), k-coloring is NP-complete on H-free graphs.
• List-3 coloring is polynomial on $P_6 + rP_3$-free and P_7-free graphs.
• Summary for P_t-free graphs:

	k-COLOURING				LIST k-COLOURING		
t	$k=3$	$k=4$	$k=5$	$k \geq 6$	$k=3$	$4 \leq k \leq 5$	$k \geq 6$
$t \leq 5$	P	P	P	P	P	P	P
$t = 6$	P	P	NP-c	NP-c	P	NP-c	NP-c
$t = 7$	P	NP-c	NP-c	NP-c	P	NP-c	NP-c
$t \geq 8$	?	NP-c	NP-c	NP-c	?	NP-c	NP-c

References:
• Maria Chudnovsky, Shenwei Huang, Sophie Spirkl, Mingxian Zhong:
List-three-coloring graphs with no induced P6+rP3, 2018.
• Klimošová, Malík, Masařík, Novotná,
• Flavia Bonomo, Maria Chudnovsky, Peter Maceli, Oliver Schaudt, Maya Stein, and Mingxian Zhong: Three-coloring and list three-coloring of graphs without induced paths on seven vertices, 2017.

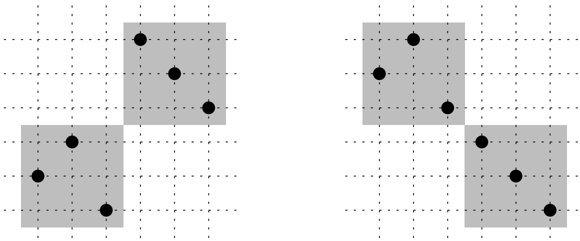
Problem 11. *Permutation superpatterns (suggested by Michal Opler)*

Source: Several variants mentioned by Engen and Vatter [4] and Bannister et al. [3].

Definitions.
• *A permutation is a sequence* $\pi = \pi_1, \pi_2, \ldots, \pi_n$ *in which each number from the set* $[n] = \{1, 2, \ldots, n\}$ *appears exactly once.*
• *A permutation* π *contains a permutation* σ*, if* π *has a subsequence of length* k *whose elements have the same relative order as the elements of* σ*, otherwise we say that* π *avoids* σ*, or* π *is* σ*-avoiding.*
• *A permutation class* $\mathcal{C}$ *is a down-set of permutations, i.e. if* π *is in* $\mathcal{C}$*, then all the permutations contained in* π *are in* $\mathcal{C}$ *as well.* $Av(\sigma)$ *denotes the class of* σ*-avoiding permutations.*
• *Given a permutation class* $\mathcal{C}$*, the permutation* π *is said to be* n*-universal for* $\mathcal{C}$ *if* π *contains all of the permutations of length* n *in* $\mathcal{C}$ *(the alternate term superpattern is sometimes used in the literature). A permutation is said to be simply* n*-universal if it contains all permutations of length* n*.*
• *Consider a pair of permutations* $\sigma = \sigma_1, \ldots, \sigma_k$ *and* $\tau = \tau_1, \ldots, \tau_\ell$*. The direct sum of* σ *and* τ*, denoted* $\sigma \oplus \tau$*, is the permutation* $\pi = \sigma_1, \ldots, \sigma_k, k+\tau_1, k+\tau_2, \ldots, k+\tau_\ell$*. Similarly, their* skew sum*, denoted* $\sigma \ominus \pi$*, is the permutation* $\ell+\sigma_1, \ldots, \ell+\sigma_k, \tau_1, \tau_2, \ldots, \tau_\ell$*.*
• *The class of separable permutations is the smallest non-empty class closed under taking direct and skew sums.*

Question: Investigate what is the shortest n-universal permutation for various permutation classes $\mathcal{C}$, in particular for the class of separable permutations.

Related results:
• To date, the best bounds on the size of the smallest n-universal permutation are that it lies between n^2/e^2 (a consequence of Stirling's Formula) and $\lceil \frac{n^2+1}{2} \rceil$, which was established by Engen and Vatter [4].



$231 \oplus 321 = 231654$ $\qquad$ $231 \ominus 321 = 564321$

Figure 1: Example of direct and skew sum.

• The exact formula for smallest n-universal permutation for the class of layered permutations, i.e. $\mathrm{Av}(231, 312)$ was given by Albert et al [1].
• Banninster et al. [2] constructed n-universal permutation for $\mathrm{Av}(213)$ of size $n^2/4 + \Theta(n)$ and n-universal permutation of size $O(n \log^{O(1)} n)$ for any proper subclass of $\mathrm{Av}(213)$.
• Moreover, Bannister et al. [2] proved a connection between the superpatterns of $\mathrm{Av}(213)$ and *universal point sets*, i.e. sets of points that can be used as vertices for straight-line drawings of all n-vertex planar graphs. Consequently, they obtained the current best upper bound for a size of universal point set of $n^2/4 - \Theta(n)$.
• Banninster et al. [3] constructed n-universal superpattern of size $O(n^{3/2})$ for $\mathrm{Av}(321)$.

References:

[1] Albert, M. H., Engen, M. T., Pantone, J. T., and Vatter, V. Universal layered permutations. *Electron. J. Combin.*, Vol. 25, 3 (2018).

[2] Bannister, M. J., Cheng, Z., Devanny, W. E., and Eppstein, D. A. Superpatterns and universal point sets. *J. Graph Algorithms Appl.*, Vol. 18, 2 (2014).

[3] Bannister, M., Devanny, W., and Eppstein, D. Small superpatterns for dominance drawing. In *ANALCO14 — Meeting on Analytic Algorithmics and Combinatorics*, (2014).
[4] Engen, M., and Vatter, V. Containing all permutations. arXiv:1810.08252 [math.CO].

Problem 12. *Characterizing class of directed graphs closed on $\chi'(D) = \Delta^{cycle}(D)$ (suggested by Aneta Šťastná)*

Source: Proposed by Sebastian Widerrecht in 2019.

Definitions.
• *D is a directed graph with digons [1] allowed.*
• $\deg^{cycle}(v)$ *is number of directed cycles containing v.*
• $\Delta^{cycle}(v) = \max\{\deg^{cycle}(v) \mid v \in V(G)\}$
• *$\chi'(D)$ is minimal number of colors to color the directed cycles of D such that no two intersecting cycles share a colour.*
• *$\mathcal{C}_D$ is a hypergraph derived from D,*

$$\mathcal{C}_D = (V(D), \{v \in C \mid C \textit{ directed cycle in } D\}).$$

• A *vertex shrink* is operation on hypergraph H where we remove a vertex v by removing it from all hyperedges.
• A hypergraph H is *balanced* iff $\chi'(H') = \Delta(H')$ for all hypergraphs H' obtained from H by edge deletions and vertex shrinks.
• A hypergraph H is *normal* iff $\chi'(H') = \Delta(H')$ for all hypergraphs H' obtained from H by edge deletions.
• A directed graph is *non-even* iff arcs can be 0-1-weighted such that each directed cycle has odd weight.

Question: Characterize class of graphs $\mathcal{D}$ such that directed graph D belongs to $\mathcal{D}$ iff $\chi'(D') = \deg^{\text{cycle}}(D')$ for all $D' \subseteq D$.

Related results:
• It is easy to observe that

$$\Delta^{\text{cycle}}(D) = \Delta\mathcal{C}_D \qquad (1)$$

$$\chi'(D) = \chi'(\mathcal{C}_D) \qquad (2)$$

• $\mathcal{D} \subsetneq$ non-even digraphs
• $\mathcal{D}$ is a minor-closed class.

[1] A *digon* is a pair of arcs $(u, v), (v, u)$ between $u, v \in V(D)$.

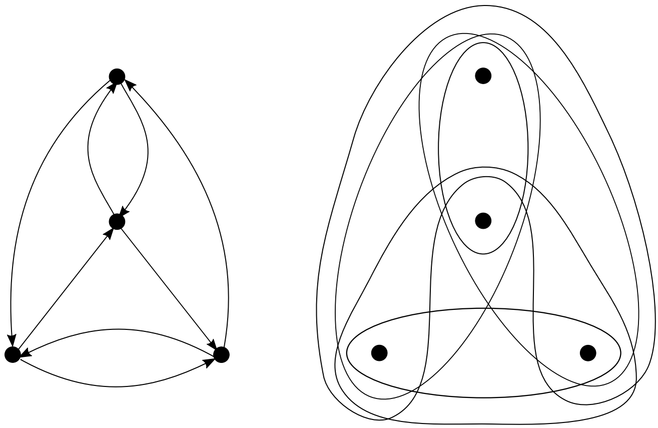


Figure 2: Digraph A_4 and it's corresponding cycle hypergraph $\mathcal{C}_{\mathcal{A}_\triangle}$

• Graphs A_i for $i \in 4, 6, \ldots$ are not in $\mathcal{D}$ and A_4 is a forbidden minor. The graph A_4 is shown on first figure of this problem together with it's cycle hypergraph. Other A_i's are obtained by replacing a bidirected edge at the bottom part of the figure by path containing edges of both orientation of length $i - 3$.
• $\mathcal{D}$ contains no odd bicycles. The smallest odd bicycle is shown on the second figure of this problem.
• Any results already known for balaced and normal hypergraphs can be of use.
• Directed treewidth is bounded for $\mathcal{D}$ (by directed grid theorem, see references).

References:

K. Kawarabayashi, K. Stephan: The Directed Grid Theorem, Proceedings of the Forty-seventh Annual ACM Symposium on Theory of Computing (2015).

Problem 13. *Chromatic number of cycle digraphs (suggested by Aneta Šťastná)*

Source: Proposed by Sebastian Widerrecht in 2019.

Definitions.
• *D is a directed graph with digons* [2] *allowed.*
• $\deg^{cycle}(v)$ *is number of directed cycles containing v.*
• $\Delta^{cycle}(v) = \max\{\deg^{cycle}(v) \mid v \in V(G)\}$
• $\chi'(D)$ *is minimal number of colors to color the directed cycles of D such that no two intersecting cycles share a colour.*
• $\mathcal{C}_D$ *is a hypergraph derived from D. $\mathcal{C}_D$ is defined as*

$$(V(D), \{v \in C \mid C \text{ directed cycle in } D\}).$$

Example:
On first figure of this problem we can observe a digraph A_4 and its corresponding cycle hypergraph $\mathcal{C}$. We can observe that hypergraph $\mathcal{C}$ is not colorable by 4 colors. A coloring using 5 colors is shown on the second figure of this problem.

Question: Prove $\chi'(\mathcal{C}_D) \leq \lfloor \frac{3}{2}\Delta(\mathcal{C}_D)\rfloor$ or a some other bound better than general using $\mathcal{C}_D \subsetneq \mathcal{H}$ for $\mathcal{H}$ class of all hypergraphs.

Given a hypergraph H, how can we decide if there exists a directed graph D such that $H = \mathcal{C}_D$?

Related results:
• It holds that $\chi'(H) \leq 2\Delta(H) - 1$ for any hypergraph H.
• There exists a conjecture that $\chi'(H) \leq \lfloor \frac{3}{2}\Delta(H)\rfloor$ for any hypergraph H.
• It is easy to observe that

$$\Delta^{\text{cycle}}(D) = \Delta\mathcal{C}_D \tag{3}$$
$$\chi'(D) = \chi'(\mathcal{C}_D) \tag{4}$$

[2] A *digon* is a pair of arcs $(u, v), (v, u)$ between $u, v \in V(D)$.

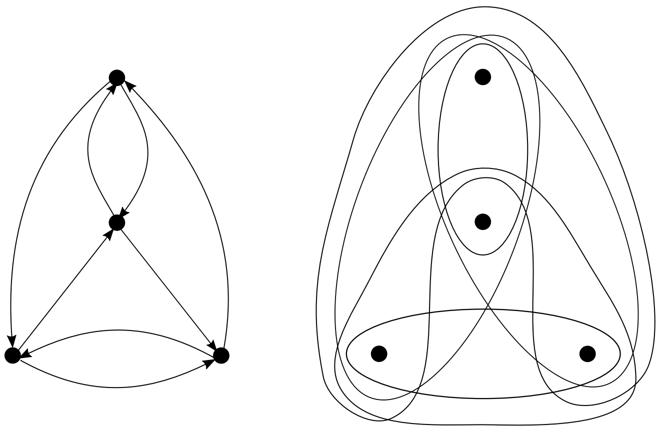


Figure 3: Digraph A_4 and it's corresponding cycle hypergraph $\mathcal{C}$

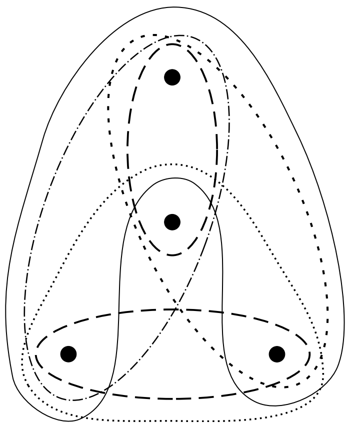


Figure 4: A coloring of $\mathcal{C}$ using 5 colors.