KAMAK 2018

Penzion Stará Škola, Strážné

September 16.–21.

Charles University, Prague

Organizers:

Pavel Dvořák
Robert Šámal

Brochure of open problems,
Prague, 2018.

Contents

Program

8:00 breakfast

9:00 morning session I

10:30 break

11:00 morning session II

12:30 lunch

15:00 afternoon session I

16:30 break

17:00 afternoon session II

18:30 progress reports

19:00 dinner

Wednesday is planned to be free to make an excursion in the neighborhood with everyone who would like to come.

OPEN PROBLEMS

Problem 1. *Unions of k-convex point sets (suggested by Martin Balko)*

Source: Proposed by Aichholzer et al. [1] in 2014.

Definitions.
• *For $k \geq 1$, a* k-convex polygon *is a simple polygon that is intersected by every line in at most k-connected components.*
• *For $k \geq 1$, a point set S in the plane in general position (that is, no three points of S lie on a common line) is* k-convex *if there is a k-convex polygon with vertex set S.*

Clearly, 1-convex polygons correspond to convex polygons and 1-convex sets to point sets in convex position. Note that every k-convex set is also $(k+1)$-convex. It is known that every subset of a k-convex point set is k-convex [1]. Aichholzer et al. [1] also showed that every set of n points in the plane in general position is k-convex for some $k \leq O(\sqrt{n})$ and this is tight for some point sets (a perturbed $\sqrt{n} \times \sqrt{n}$ grid, for example).

Question: Are there examples for general k and j such that the union of a k-convex point set and a j-convex point set is not $(k+j)$-convex?

Related results:
• Aichholzer et al. [1] gave an example of 1-convex sets S and T such that their union $S \cup T$ is not 2-convex; see the figure. They also showed that a union of a k-convex set and a j-convex set is always $(k+j+1)$-convex.
• Balko et al. [2] found examples of k-convex sets and j-convex sets such that their union is not $(k+j-1)$-convex.

References:

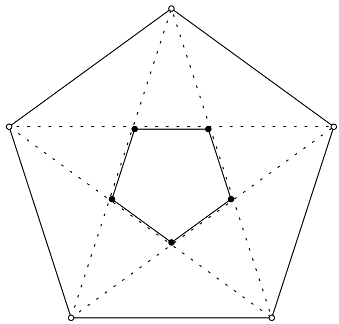

Figure 1: Union of two 1-convex point sets, each on five points, that is not 2-convex. The inner point set is contained in the open region bounded by the five dotted diagonals.

[1] Aichholzer O., Aurenhammer F., Hackl T., Hurtado F., Pilz A., Ramos P. et al. On k-Convex Point Sets. Computational geometry. 2014;47(8):809-832.

[2] Balko M., Bhore S., Sandoval L.M., Valtr P. On Erdős–Szekeres-type problems for k-convex point sets. Submitted.

Problem 2. *Online Ramsey number of trees (suggested by Václav Blažej)*

Definitions.
• *An online Ramsey game with the goal graph H is a game between Builder and Painter, alternating in turns. In each round Builder draws an edge and Painter colors it either red or blue. Builder wins if after some round there is a monochromatic copy of the graph H, otherwise Painter is the winner.*
• *An online Ramsey number $\widetilde{r}(H)$ is the minimum number of rounds such that Builder has a winning strategy in the online Ramsey game.*

Example:
• It is possible to create a monochromatic triangle in 8 moves by creating a S_5 and connecting leaves of the monochromatic S_3 with three edges.
• We can create a monochromatic path P_n by induction by having a red and blue path (possibly with length 0), joining them with an edge and creating another edge from a common endpoint. It's not hard to see that sum of their lengths increases which eventually causes one of them to have n edges.

Question: Show bounds on the $\widetilde{r}(T)$ where T is an arbitrary tree.

Related results:
• It is possible to create paths, brooms, spiders, centipedes, and few other tree classes in optimal time, however there is no polynomail time strategy for general trees.
• A general strategy for trees shown in 2004 by Grytczuk, Haluszczak and Kierstead shows that it is possible to create all forests on a class of forest background graphs. This strategy needs exponential number of rounds.

References:

• Jaroslaw Grytczuk, Mariusz Hałuszczak, and Hal A. Kierstead. On-line ramsey theory. *Electronic Journal of Combinatorics*, 11, 2004.

Problem 3. *Spanning Trees in Multipartite Geometric Graphs (suggested by Miloslav Brožek)*

Definitions.

• *We are given multiple points on plane. Each point has one of* $\boldsymbol{K}$ *colors assotiated with it. We have to connect the points by segments in such way, that it forms minnimum spanning tree (where edges are segments between points and their weight is the distance) while only points of different colors can be connected.*

Question: Is is possible to solve the problem faster than posed in results? Is the result also lower bound?

Related results:

• It was shown it is possible to solve such problem in $\mathcal{O}(Nlog(N)log(K))$ time.

References:

https://arxiv.org/pdf/1611.01661.pdf

Problem 4. *Inapproximability of Edit Distance (suggested by Pavel Dvořák)*

Let x, y be two strings over an alphabet Σ of length n.

Definitions.
• *Edit distance (ED) of x and y is the minimum number of insertion, deletion and substitution that transform x to y.*
• *Longest common subsequence (LCS) of x and y is the longest string s such that s is a substring of x and y.*

In 2017 Abboud et al.[1] showed that under SETH there is no $\left(2^{(\log n)^{1-o(1)}}\right)$-approximation for bichromatic version of LCS running in truly subquadratic time (i.e., $n^{2-\varepsilon}$).

Question: Show inapproximability result for edit distance.

Related results:
• For exact algorithms edit distance and LCS is the same problem, because $\mathrm{LCS}(x, y) = n - \mathrm{ED}(x, y)$.
• Under SETH, there is no exact algorithm for edit distance which runs in truly subquadratic time [2].
• There is a constant approximation for edit distance running in time $\tilde{O}\left(n^{2-2/7}\right)$ [3].

References:
[1] Amir Abboud, Aviad Rubinstein, R. Ryan Williams. Distributed PCP Theorems for Hardness of Approximation in P. FOCS 2017.
[2] Arturs Backurs, Piotr Indyk. Edit Distance Cannot Be Computed in Strongly Subquadratic Time (unless SETH is false). STOC 2015.
[3] Diptarka Chakraborty, Debarati Das, Elazar Goldenberg, Michal Koucký, Michael Saks. Approximating Edit Distance Within Constant Factor in Truly Sub-Quadratic Time, FOCS 2018.

Problem 5. *Subquadratic Conondeterministic Algorithm for Orthogonal Vector Problem (suggested by Pavel Dvořák)*

Definitions.
• *Let* $\langle \cdot, \cdot \rangle$ *be a standard inner product of vectors over* $\mathbb{R}$*.*
• *Nondeterministic strong exponential time hypothesis (NSETH): For every* $\varepsilon > 0$ *there exists* k *such that there is no nondeterministic algorithm for* k-TAUT *running in time* $2^{(1-\varepsilon)n}$*.*

Problem:	ORTHOGONAL VECTOR PROBLEM (OVP)
Instance:	$A, B \subseteq \{0,1\}^d, \vert A\vert = \vert B\vert = n, d = O(\log n)$
Question:	Are there orthogonal $x \in A$ and $y \in B$, i.e., $\langle x, y \rangle = 0$?
Problem:	k-TAUT
Instance:	k-CNF formula φ
Question:	Is φ a tautology?

Question: Find a truly subquadratic conondeterministic algorithm for OVP, i.e., a nondeterministic algorithm which runs in time $O(n^{2-\varepsilon})$ and answer yes if all pairs of vectors $x \in A$ and $y \in B$ are orthogonal.

Related results:
• An existence of truly subquadratic conondeterministic algorithm for OVP would refute NSETH [1].

References:
[1] Marco L. Carmosino, Jiawei Gao, Russell Impagliazzo, Ivan Mihajlin, Ramamohan Paturi, Stefan Schneider. Nondeterministic Extensions of the Strong Exponential Time Hypothesis and Consequences for Non-reducibility. ITCS 2016.

Problem 6. *Intersection of longest paths (suggested by Adam Kabela)*

We might study the intersecting of longest paths (that is, paths of maximum length). Clearly, every two longest paths of a connected graph have a vertex in common. In 1966, Gallai asked whether all longest paths of a connected graph share a vertex. This question was answered in the negative (for instance, see the graphs depicted in the figure below). However, many related questions remain open (some of them are listed below).

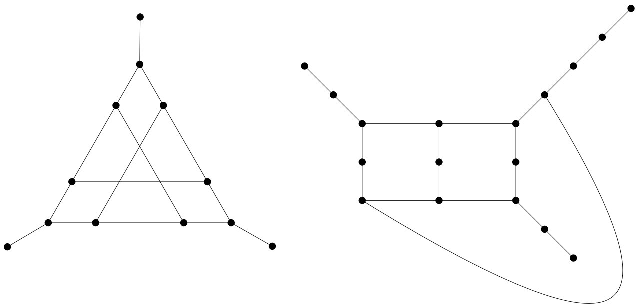

Figure 2: Two examples of connected graphs whose every vertex is missed by some longest path. Smallest such graph (left) found independently by Walther and Voss, 1974 and by Zamfirescu, 1976. Planar such graph (right) found by Schmitz, 1975 (in this graph, we can choose seven longest paths such that no vertex belongs to all of them).

Questions:

• Do all longest paths in a connected chordal graph have a common vertex? (Balister et al., 2004)

• What about three longest paths in a connected chordal graph?

• Do every three longest paths of a connected graph share a vertex? (Zamfirescu, 2001)

• What about six longest paths in a connected graph?
• Is there a 4-connected graph such that every vertex is missed by some longest path? (Zamfirescu, 2001)
• Is there a connected graph such that for every set of three of its vertices, there is a longest path missing all three? (Zamfirescu, 2001)

Related results: For more details, see the survey of Shabbir et al. and the references threrein.

References: A. Shabbir, C. T. Zamfirescu, T. I. Zamfirescu: Intersecting longest paths and longest cycles: a survey, Electronic Journal of Graph Theory and Applications 1 (2013), 56–76.

Problem 7. *Bipartite induced subgraphs (suggested by Tereza Klimošová)*

Source: Proposed by Louis Esperet, Ross J. Kang, Stéphan Thomassé [EKT].

The problem is motivated by a connection between minimum degree and *separation choosability* which I will not discuss here.

Conjecture 1. *There are functions f_1 and f_2 such that $f_1(d) \to \infty$ and $f_2(d) \to \infty$ as $d \to \infty$ such that any graph with minimum degree at least $d <$ contains a complete subgraph on $f_1(d)$ vertices or a bipartite induced subgraph with minimum degree at least $f_2(x)$.*

An interesting subcase of the conjecture are triangle-free graphs:

Problem 8. *Is there a function f such that every triangle-free graph with minimum degree d has an induced bipartite subgraph with minimum degree $f(d)$? (Where $f(d) \to \infty$ as $d \to \infty$.)*

In [EKT], it is shown that if f exists, $f(d) = O(\log d)$. Another question to ask in this setting is a connection to girth.

Conjecture 2. *There exist d_0 and g_0 such that any graph of girth at least g_0 with minimum degree at least d_0 contains a bipartite induced subgraph of minimum degree at least* 3*.*

Related results:
• For 3 is replaced by 2 in Conjecture 2, the statement is true with $g_0 = 4$ and $d_0 = 3$ [RV].
• Conjecture 1 holds for regular graphs and graphs with high minimum degree. This is implied by the following result from [EKT].

Theorem 1. *Any graph with chromatic number at most k and minimum degree d has a bipartite induced subgraph of minimum degree at least $d/2k$.*

References:

[EKT] Louis Esperet, Ross J. Kang, Stéphan Thomassé: Separation choosability and dense bipartite induced subgraphs. arXiv:1802.03727.
[RV] M. Radovanović and K. Vušković: A class of three-colorable triangle-free graphs. Journal of Graph Theory, 72(4):430–439, 2013.

Problem 9. *Difference between flow and circular flow number of signed graphs (suggested by Anna Kompišová)*

Source: Proposed by Raspaud and Zhu in 2011.

Definitions.
• *A* signed graph (G, σ) *is a graph* G *with signes* $(+/-)$ *on the edges (signature* σ*). Edges with plus sign are* positive *and edges with minus sign are* negative.
• *An* oriented *signed graph is bidirected graph. That means that every edge consists of two half edges oriented separately. Orientation has to fulfill a rule that half edges of positive and negative edge are oriented to the same or opposite directions, respectively.*
• *Let* A *be an additive group with zero. An* A-flow *is pair of orientation* O *and flow function* $f\colon E \to A$ *such that for every vertex* v*:*

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

Where $E^+(v)$ *and* $E^-(v)$ *are outgoing resp. incoming edges to* v.
• *A* nowhere-zero k-flow (O, f) *of the signed graph* (G, σ) *is an integer flow such that for every edge* e*:* $1 \leq |f(e)| \leq k-1$.
• *A* flow number $\Phi(G, \sigma)$ *of* (G, σ) *is the smallest* k *such that* (G, σ) *has nowhere-zero* k*-flow.*
• *A* circular r-flow (O, f) *of the signed graph* (G, σ) *is an* $\mathbb{R}$*-flow such that for every edge* e*:* $1 \leq |f(e)| \leq r-1$.
• *A* circular flow number $\Phi_c(G, \sigma)$ *of* (G, σ) *is the infimum of values* r *such that* (G, σ) *has circular* r*-flow.*
• *A* switching *at a vertex* v *is the operation which changes the signes of edges incident with* v *to the opposite signs.*
• *Signed graphs are* switching equivalent *if they have identical base graph and the signature of one graph can be switched to the signature of the other.*

Switching equivalent graphs have the same flow and circular flow number.

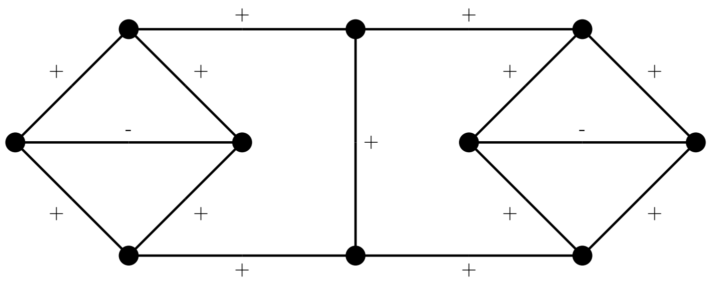

Figure 3: Signed graph (G, σ)

Example: Consider a signed graph (G, σ) at the figure 3.

This graph has circular 4-flow (figure 4) and does not have circural r-flow for $r < 4$. Therefore the circular flow number $\Phi_c(G, \sigma) = 4$.

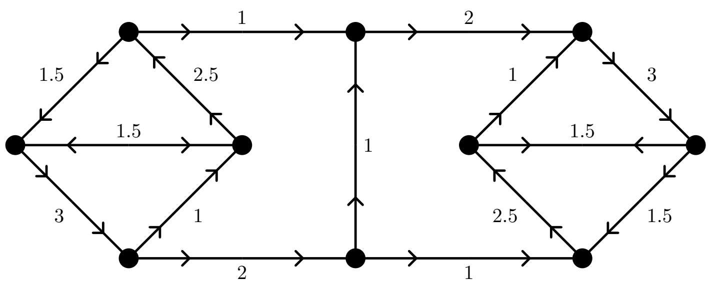

Figure 4: Circular 4-flow of (G, σ)

Similarly, the graph has nowhere-zero 5-flow (figure 5) and does not have nowhere-zero k-flow for $k < 5$. Therefore the flow number $\Phi(G, \sigma) = 5$.

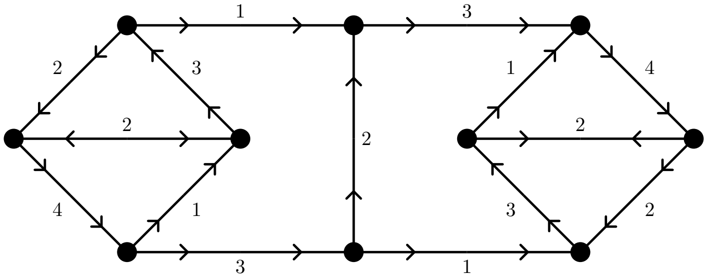

Figure 5: Nowhere-zero 5-flow of (G, σ)

The difference between flow and circular flow number can be 1. And this is the biggest number proved for signed cubic graphs.

Question: How big can the difference $\Phi(G,\sigma) - \Phi_c(G,\sigma)$ be, if the signed graph (G,σ) is cubic and $\Phi(G,\sigma) = 6$?

Related results:

• Let (G,σ) be a signed cubic graph:
  - $\Phi(G,\sigma) = 3 \Leftrightarrow \Phi_c(G,\sigma) = 3$.
  - $\Phi(G,\sigma) = 4 \Rightarrow \Phi_c(G,\sigma) = 4$
  - $\Phi(G,\sigma) = 5 \Rightarrow \Phi_c(G,\sigma) \in [4,5]$

For every combination of flow number and circular flow number allowed by these constraints there are infinitely many signed cubic graphs belonging to this category.

• Let (G,σ) be a signed cubic graph then $\Phi_c(G,\sigma) \notin (3,4)$.

• Known graphs with flow number 6 have circular flow number 6 as well.

• For every signed graph (G,σ): $\Phi_c(G,\sigma) \leq \Phi(G,\sigma)$.

• For every signed graph (G,σ): $\Phi(G,\sigma) \leq 2\lceil\Phi_c(G,\sigma)\rceil - 1$.

References:

• E. Máčajová, E. Steffen, The difference between the circular and the integer flow number of bidirected graphs, Discrete Mathematics

338 (2015), 866–867.
• A. Raspaud, X. Zhu, Circular flow on signed graphs, Journal of Combinatorial Theory, Series B 101 (2011), 464–479.
• M. Schubert, E. Steffen, Nowhere-zero flows on signed regular graphs, European Journal of Combinatorics 48 (2015), 34–47.

Problem 10. *Bound on the inducibility of cycles (suggested by Karel Král)*

Source: Dan Kráľ, personal communication.

Definitions.
• *Pippenger and Golumbic 1975 conjectured that every n-vertex graph has at most $n^k/(k^k - k)$ induced cycles of length $k \geq 5$.*

Question: Make the bound tighter.

Related results:
• Pippenger and Golumbic showed that the inducibility of every k-vertex graph H is at least $k!/(k^k - k)$. Where inducibility of a graph H is the limit of the maximum induced densities (number of induced copies of H in G divided by $\binom{|V(G)|}{|V(H)|}$) of H in an n-vertex graph.
• D. Kráľ, S. Norin, and J. Volec 2018 prove that every n-vertex graph has at most $2n^k/k^k$ induced cycles of length $k \geq 5$.

References:

The inducibility of graphs, N. Pippenger, M. C. Golumbic, J. Combinatorial Theory Ser. B 19 (1975), 189-203.

A bound on the inducibility of cycles, Kráľ, Norin, Volec, Arxiv https://arxiv.org/abs/1801.01556

Problem 11. *Rainbow parity matching (suggested by Martin Loebl)*

Source: Proposed by Martin Loebl in 2017.

Definitions.
• *Given pairs* $p_1, \ldots, p_k$ *of edges, a perfect matching is called* rainbow even *(REM) if it has an even number of edges from each* p_i*.*

Question: 1. Is there an algorithm for planar graphs with the running time asymptotically better than 2^g?

2. Are there classes of planar graphs (e.g. outer-planar graphs) for which deciding whether the graph has a REM is polynomial?

3. Relate REM problem to the Strong Exponential time hypethesis.

Related results:
• It is not difficult to show that existence of REM is NP-complete for the planar graphs.
• It is not difficult to relate REM in planar graphs to the Exponential time hypothesis.

Problem 12. *Exact algorithm for 3-edge coloring (suggested by Robert Lukoťka)*

Question: Find an exact algorithm for deciding 3-edge-colorability of a graph with asymptotically as good time as possible.

Question: Find an exact algorithm for deciding 3-edge-colorability of a cubic graph with asymptotically as good time as possible.

Related results:
• Eppstein's algorithm [1] uses a reduction to special CSP and has running time $O(2^{n/2})$ (which is $O(1.415^n)$).
• Kowalik's algorithm is asymptotically best up to date and has running time $O(1.334^n)$.
• On this year spring school I presented two approaches, how to attain running time $O(2^{n/2})$ relatively easily on cubic graph. Thus probably if we try hard, we should be able to improve Kowalik's algorithm's running time.

References:
[1] D. Eppstein: Improved Algorithms for 3-Coloring, 3-Edge-Coloring, and Constraint Satisfaction, SODA 2001.
[2] L. Kowalik: Improved edge-coloring with three colors, Theoretical Computer Science 410 (2009), 3733–3742.

Problem 13. *Tic-tac-toe on custom boards (suggested by Robert Lukoťka)*

Definitions.
• *A* **board** *is a finite subset of* $\mathbb{Z}^2$*. The elements of* B *are* **fields**
• *A* **three-in-a-row on a board** B *is a set of fields* $\{a, a+x, a+2x\}$*, where* $a, x \in \mathbb{Z}^2$ *and* $||x||_\infty = 1$ *(this is as in an ordinary tic-tac-toe game).*
• *A* **simple tic-tac-toe game on board** B *is a game played by two players. Each player has a set of fields he* **owns***. At the start of the game each player owns* $\emptyset$*. Players take turns. In his turn a player chooses a field not owned by either player and adds it to the set of fields he owns. The game ends when each field is owned by one of the players. First player wins if he owns a three-ina-a-row, otherwise second player wins.*
• *A* **full tic-tac-toe game on board** B *has the same rules as the simple tic-tac-toe game. However the game ends also whenever a player owns three-in-a-row.*

Question: Having a board B as an input, how hard is it to determine, whether the first player has a winning strategy in the simple tic-tac-toe game?

Question: Having a board B as an input, how hard is it to determine, whether the first player has a winning strategy in the full tic-tac-toe game?

Related results:
• There are many generalizations of tic-tac-toe game studied. But they are either played on a rectangular board [2] or a more general board [1]. One would guess that such a decision problem is either easy or hard, and it should be easy to distinguish.
• In 2016 (I guess) we spent some time on these problems during the Spring school. But, to our surprise, we were not able to find neither a simple algorithm nor a proof that these problems are NP-complete.

References:

[1] J. Beck: Games, Randomness and Algorithm, In: The Mathematics of Paul Erdös I, Springer, 2013.

[2] M. Y. Hsieh, S.-Ch. Tsai: On the fairness and complexity of generalized k-in-a-row games, Theoretical Computer Science 385 (2007), 88–100.

Problem 14. *Restricted Pattern Permutation Matching (suggested by Michal Opler)*

Source: Proposed by Albert et al. [1] in 2016.

Definitions.
• *A permutation is a sequence* $\pi = \pi_1, \pi_2, \dots, \pi_n$ *in which each number from the set* $[n] = \{1, 2, \dots, n\}$ *appears exactly once.*
• *A permutation* π *contains a permutation* σ*, if* π *has a subsequence of length* k *whose elements have the same relative order as the elements of* σ*, otherwise we say that* π *avoids* σ*, or* π *is* σ*-avoiding.*
• *A permutation class* $\mathcal{C}$ *is a down-set of permutations, i.e. if* π *is in* $\mathcal{C}$*, then all the permutations contained in* π *are in* $\mathcal{C}$ *as well.* $Av(\sigma)$ *denotes the class of* σ*-avoiding permutations.*

Permutation Pattern Matching (PPM)	
Input:	*A text permutation* τ *of size* n *and a pattern* π *of size* k*.*
Question:	*Does* τ *contain* π*?*

PPM and C-PPM

$\mathcal{C}$ Permutation Pattern Matching ($\mathcal{C}$-PPM)	
Input:	*A text permutation* τ *of size* n *and a pattern* π *of size* k*, both belonging to a fixed permutation class* $\mathcal{C}$*.*
Question:	*Does* τ *contain* π*?*

Example:
Question: For which permutations σ can Av(σ)-PPM be solved in polynomial time?

Related results:
• Bose, Buss, and Lubiw [2] showed in 1998 that the PPM problem

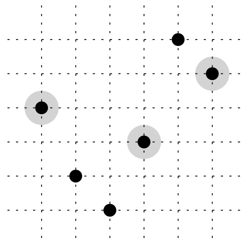

Figure 6: An occurence of pattern 213 in a permutation 421365.

is NP-complete. There is an FPT algorithm parametrized by the length of the pattern due to Marx and Guillemot [3].
• Guillemot and Vialette [4] showed that Av(321)-PPM is polynomially solvable and better algorithm was later provided by Albert et al. [1] who also proved the tractability of $\mathrm{Av}(2143, 3412)$-PPM.
• It was shown by Jelínek and Kynčl [5] that Av(4321)-PPM is NP-hard. Furthermore, their results imply that $\mathrm{Av}(\sigma)$-PPM is NP-hard for any σ of size at least 10.

References:

[1] Michael H. Albert, Marie-Louise Lackner, Martin Lackner, and Vincent Vatter. The Complexity of Pattern Matching for 321-Avoiding and Skew-Merged Permutations. *Discrete Mathematics & Theoretical Computer Science*, Vol. 18 no. 2, Permutation Patterns 2015, December 2016.
[2] Prosenjit Bose, Jonathan F. Buss, and Anna Lubiw. Pattern matching for permutations. In *Algorithms and data structures (Montreal, PQ, 1993)*, volume 709 of *Lecture Notes in Comput. Sci.*, pages 200–209. Springer, Berlin, 1993.
[3] Sylvain Guillemot and Dániel Marx. Finding small patterns in permutations in linear time. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 82–101. ACM, New York, 2014.

[4] Sylvain Guillemot and Stéphane Vialette. Pattern matching for 321-avoiding permutations. In *Algorithms and computation*, volume 5878 of *Lecture Notes in Comput. Sci.*, pages 1064–1073. Springer, Berlin, 2009.
[5] Vít Jelínek and Jan Kynčl. Hardness of permutation pattern matching. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 378–396. SIAM, Philadelphia, PA, 2017.

Problem 15. *A matching inequality (suggested by Robert Šámal)*

Source: Proposed by Eberhard Triesch at MCW 1997. Also by Jan Volec at MCW 2017.

Question: Suppose that $G = (V, E)$ and $G' = (V, E')$ are bipartite graphs on the same vertex set V and with the same 2-colouring $V = U \cup W$ where both colour classes U and W contain n elements. Assume further that for all $A \subset U$ the number of neighbours in G is at least as large as in G':

$$|N_G(A)| \geq |N_{G'}(A)| \quad \text{for all } A \subset U.$$

Then the number of perfect matchings in G is at least as large as the number of perfect matchings in G'.

Related results:

• The case where G' is 1-regular is just the Hall's "marriage theorem".

• The result would follow from a positive solution to Aharoni–Keich conjecture on a certain generalization of determinants.

References:

• `kam.mff.cuni.cz/~kamserie/serie/clanky/1997/s339.ps`

• and references therein.

• R. Aharoni, U. Keich, *A generalization of the Ahlswede–Daykin inequality*, Discrete Math. 152 (1996), pp. 1–12.

Problem 16. *Induced bipartite subgraph (suggested by Robert Šámal)*

Source: Proposed by Louis Esperet, Ross J. Kang, and Stéphan Thomassé
Question: There is a constant $C > 0$ such that any triangle-free graph with minimum degree at least d contains a bipartite induced subgraph of minimum degree at least $C \log d$.

Related results:
• Triangle-free graph with minimum degree at least 3 contains a bipartite induced subgraph of minimum degree at least 2. [RV]
• It is open, whether large girth and large minimum degree forces a bipartite induced subgraph of minimum degree at least 3.
• Triangle-free graph with n vertices and minimum degree at least d contains a bipartite induced subgraph of minimum degree at least $d^2/(2n)$. [BKVP]

References:
• [RV] Radovanović and Vušković: A class of three-colorable triangle-free graphs. Journal of Graph Theory, 72(4):430–439, 2013.
• [BKVP] Wouter Cames van Batenburg, Ross J. Kang, Rémi de Joannis de Verclos, and François Pirot: Bipartite induced density in triangle-free graphs, arXiv:1808.02512
• [EKT] Louis Esperet, Ross J. Kang, and Stéphan Thomassé: Separation choosability and dense bipartite induced subgraphs, arXiv:1802.03727

Problem 17. *Improving the best known algorithm for maximum independent set (suggested by Jakub Tětek)*

Definitions.

• *Independent set is a subset of vertices $I \subseteq V(G)$ of a graph G such that no two vertices in I are adjacent in G.*

• Asymptotic notation with * means that it is ignoring polynomial factors. For example $\mathcal{O}^*(2^n) = \bigcup_c \mathcal{O}(n^c 2^n)$.

The best currently known algorithm for finding a maximum independent set runs in time $\mathcal{O}^*(1.1996^n)$. All known algorithms for maximum independent set rely on the branch-and-reduce paradigm, meaning that reduction rules are applied and when no reduction rule can be applied, the algorithm branches on some specified vertex, recursively solving two subproblems (therefore leading to exponential running time). New ideas are needed if speedups are to be obtained without introducing overly complicated reduction and branching rules.

The goal would be to come up with a novel approach, most likely based on the branch-and-reduce paradigm, that would allow for faster algorithms or tighter analysis. It would also be very interesting to come up with an algorithm using some novel approach (not branch-and-reduce) even if it the running time would not be state-of-the-art.

Problem 18. *Circular colorings of cubic graphs with large girth (suggested by Jan Volec)*

Source: Hamed Hatami, 2005

Circular colorings and circular chromatic number.

For positive integers k and d satisfying $k \geq 2d$, let $K_{k/d}$ be the graph with the vertex-set $\{0, 1, \ldots, k-1\}$ so that x and y are adjacent if and only if $d \leq |x - y| \leq k - d$. Note that $K_{5/2}$ is isomorphic to C_5, and more generally, $K_{(2d+1)/d}$ is isomorphic to C_{2d+1}. Also note that if G is homomorphic to $K_{k/d}$, then it is also homomorphic to $K_{k'/d'}$ for all k' and d' satisfying $k'/d' \geq k/d$.

For a rational $x = k/d$, a circular x-coloring of G can be viewed as a homomorphism from G to $K_{k/d}$. The circular chromatic number $\chi_c(G)$ of a graph G is defined as the infimum of k/d such that G is homomorphic to $K_{k/d}$. It is known that for any graph G the infimum is attained, and $\chi(G) = \lceil \chi_c(G) \rceil$.

Question: Is it true that any 3-regular graph G with sufficiently large girth has $\chi_c(G) < 3$?

Conjecture *(Nešetřil's Pentagon Problem)*: Is it true that any 3-regular graph with sufficiently large girth is homomorphic to C_5?

Related results:

- There are 3-regular graphs with arbitrary large girth that are not homomorphic to C_7.

References:

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Problem 19. *Fractional Chromatic Number vs. Hall Ratio (suggested by Jan Volec)*

Fractional colorings and fractional chromatic number.

Let G be a graph and k a positive real. A *fractional k-coloring* is an assignment of measurable subsets of the interval $[0, k)$ to the vertices of G such that each vertex is assigned a subset of measure one and the subsets assigned to adjacent vertices are disjoint. The *fractional chromatic number of* G is the infimum over all positive real numbers k such that G admits a fractional k-coloring. Note that for finite graphs, such a real k always exists, the infimum is in fact a minimum, and its value is always rational. We let $\chi_f(G)$ be this minimum.

Let us mention a different way how to define χ_f. A simple averaging argument shows that $\chi_f(G) \geq n/\alpha(G)$. However, the argument stays valid even in the setting where the vertices have weights, and we measure the size of the independent set by the weight it takes rather than its cardinality.

Formally, let $G = (V, E)$ be a graph and $w : V \to \mathbb{R}$ a weight function. Let $\alpha(G, w)$ be the maximum sum of the weights of the vertices that form an independent set, i.e.,

$$\alpha(G, w) := \max_{I \in \mathcal{I}} \sum_{v \in I} w(v).$$

It holds that

$$\chi_f(G) = \max_{w:V\to[0,1]} \frac{\sum_{v\in V} w(v)}{\alpha(G, w)}.$$

Hall ratio.

As mentioned above, the ratio $v(G)/\alpha(G)$ is a lower-bound on $\chi_f(G)$. However, since a fractional k-coloring of G is also a fractional k-coloring of any $H \subseteq G$, we have that

$$\chi_f(G) \geq \frac{v(H)}{\alpha(H)}.$$

We define $\rho(G)$ — the *Hall ratio* of a graph G — to be the best lower-bound obtained in this way, i.e.,

$$\rho(G) := \max_{H \subseteq G} \frac{v(H)}{\alpha(H)}.$$

Clearly, $\rho(G)$ is attained by some induced subgraph $H \subseteq G$. Therefore,

$$\rho(G) = \max_{w:V\to\{0,1\}} \frac{\sum_{v\in V} w(v)}{\alpha(G,w)}.$$

Question 1: Is there a function $g : \mathbb{R} \to \mathbb{R}$ such that $\chi_f(G) \leq g(\rho(G))$ for every graph G?

Question 2: Is $\chi_f = O(\rho)$ for every Δ-free graphs?

Question 3: Is $\chi_f = O(\rho)$ for every K_4-free graphs?

Question 4: Is $\chi_f = O(d/\log d)$ for d-degenerate Δ-free graphs?

Question 5: Every Δ-free G with $\delta(G) \geq d$ contains an induced bipartite subgraph H with $\delta(H) = \Omega(\log d)$.

Related results:

• There exists $C_0 > 0$ such that for all $C \geq C_0$ there exists a graph G with $\rho(G) \leq C$ and $\chi_f(G) \geq e^{\log^2(C)/5}$.

• There exists $C_1 > 0$ such that for every $C \geq C_1$ there exists a K_5-free graph G with $\rho(G) \leq C$ and $\chi_f(G) \geq (C/20)^2$.

• If Question 2 holds, then Question 4 holds.

• If Question 4 holds, then Question 5 holds.

References:

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