KAMAK 2015

Hájenka na Rezku, Rezek

September 20.–25.

Charles University, Prague

Organizers:

Dušan Knop
Robert Šámal

Brochure of open problems,
Prague, 2015.

Contents

Program

8:30 breakfast

9:30 morning session I

10:30 break

11:00 morning session II

12:30 lunch

15:00 afternoon session I

16:30 break

17:00 afternoon session II

18:30 progress reports

19:00 dinner

Wednesday is planned to be free to make an excursion in the neighbourhood with everyone who would like to come.

OPEN PROBLEMS

Problem 1. *A swarm explores a tree (suggested by Martin Böhm)*

Source: Proposed by Sándor P. Fekete at Euro Summer Institute 2015, Szeged, Hungary.

A swarm of r robots starts at a root of an unknown tree with (unknown) non-negative lengths on the edges. Their task is to explore the tree. Whenever a robot arrives at a vertex, it learns the degree of this vertex, but the length of an edge is made known only when a robot traverses it completely (i.e. when it discovers the other vertex). The robot can change its direction whenever it wishes, even in the middle of an edge.

Every robot has its own unlimited energy source for their movement. We are interested the energy use of the robot that travelled the most distance. The reason we care about this performance measure is because in practice, every robot in the swarm will be outfitted with the same battery, and the energy use of the most-used robot dictates the battery capacity needed.

We compare our energy use to the best offline algorithm, i.e., the best possible utilization of the same number of robots when the tree is known in advance but needs to be traversed anyway.

Question: What is the best competitive ratio for the problem of r robots exploring a tree? Can we design a better lower bound than $3/2$ and a better algorithm than a greedy one? Can we design a good algorithm when the number of robots is fixed, such as $r = 3, 4$? Can we design a good algorithm on stars or other classes of trees?

Notes: There is a simple lower bound of $3/2$ on the competitive ratio, for general $r \geq 2$. There is also a tight strategy in the case of $r = 2$: think of two robots, each having its own colour and marking edges.

It was mentioned at the lecture that a greedy algorithm is 8-competitive but a proof was not given.

In this model, we assume robots can communicate with each other wirelessly (i.e. a robot can make a decision based on the entire explored tree so far and the state of all other robots).

Problem 2. *The k-defensive domination on cographs (suggested by Jiří Fiala)*

Source: Proposed by Jiří Fiala and Art Farley in 2015 year.

Definitions.
• *D_k is a* k-defensive set *in a graph G, if for every subset A of vertices of size k there exists an injective map $f : A \to D_k$ such that for all $u \in A$ either $u = f(u)$ or $u is a neighbor of f(u)$ (i.e. for k Attackers, the Defenders are on the attacked vertices or could move in there in one step).*
• *A graph is a* cograph *if it can be constructed recursively using the following rules:*

1. *single vertex graph is a cograph,*
2. *if $G = (V, E), H = (W, F)$ are cographs, then so is their disjoint union, that is the graph $(V \cup W, E \cup F)$,*
3. *if $G = (V, E), H = (W, F)$ are cographs, then so is their complete join, that is the graph $(v \cup W, E \cup F \cup \{\{v, w\} : v \in V, w \in W\})$.*

Question: The problem is to compute the minimum size of a set D_k for given cograph G and k.

Related results:
• For the cograph decomposition, it is easy to handle the disjoint union, since the minimum defense set for the disjoint union of G and G' is just simply the disjoint union of the defense sets of G and of G'. The tricky part is when performing the join between G and G'.
• The size of the defense set might drop significantly as there is an upper bound $2k$, if both G and G' have at least k vertices. It could be even less than the minimum of the two.
• When the size of one of the two parts is smaller than k, say $|G| = l < k$. It seems that the upper bound on is:

$$D_k(G \text{ join } G') \leq l + D_{k-l}(G').$$

Problem 3. *$\mathcal{A}$-defensive domination (suggested by Tomáš Gavenciak)*

Source: Extension of k-def. dom. concept of A. Farley and A. Proskurowski.

Definitions.
• *A (multi)set D defends against an attack (a multiset) A in graph G if there is an injective map $f : A \to D$ mapping vertices to distance at most 1. D*

is a defense agains a family $\mathcal{A}$ of attacks if it defends every $A \in \mathcal{A}$. The problem of $\mathcal{A}$-defensive domination is to find smallest such D for given G and $A \in \mathcal{A}$.
• *With control over multisets: Aditionally, D_{min} (pre-placed defenders) and D_{max} (vertex defender capacities) are given, and D has to satisfy $D_{min} \subseteq D \subseteq D_{max}$ (as multisets).*
• *A comparability graph of a poset has the elements as vertices and an edge between comparable elements. Co-comparability graph is a complement of a comparabiliy graph. This includes interval graphs, cographs, permutation graphs e.t.c.* •

Question: What is the complexity of $\mathcal{A}$-defensive domination on the following graph classes: bounded tree-width, bounded path-width, interval graphs, cographs and cocomparability graphs. Optionally with specified D_{min}. Optionally with $D_{max} \equiv 1$ (simple sets) or arbitrary given D_{max}.

Related results:

• $\binom{V}{k}$-def. dom. with $D_{max} \equiv 1$ is polynomial on trees [1].

• $\mathcal{A}$-def. dom. is polynomial on interval graphs with $D_{max} \equiv \infty$ [2].

• Cocomparability graphs are characterised by existence of cocomparability order $\preceq$: for any $a \preceq b \preceq c$ if $ac \in E$ then $ab \in E$ or $bc \in E$ (or both). For interval graphs this is: if $ac \in E$ then $ab \in E$. For unit interval graphs this is: if $ac \in E$ then both $ab \in E$ and $bc \in E$. See [3].

References:

[1] A. Farley, A. Proskutowski: *Defensive domination.*

[2] D. Dereniowski, T. Gavenciak, J. Kratochvil: *Cops, a fast robber and defensive domination on interval graphs.*

[3] D. G. Corneil, R. M. Krueger: *A unified view of graph searching.*

Problem 4. *Enumerating of Ramsey color avoiding numbers (suggested by Jaroslav Hančl)*

Source: Proposed by Po-Shen Loh in 2015.

Definitions.
• *Let $f(n)$ be the maximum number such that every 3-coloring of the edges of the n-vertex transitive tournament contains a directed path with at least $f(n)$ vertices, which avoids at least one of the colors.*

Question: Determine $f(n)$.

Related results:
• By merging two of the three color classes, one may obtain a bound $f(n) \geq \sqrt{n}$.
• Standard construction achieves $f(n) \leq n^{2/3}$ when n is a perfect cube.
• Best lower bound so far is due to Po-Shen Loh, who used close connection to Ruzsa-Szemerédi induced matching problem and obtained bound $f(n) \geq \sqrt{n}e^{\log^* n}$, where $\log^* n$ is the inverse of the tower function $T(n) = 2^{T(n-1)}$, $T(0) = 1$.

References:

Po-Shen Loh, *Directed Paths: from Ramsey to Ruzsa and Szemerédi*, arxiv.org/pdf/1505.07312.pdf

Problem 5. *The $(2,1)$ consecutive ones property testing (suggested by Dušan Knop)*

Source: Proposed by Murray Patterson at IWOCA 2013 open problem session.

Definitions.
• *We call a matrix* binary *if it is over the field* $\mathbb{Z}_2$.
• *Let A be an $m \times n$ matrix and $\pi : [n] \to [n]$ be a permutation. By a_i we denote the i-th column of matrix A. By A^π we denote the matrix whose i-th column is the column $a_{\pi(i)}$ (i.e. A^π is the matrix that arises from A by permuting the columns according to the permutation π).*
• *Let A be an $m \times n$ binary matrix. The matrix A has the* consecutive ones *property, if there exists a permutation (of columns of A) $\pi : [n] \to [n]$ such that the matrix A^π is of the form* $0^*1^*0^*$ *(i.e. at most one block of ones).*
• *(A generalization of the previous) Let A be an $m \times n$ binary matrix. The matrix A has the (p, q)* consecutive ones *property, if there exists a permutation (of columns of A) $\pi : [n] \to [n]$ such that the matrix A^π is of the form*

$$0^*(1^*0^{\leq q})^{\leq p-1}1^*0^*$$

(i.e. at most p blocks of ones with at most q zeroes between any two blocks).

Question: What is the time complexity of deciding the $(2,1)$ consecutive ones property?

Related results:

• There is a polynomial time algorithm [BoothLueker] that decides the consecutive ones property. Moreover there is a compact representation of all possible permutations π.
• Deciding the (p, q) consecutive ones property is NP-hard [Maňuch et al.] for all $p \geq 2, q \geq 1$ with the only (possible) exception $p = 2, q = 1$.

References:

[BoothLueker] K. S. Booth and G. S. Lueker. *Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms*, Journal of Computer and System Sciences, 13(3): 335–379, 1976.

[Maňuch et al.] J. Maňuch, M. Patterson and C. Chauve. *Hardness results for the gapped consecutive-ones property*, Discrete Applied Mathematics, 160(18): 2760–2768, 2011.

Problem 6. *The maximum edge q-coloring problem (suggested by Dušan Knop)*

Source: Proposed by Tommi Larjomaa and Alexandru Popa at IWOCA 2014 open problem session.

Definitions.
• *A* q edge coloring *of a graph* $G = (V, E)$ *is a function* $c : E \to [q]$.
• *A* maximum q edge coloring *of a graph* $G = (V, E)$ *is an edge coloring of the graph* G *such that for every vertex* $v \in V$*, all the edges incident with* v *have to be colored with at most* q *colors (i.e. the set* $C(v) = \{c(e) \colon v \in e\}$ *has at most* q *elements).*
• *A* min-max q edge coloring *of a graph* $G = (V, E)$ *is an maximum* q *edge coloring such that the* $\max_{v \in V} |C(v)|$ *is minimized.*

Questions:
• Is there an approximation algorithm for the maximum edge q coloring with competitive ratio better that 2?
• Find a lower bound on approximation ratio for the maximum edge q coloring assuming P$\neq$ NP (based on the PCP-theorem).
• Design an approximation algorithm for the min-max edge q coloring problem.

Related results:
• The maximum edge q coloring problem is NP-hard, and moreover, hard to approximate within a factor of 3/2 assuming the Unique Games Conjecture.
• No approximation algorithm is known for the min-max edge q coloring problem.

References:

• T. Larjomaa and A. Popa. *The Min-Max Edge q-Coloring Problem*, IWOCA, 2014. (Available on Arxiv: arXiv:1302.3404)

Problem 7. *The Pancake Problems (suggested by Peter Korcsok)*

Source: Proposed by Harry Dweighter in 1975.

Motivation.
We are given a stack of (possibly burnt) pancakes and we'd like to sort them by their sizes (and the burnt ones should be burnt-side down). The only allowed operation if to flip several top-most pancakes.

Definitions.
• *A* prefix *of a permutation or string is an initial segment.*
• *A* flip *is the reversal of a prefix.*
• *In a signed permutation, each element i occurs as i or i', and i' is treated as a negated element. When reversing a prefix of a signed permutation, the signs also reverse.*

Questions.
• What is the maximum number $f(n)$ of flips needed to sort a permutation of $[n]$ into ascending order?
• What is the maximum number $g(n)$ of flips needed to sort a signed permutation of $[n]$ into ascending order with all positive signs?
• Which permutation of $[n]$ needs the most flips to sort?

Related results.
• There are known exact values of $f(n)$ up to $n = 19$ and of $g(n)$ up to $n = 17$.
• In an average case, unsigned permutation can be sorted with at most $17n/12 + O(1)$ flips.

References.
• Harry Dweighter, Problem E2569, Amer. Math. Monthly 82 (1975) 1010.
• William H. Gates, Christos H. Papadimitriou, Bounds for sorting by prefix reversal, Discrete Math. 27 (1979) 45–57.
• Josef Cibulka, On average and highest number of flips in pancake sorting, Theoret. Comput. Sci. 412 (2011) 822–834.

Problem 8. *A Metric Group Product (suggested by Karel Král)*

Source: Proposed by Dylan McKay in 2015.

Question: Is there a function $f\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ ($\mathbb{R}_{\geq 0}$ are non-negative real numbers), such that $(\mathbb{R}_{\geq 0}, f)$ is both a group and a metric space?

Definitions.

- *The group axioms are:*
  - *Closure:* $f(a,b) \in \mathbb{R}_{\geq 0}$
  - *Identity: there exists* e *such that for all* $a \in \mathbb{R}_{\geq 0}\colon f(a,e) = f(e,a) = a$
  - *Associative: for all* $a,b,c \in \mathbb{R}_{\geq 0}\colon f(f(a,b),c) = f(a,f(b,c))$
  - *Inverse: for all* $a \in \mathbb{R}_{\geq 0}$ *there is* $b \in \mathbb{R}_{\geq 0}$ *such that* $f(a,b) = e$.
- *The metric axioms are:*
  - $f(a,b) = 0$ *if and only if* $a = b$
  - $f(a,b) = f(b,a)$
  - $f(a,b) \leq f(a,c) + f(c,b)$

Example:
Xor of binary representations does not work because the representation is not unique (see the reference for more information).

Related results:
- Xor of binary representations works for natural numbers.

References:
`http://blog.computationalcomplexity.org/2015/06/a-metric-group-product.html`

Problem 9. *Parity Matching problem (suggested by Martin Loebl)*

Source: Proposed by Marcos Kiwi and Martin Loebl in 2015.

Definitions.
- *Parity Matching problem* $P(G, w, p_1, \ldots, p_g, k)$ *is as follows: Given integer* k*, graph* G *with* n *vertices, positive integer weights* $w(e)$ *on edges and* g *disjoint pairs of edges* $p_1, \ldots, p_g$*, find out if there is a feasible perfect matching or total weight at least* k*; perfect matching* M *is feasible if it has an even number of edges from each* $p_i, i = 1, \ldots, g$.

Question: No deterministic or probabilistic algorithm for $P(G, w, p_1, \ldots, p_g, k)$ has complexity less than $poly(n)2^g$.

Related results: I can explain that MaxCut problem for G with crossing number g can be reduced to the Parity Matching problem for $2g$. That can be solved by calculating linear combination of 2^{2g} Pfaffians. After attempts to find more straightforward algorithm, we start to think that no significantly better way exists. The Parity Matching problem pinpoints the complexity of MaxCut for graphs embedded on surfaces.

There is a straightforward reduction of $P(G, w, p_1, \ldots, p_g, k)$ to 2^g weighted perfect matching problems. I find it very surprising (but still we conjecture it) that no better algorithm should exist.

We have some very pleriminary results very weakyl supposting the conjecture.

Problem 10. *Reduction networks (suggested by Martin Loebl)*

Source: Proposed by Andrea Jimenez and Martin Loebl in 2014.

A reduction network is a communication network consisting of interconnected parts where each part has to perform some task. The elements of each part need to communicate in order to successfully perform the task. The parts are linearly ordered, and after a part performs its task, the part is no more directly functional and it is reduced; that is, the communication network is updated in such a way that the reduced part is removed and only its residue remains. The goal is to make reductions whose residues help communication in yet functional parts.

Definitions.

• *The reduction network is modeled by an undirected cubic graph G, and its linearly ordered parts by the ears of an ear-decomposition of G. The order of the parts is reversed order of the ears. Each ear decomposition starts with an induced cycle and each ear is a path of 2, 3 or 4 edges or a star of 3 edges; in particular, an edge is not an ear. Let us describe reductions of a given reduction network.*

At the initial step*, the first part (last ear: must be a star since the graph is cubic) is reduced and the updated communication network becomes a mixed graph; that is, we delete from G the inner vertices of the last ear, and add between remaining vertices directed edges (arcs) which form the residue of the initial reduction. We also identify pairs of directed edges which 'do not like each other' and we put them into set R which is updated during the reduction*

of the reduction network. Initially, $R = \emptyset$. Hence, mixed graph is 4-tuple (V, E, A, R) where A is the set of its arcs.
At each later step, *given current mixed graph and current ear L to be reduced, we do the following:*
(1) If L is path of 4 edges and the current mixed graph has no arc between its inner vertices then we ALARM: all work and hope is lost, the network will explode in five seconds and destroy the Earth.
(2) Otherwise we reduce the ear, if it has a reduction (to be described later). If there is no possible reduction, we leave the network to its fate and collect the money.
(3) If the current mixed graph consists of the initial cycle and some arcs only, we also collect the money.
We need to say what is reduction of an ear in a mixed graph (V, E, A, R).
• *Let H be the set of all the orientations of the edges of the ear L we want to reduce. Hence H has each edge of L oriented twice. We first update R by adding all pairs of new opositely directed arcs. Let H' consists of the arcs of H along with the arcs of A incident to an inner vertex of L. We partition H' into directed paths and directed cycles so that no pair of R belongs to the same path or cycle. Finally, we replace each directed path by the arc from its initial vertex to its terminal vertex, and delete the inner vertices of the ear. We note that a reduction of an ear need not exist.*
• *The absolute hope would be that each reduction network has a reduction which ends with money collection, but this is not true. Luckily, we can relax this hope considerably.*
(1) We assume that the descend *of each ear L of 4 edges is connected; the descend of L is induced subgraph on the subset W of inner vertices of later ears; we put v to W if it is inner vertex of a later ear, and there is a path of edges of later ears that connects v to an inner vertex of L.*
(2) In fact, we can even change the input graph: Let $L = L_0$ be an ear of 4 edges and let $L_1, \ldots, L_k$ be a sequence of later ears which are paths of 3 edges and for each $i = 1, \ldots, k$, the terminal vertices of L_i are inner vertices of L_{i-1}. Let V_L be the set of the inner vertices of the ears later than L_k. The inner vertices of $\cup L_i$ are connected by exactly three edges to V_L, since G is cubic. Now, look at this: we can arbitrarily change the end-vertices of these three edges along the inner vertices of degree 2 of $\cup L_i$.

Question: Does every network have a money collection reduction?

Related results:
• This comes from the study of the directed cycle double cover conjecture.
• It is true, even without assuming (2), for planar networks: this I can explain.

Problem 11. *Disjoint segments (suggested by Viola Mészáros)*

Source: J. Pach's website.

There is a collection of n pairwise disjoint line segments in the plane in general position. We want to build a path using some of these segments. The path is built so that between chosen endpoints of the given segments we add rectilinear edges. Hence, in the path every other edge belongs to the given segments and every other edge is added later.

Question: What is the maximum $k = k(n)$ that it is possible to build a noncrossing path of size k in any given set of n pairwise disjoint line segments?

Related results by J. Pach and R. Pinchasi:
• Upper bound: there is a set of n segments where no subset of size $2n^{1/2}$ or more can be completed to such a path.
• Lower bound: in any set of n pairwise disjoint line segments one can find a subcollection of $\Omega(n^{1/3})$ segments that can be completed to a noncrossing path.

References:

Paper 191 at `http://cs.nyu.edu/~pach/`

Problem 12. *Woodall's Conjecture (suggested by Viola Mészáros)*

Source: Open Problem Garden, posted in 2007 by M. DeVos.

Let G be a directed graph.

Definitions.
• *A directed cut of* G *is a cut where all edges are directed in the same way.*
• *A dijoin is a set of edges that intersects every directed cut.*
• *A strongly connected digraph is a directed graph in which it is possible to reach any node starting from any other node by traversing edges in the direction in which they point.*

Conjecture: If G is a directed graph with smallest directed cut of size k, then G has k disjoint dijoins.

Related results:
• There seem to be few positive results towards Woodall's conjecture for general digraphs. Seymour and DeVos observed that the conjecture is true for $k = 2$. To see this, note that the underlying graph is 2-edge-connected,

so it may be oriented to give a strongly connected digraph H. Now partition the edges into two sets $\{X, Y\}$ where X consists of those edges which have the same orientation in both H and G, and Y is the rest. It is immediate that both X and Y meet every directed cut, so each is a dijoin. Extending this to $k = 3$ appears to be difficult.

References:

`http://www.ime.usp.br/~pf/dijoins/woodall/survey1-en.pdf`
`http://integer.tepper.cmu.edu/webpub/dijoin.pdf`

Problem 13. *First Selection Theorem for Boxes (suggested by Saurabh Ray)*

Source: This is one of the problems studied in the reference below that has not been completely resolved.

Question: Any two points $p, q \in \mathbb{R}^d$ induce an axis parallel box of which p and q are diagonally opposite corners. Thus n points in $\mathbb{R}^d$ induce $\binom{n}{2}$ such boxes - one for each pair of points. What is the largest constant α_d (assuming d is fixed), so that given any set of n points in $\mathbb{R}^d$, at least $\alpha_d n^2 - o(n^2)$ of the $\binom{n}{2}$ boxes induced by them have a common intersection? Even for $d = 3$, the right bound is not known.

Related results:

- In two dimensions the exact answer is known, $\alpha_2 = \frac{1}{8}$. Given n points in the plane, we can take a vertical line and a horizontal line, each of which bisect the point set. It can be easily show that the intersection point of these two lines is in at least $n^2/8$ of the induced rectangles. On the other hand, it can be shown that if the n points are placed at the corners of a regular n-gon, then no point in the plane is contained in more than $n^2/8$ of the induced rectangles.
- In higher dimensions, there is a large gap in the bounds known. It is known that $\frac{1}{2^{2^d-1}} \leq \alpha_d \leq \frac{1}{2^{d+1}}$. Even in three dimensions, the right bound is not known.
- A well studied problem in this area asks the same question for simplices spanned by $(d + 1)$ points in $\mathbb{R}^d$. It is conjectured that among the $\binom{n}{d+1}$ d-simplices (convex hull of $d + 1$ points) induced by n points in $\mathbb{R}^d$, at least $\left(\frac{n}{d+1}\right)^{d+1}$ have a common intersection. It is know that this bound cannot be improved. The statement is known to be true in two dimensions: the center-point of any set of n points is contained in $n^3/27$ of the $\binom{n}{3}$ triangles induced by the points and there are point sets where no point in the plane is contained in more than these many induced triangles. Already in three dimensions, the

problem becomes very challenging and has so far defied solution despite the development of sophisticated topological machinery for improving the lower bound.

References: P. Ashok, N. Rajgopal, S. Govindarajan: *Selection Lemmas for various geometric objects* (`http://arxiv.org/pdf/1401.0443v1.pdf`), 2014.

Problem 14. *Counting generalized permutations (suggested by Robert Šámal)*

Source: Proposed by Linial in 2012

We identify permutations with the corresponding permutation matrix: an $n \times n$ array of 0s and 1s, with exactly one 1 per every line (row or columns). We generalize this concept as follows: a d-dimensional permutation is an array of $n \times \cdots \times n = n^{d+1}$ zeros and ones, with exactly one 1 per every line (a set of n entries with d indices fixed and the remaining one going over all possible n values).

Conjecture: The number of d-dimensional permutations of order n is

$$\left((1 + o(1))\frac{n}{e^d}\right)^{n^d}.$$

Related results:

• The result is true for $d = 1$ (Stirling formula).
• The result is true for $d = 2$ (Latin squares).
• It is true that the number of d-dimensional permutations is at most the given quantity. [LL] Thus the goal is to provide a matching lower bound.

References: [LL] Linial, Luria: An upper bound on the number of high-dimensional permutations, Combinatorica, 2014

Problem 15. *Four-peg Hanoi towers (suggested by Robert Šámal)*

Source: Proposed by H.E. Dudeney in 19008.

The puzzle of Hanoi towers is usually played with three pegs and n disks of distinct sizes. Originally, all disks are on the first peg, they are to be moved to the last one in as small number of moves as possible, while at no time is a larger disk put over a smaller one. It is a simple exercise that the minimum number of moves is exactly $2^n - 1$. The question is what happens, when we have four pegs; we let Q_n denote the minimum number of moves that are required.

Problem: Determine Q_n.
Conjecture: An optimal sequence of moves is given by a recursive procedure, described below (so-called Frame–Stewart algorithm.)
Related results:
• [Sz] $Q_n \leq 2^{(1+o(1))B_k n^{1/(k-2)}}$ (k is the number of pegs, i.e., $k = 4$)
• [Sz] $Q_n \geq 2^{(1+o(1))C_k n^{1/(k-2)}}$
• [F],[S] An upper bound Q_n: pick $m \leq n$. Perform Q_m moves to transfer the top m disks to the second peg, then do the usual 3-peg Hanoi tower algorithm on the remaining three pegs and finally use another Q_m moves to finish. It is conjectured, that this (for some m) is actually an optimal way. (But not a unique one! Even the best b is not always unique.)

References: [Sz] Mario Szegedy, In How Many Steps the k Peg Version of the Towers of Hanoi Game Can Be Solved? STACS'99, LNCS 1563, pp. 356–361, 1999.

[F] J.S. Frame, A Solution to AMM Problem 3918 (1939), American Mathematical Monthly, vol. 48, pp.216–217, 1941.

[S] B.M. Stewart, Solution to Problem 3918, American Mathematical Monthly, vol. 48, pp. 217–219, 1941.

[KMP] Sandi Klavžar, Uroš Milutinović, Ciril Petr, On the Frame–Stewart algorithm for the multi-peg Tower of Hanoi problem, Discrete Applied Mathematics, Volume 120, Issues 1–3, pp. 141–157

Problem 16. *Independent spanning trees (suggested by Robert Šámal)*

Source: Proposed by Itai and Rodeh 1984
Conjecture 1: Given a vertex k-connected graph G and a vertex $d \in V(G)$, there exists spanning trees $T_1, \ldots, T_k$ in G such that for every $v \in V(G)$, all k paths from v to s (in $T_1, \ldots, T_k$) are *internally vertex disjoint.*
Conjecture 2: Given an edge k-connected graph G and a vertex $d \in V(G)$, there exists spanning trees $T_1, \ldots, T_k$ in G such that for every $v \in V(G)$, all k paths from v to s (in $T_1, \ldots, T_k$) are *edge disjoint.*

For the last conjecture we need to explain a concept of local routing tables. Given a graph G, routing tables consist of a sequence of mappings. For each vertex $v \in G$ we have a mapping $f_v : \delta(v) \to (\delta(v))^k$; that is, f_v assigns to each edge incident to v a sequence of k such edges. To *use the routing tables*, we construct a walk in the graph. Given a vertex v and edge e by which we enter v, we decide to leave v by the first available edge in $f_v(e)$ – an edge may not be available, we are assuming some edges were damaged. Then we repeat, until we (hopefully) reach d.

Conjecture 3: Given an edge k-connected graph G and a vertex $d \in V(G)$, there exist routing tables that lead from every start to d, provided $< k$ edges are damaged.

Related results:

- [IR] Conjecture 1 is true for $k = 2$.
- [ZI] Conjecture 1 is true for $k = 3$.
- [N] Conjecture 2 is true for $k \leq 4$.
- Conjecture 1 implies Conjecture 2.

References: [IR] Itai, Rodeh, The multi-tree approach to reliability in distributed networks, 25th FOCS, 1984, 137–147, (also in Info. and Comput.)

[ZI] Zehavi, Itai: Three tree-paths. J. Graph Theory 13 (1989), no. 2, 175–188.

[N] Jitka Novotná: diploma thesis.

Problem 17. *Free flows (suggested by Robert Šámal)*

Source: Proposed by Nešetřil and Šámal

Let G be a digraph, M an abelian group, $\varphi : E(G) \to M$ a flow. We say φ is p-free if $\varphi(e_1)+\cdots+\varphi(e_k)$ is never zero for $1 \leq k \leq p$ and $e_1, \ldots, e_k \in E(G)$.

Conjecture For every $p \geq 1$ exists a k_p such that any orientation of a $(p+1)$-edge-connected graph has a p-free flow $\varphi : E(G) \to M$ where M is a group of order $\leq k_p$.

Related results:

- It is true for $p = 1$ (Jaeger) and for $p = 2$ (DeVos, Johnson and Seymour).
- It is true for every p if we assume the graph is $(2p + 1)$-edge connected.

Problem 18. *Magic square of squares (suggested by Robert Šámal)*

Source: This question was first asked in 1984 by Martin LaBar and popularized in 1996 by Martin Gardner, who offered \$100 to the first person to construct such a square.

A 3×3 square is called *magic* if all three columns, all three rows, and both diagonals have the same sum.

Question Is there a magic square in which all elements are squares of distinct positive integers?

Reference:

- Christian Boyer. Some notes on the magic squares of squares problem. The Mathematical Intelligencer 27 (2005), 2, 52–64.
- Paul Pierrat, Franc Ois Thiriet, Paul Zimmermann: Magic squares of squares. `http://www.loria.fr/~zimmerma/papers/squares.pdf`

Problem 19. *Regular systems of vectors (suggested by Robert Šámal)*

Source: Proposed by William Martin at CanaDAM (and probably famous in the crypto community).
Challenge 1 Find, as many as you can, equiangular lines in $\mathbb{C}^d$ (unit vectors whose inner products have constant absolute value). Find d^2, if possible.
Challenge 2 Find, as many as you can, equiangular lines in $\mathbb{R}^d$ (unit vectors whose inner products have constant absolute value). Find $\binom{d+1}{2}$, if possible.
Challenge 3 Find, as many as you can, orthonormal bases in $\mathbb{C}^d$ where unit vectors from distict bases have inner product with constant absolute value. Find $d + 1$, if possible.
Challenge 4 Find, as many as you can, orthonormal bases in $\mathbb{R}^d$ where unit vectors from distict bases have inner product with constant absolute value. Find $\frac{d}{2} + 1$, if possible.
Related results:
• Challenge 1 is solved for $d = 2, 3, 8$ using Hadamard matrices. (Jedwab and Wiede, 2014)

Problem 20. *Dissecting the square into congruent copies (suggested by Martin Tancer)*

Source: Proposed by Yuan, Zamfirescu and Zamfirescu [YYZ]; goes back to questions of Danzer in 1980's .

Question: Let p be an odd prime number. Is there any other dissection of the square into p congruent convex polygons apart from the standard one? (See Figure 1.)

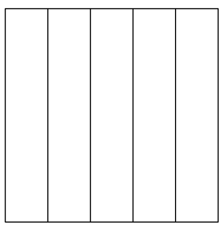


Figure 1: The standard dissection for $p = 5$.

Related results:

• The answer is known to be 'no' for $p = 3$ [Mal] and for $p = 5$ [YYZ]. The first open case is therefore for $p = 7$. It might also be of interest to avoid a complicated case analysis for $p = 5$.

References:

[Mal] Samuel J. Maltby. *Trisecting a rectangle*, Journal of Combinatorial Theory, Series A, 66(1):4052, 1994.

[YYZ] Liping Yuan, Carol T. Zamfirescu, and Tudor I. Zamfirescu. *Dissecting the square into five congruent parts*, Discrete Mathematics, 339(1):288–298, 2016.

Problem 21. *No three points on a line in discrete toroidal grid (suggested by Martin Tancer)*

Source: Essentially Misiak et al. [MSS+] .

Definitions.

• *By the* discrete toroidal $m \times n$ grid *we mean the quotient of* $\mathbb{Z}^2$ *under identifications* $(x, y) \sim (x', y)$ *and* $(x, y) \sim (x, y')$ *whenever* $x \equiv x' \pmod{m}$ *and* $y \equiv y' \pmod{n}$.

• *A* line *in such a grid is an image of a line in* $\mathbb{Z}^2$ *under this quotient. See Figure 2.*

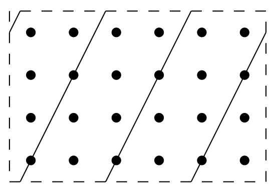


Figure 2: The discrete toroidal 6×4 grid and six points on a same line in this grid.

Question: What is the maximal number of points of the discrete toroidal $m \times n$ grid that can be selected so that no three of them are on a line?

Related results:

• Let $v(m, n)$ be the value in the question. Misiak et al. [MSS+] show that $v(m, n) \leq 2\gcd(m, n)$ (easy) and moreover $v(m, n) = \gcd(m, n)$ if $gcd(m, n) \in \{1, 2\}$ or if $gcd(m, n) = p$ where p is a prime number and at least one of the values m, n is divisible by p^2 (still quite easy). On the other hand they show that $v(m, n) = p + 1$ if $\gcd(m, n) = p$ where p is a prime number and none of the values m, n is divisible by p^2.

References:

[MSS+] Aleksander Misiak, Zofia Stepień, Alicja Szymaszkiewicz, Lucjan Szymaszkiewicz, and Maciej Zwierzchowski. *A note on the no-three-in-line problem on a torus*, Discrete Mathematics, 339(1):217–221, 2016.

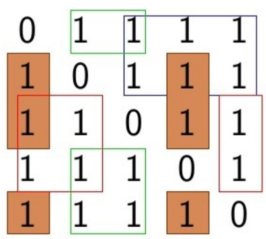
Problem 22. *Rectangle covering bound for Unique Disjointness (suggested by Hans Raj Tiwary)*

Source: See reference.

Let $n \geqslant 1$ be a natural number and let $N = 2^n$. Let M be a $N \times N$ matrix with 0/1 entries.

Definitions.
• *A* ***rectangle*** *R is submatrix of M induced by a subset $r \times c \subseteq [N] \times [N]$ of rows and columns.*
• *A rectangle is called* ***monochromatic*** *if either every entry in it is 1 or every entry is zero.*
• *A set of rectangles are said to* ***cover*** *M if every non-zero entry of M is in at least one of the rectangles.*
• *The* ***rectangle cover number*** *of M, denoted by $\mathrm{rc}(M)$, is the smallest number of monochromatic rectangles needed to cover M.*

Example: A 5×5 matrix with a cover of size 4 (Each rectangle is represented by a distinct color).



Let D be an $N \times N$ matrix defined as follows. Rows and columns of D are indexed by 0/1 vectors of length n, and the entry D_{ab} is defined to be

$$D_{ab} = \begin{cases} 1 & \text{if } a^\intercal b \neq 1 \\ 0 & \text{otw.} \end{cases}$$

Question: What is the rectangle covering number of D?

Related results:
• $\mathrm{rc}(M) \leqslant 2^n$. (This is trivial).
• $\mathrm{rc}(M) \geqslant 1.5^n$ (See reference)

References: Volker Kaibel, Stefan Wltge: A Short Proof that the Extension Complexity of the Correlation Polytope Grows Exponentially.

Problem 23. *Automorphism Groups of 3-dimensional Comparability Graphs (suggested by Peter Zeman)*

Definitions.
• *A graph that has a transitive orientation is called a* comparability graph.
• *The* Dushnik-Miller dimension *of a poset P is the least number of linear orderings $L_1, \dots, L_k$ such that $P = L_1 \cap \cdots \cap L_k$.*
• *Similarly, we define the* dimension *of a comparability graph X, denoted by* $\dim(X)$*, as the dimension of any transitive orientation of X. (Every transitive orientation has the same dimension.)*
• *We denote the class of comparability graphs by* COMP*, and the class of comparability graphs of dimension at most k by k-*DIM*. We get the following infinite hierarchy of graph classes:*

$$1\text{-DIM} \subsetneq 2\text{-DIM} \subsetneq 3\text{-DIM} \subsetneq 4\text{-DIM} \subsetneq \cdots \subsetneq \text{COMP}.$$

• *Let* $\mathrm{Aut}(X)$ *denote the* automorphism group *of the graph X. For a graph class $\mathcal{C}$, let* $\mathrm{Aut}(\mathcal{C}) = \{G : X \in \mathcal{C},\ G \cong \mathrm{Aut}(X)\}$*. The class $\mathcal{C}$ is called* universal *if every abstract finite group is contained in* $\mathrm{Aut}(\mathcal{C})$*, and* non-universal *otherwise.*

Question: Show that Aut(3-DIM) is universal.

Related results:
• Since 1-DIM consists of all complete graphs, Aut(1-DIM) $= \{\mathbb{S}_n : n \in \mathbb{N}\}$. The automorphism groups of 2-DIM are non-universal and they were completely characterized in [1].
• The automorphism groups of k-DIM, for $k \geq 4$, are universal [1]. For a given graph X there is a comparability graph $C_X \in$ 4-DIM such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(C_X)$. The construction is simple, however, the proof that $C_X \in$ 4-DIM is quite involved and technical.
• Yannakakis showed that the recognition of 3-DIM is NP-complete by a reduction from 3-coloring [2]. For a graph X, a comparability graph Y is

constructed with several vertices representing each element of $V(X) \cup E(X)$. It is proved that $\dim(Y) = 3$ if and only if X is 3-colorable. Unfortunately, the automorphisms of X are lost in Y since it depends on the labels of $V(X)$ and $E(X)$ and Y contains some additional edges according to these labels.

References:

[1] P. Klavík and P. Zeman: Automorphism groups of comparability graphs.
[2] M. Yannakakis: The complexity of the partial order dimension problem.