An Invitation to Game Comonads, day 5, advanced topics: Lovász' counting theorem

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12 August 2022

ESSLLI 2022, Galway

Outlook

Summary of previous days

$$\overset{G}{\circlearrowleft}\mathbf{Str}(\sigma) \xrightarrow{\ \mathbf{t}\ } \overset{G^I}{\circlearrowleft}\mathbf{Str}(\sigma^I) \xrightarrow{F^{G^I}} \mathbf{EM}(G^I)$$

Game comonads express logical properties of structures:

- $G(A) \to B$ iff $A \Rrightarrow^{\mathscr{L}} B$, for a fixed $\mathscr{L} \subseteq \mathrm{PP}$
- $F^G(\mathbf{t}(A))$ and $F^G(\mathbf{t}(B))$ are bisimilar iff $A \equiv^{\mathscr{L}} B$,
  for a fixed $\mathscr{L} \subseteq \mathrm{FO}$

As well as combinatorial properties:

A satisfies property Δ (for some $\Delta \subseteq \mathbf{Str}(\sigma)$)	$\iff$	A admits a coalgebra $A \to G(A)$

"To be done"

1. Designing more comonads
2. Categorifying known results
3. Building the abstract theory

Designing more comonads

Logic fragments

- local fragments
- modal μ-calculus
- dependence logic
- rank logics ($\Rightarrow$ algebraic games)
- ...

Combinatorial properties

- local tree-width, local tree-depth
- twin-width
- shrub-depth
- ...

Categorifying known results

Identify the corresponding categorical notions in

1. preservation theorems (van Benthem-Rosen, Rossman, ...) ✓
2. decomposition methods (FVM theorems) ✓
3. homomorphism-counting theorems ✓
4. 0–1 laws ??
5. locality methods ??
6. Beth definability ??
7. ... (follow Libkin's *Elements of Finite Model Theory*)

Building the abstract theory

- Develop the theory of arboreal categories.
- Study connections with Universal Coalgebra.
  (coinductive definitions, coinductive proof principle, ...)
- Connection of arboreal categories and categorical logic.

Lovász’ homomorphism-counting theorems

Counting fragments

A **counting quantifier** has the form

$$\exists^{\geq d} x \; \varphi$$

and expresses that $A \vDash \exists^{\geq d} x \; \varphi(x, \overline{b})$ iff for at least d different $a \in A$, $A \vDash \varphi(a, \overline{b})$.

We write

$$\#\mathrm{FO}$$

for the extension of FO with counting quantifiers $\exists^{\geq d}$, for every natural number d. Also, write

$$\#\mathrm{FO}_k \quad \text{and} \quad \#\mathrm{FO}^k$$

for the quantifier rank k and k-variable fragments of $\#\mathrm{FO}$, respectively.

Graded modalities

Graded modality

$$\Diamond_R^d \varphi$$

expresses that $A, a \models \Diamond_R^d \varphi$ iff there exists at least d different $b \in A$ such that $(a, b) \in R^A$ and $A, b \models \varphi$.

We write

$$\#\mathrm{ML}$$

for the extension of ML with graded modalities $\Diamond_R^d$.

Also, write

$$\#\mathrm{ML}_k$$

for the restriction of $\#\mathrm{ML}$ to formulas of modal depth k.

Lovász-type theorems

Theorem (Lovász, 1967)

For finite σ-structures A, B,

$$A \cong B \iff |\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)| \quad \forall \textit{ finite } C$$

Theorem (Dvořák, 2010)

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}^k} B \iff |\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{tw} < k$$

Theorem (Grohe, 2020)

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \iff |\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{td} \leq k$$

k-round bijection game

Theorem (Hella, 1996)

For finite A, B, we have $A \equiv^{\#\mathrm{FO}_k} B$ iff Duplicator has a winning strategy in the k-round bijection Ehrenfeucht-Fraïssé game:

- *In round n, Duplicator chooses a bijection $f : A \to B$.*
- *Spoiler chooses a_n in A and Duplicator $b_n = f(a_n)$ in B.*
- *Duplicator wins round n if $\{(a_i, b_i) \mid i \leq n\}$ is a partial isomorphism.*

Remark: If the cardinalities of A and B differ, Duplicator loses immediately.

k-pebble bijection game

Theorem (Hella, 1996)

For finite A, B, we have $A \equiv^{\#\mathrm{FO}^k} B$ iff Duplicator has a winning strategy in the k-pebble bijection game:

- *In round n, Spoiler chooses a pebble p and then Duplicator chooses a bijection $f : A \to B$.*
- *Spoiler puts pebble p on a_p in A and Duplicator puts pebble p on $b_p = f(a_p)$.*
- *Duplicator wins round n if $\{(a_p, b_p) \mid p \in S\}$ is a partial isomorphism, where S is the set of pebbles played so far.*

k-round graded bisimulation game

Theorem (Rijke, 2000)

For finite (A, a) and (B, b), we have $(A, a) \equiv^{\#\mathrm{ML}_k} (B, b)$ iff Duplicator has a winning strategy in the k-round graded bisimulation game:

- *In round n, Spoiler chooses a binary $R \in \sigma$ and then Duplicator chooses a bijection*

$$f \colon \{x \in A \mid (a_{n-1}, x) \in R^A\} \to \{y \in B \mid (b_{n-1}, y) \in R^B\}$$

- *Spoiler chooses x in A such that $(a_{n-1}, x) \in R^A$ and Duplicator chooses $b_n = f(a_n)$.*
- *Duplicator wins round n if $\{(a_i, b_i) \mid i \leq n\}$ is a partial isomorphism.*

Comonadic formulations

Theorem (Abramsky–Dawar–Wang & Abramsky–Shah)
For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \quad \Longleftrightarrow \quad F^{\mathbb{E}_k}(\mathbf{t}(A)) \cong F^{\mathbb{E}_k}(\mathbf{t}(B))$$
$$A \equiv^{\#\mathrm{FO}^k} B \quad \Longleftrightarrow \quad F^{\mathbb{P}_k}(\mathbf{t}(A)) \cong F^{\mathbb{P}_k}(\mathbf{t}(B))$$

For finite pointed σ-structures (A, a) and (B, b) in a modal signature σ,

$$A \equiv^{\#\mathrm{ML}_k} B \quad \Longleftrightarrow \quad F^{\mathbb{M}_k}(A, a) \cong F^{\mathbb{M}_k}(B, b)$$

Our strategy

(1) Equivalence in $\#\mathscr{L}$ is an isomorphism of cofree coalgebras.

$\Rightarrow$ Show that isomorphism in $\mathbf{EM}(G)$ is determined by counting.

(2) Combinatorial property Δ given by coalgebras:

$$A \text{ has property } \Delta \iff \exists \text{ coalgebra } A \to G(A)$$

$\Rightarrow$ exploit the interaction between forgetting the coalgebra structure and creating the cofree coalgebra structure.

Adjunctions in Category Theory

Given functors

$$\mathscr{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}$$

We say that L and R are **adjoint**, with L to the left and R to the right, written $L \dashv R$, if there is a <u>bijection</u>

$$b_{A,B} : \mathscr{C}(L(A), B) \xrightarrow{\cong} \mathscr{D}(A, R(B))$$

for every $A \in \mathscr{D}$ and $B \in \mathscr{C}$, such that

$$b_{A,B'}(L(A) \xrightarrow{h} B \xrightarrow{h'} B') \;=\; A \xrightarrow{b_{A,B}(h)} R(B) \xrightarrow{R(h')} R(B')$$

$$b^{-1}_{A',B}(A' \xrightarrow{f'} A \xrightarrow{f} R(B)) \;=\; L(A') \xrightarrow{L(f')} L(A) \xrightarrow{b^{-1}_{A,B}(f)} B$$

Example adjunction

For a comonad $(G, \varepsilon, (\cdot)^*)$ on category $\mathscr{C}$, we have

$$\begin{array}{c} \mathbf{EM}(G) \\ U^G \big\downarrow \;\; \big\uparrow F^G \\ \mathscr{C} \end{array}$$

Where

$$F^G : \mathscr{C} \to \mathbf{EM}(G), \qquad A \mapsto (G(A), \delta_A)$$

and

$$U^G : \mathbf{EM}(G) \to \mathscr{C}, \qquad (A, \alpha) \mapsto A$$

These functors are adjoint to each other!

Example adjunction, continuation

We have $U^G \dashv F^G$ that is, for $(A, \alpha) \in \mathbf{EM}(G)$ and $B \in \mathscr{C}$,

$$\mathscr{C}(\underbrace{U(A,\alpha)}_{A}, B) \quad \cong \quad \mathbf{EM}(G)((A,\alpha), \underbrace{F(B)}_{(G(B),\delta_B)})$$

- Given $f \colon A \to B$ compute $f^{\#} \colon (A, \alpha) \to (G(B), \delta_B)$ by setting $G(f) \circ \alpha$.
- Given $g \colon (A, \alpha) \to (G(B), \delta_B)$, set $g^{\flat} \colon A \to B$ as $\varepsilon_B \circ g$.

Exercise

Show that $(\cdot)^{\flat}$ and $(\cdot)^{\#}$ are well-defined, inverse to each other and witness that $U^G \dashv F^G$.

Combinatorial categories

A category $\mathscr{C}$ is **combinatorial** if

$$A \cong B \iff |\mathscr{C}(C, A)| = |\mathscr{C}(C, B)| \quad \forall C \in \mathscr{C}$$

Examples: sets, finite graphs or σ-structures, finite groups, finite Boolean algebras, finite inverse semigroups, ...

Abstract Lovász' theorem, first steps

Lemma

For a comonad G on a category $\mathscr{C}$, if $\mathbf{EM}(G)$ is combinatorial then

$$F^G(A) \cong F^G(B) \iff |\mathscr{C}(C,A)| = |\mathscr{C}(C,B)| \quad \forall C \in \Delta_G$$

where Δ_G consists of all C in $\mathscr{C}$ which admit a G-coalgebra.

Proof.

$$\begin{aligned} F^G(A) \cong F^G(B) &\iff |\mathbf{EM}(G)(X, F^G(A))| = |\mathbf{EM}(G)(X, F^G(B))| \\ &\qquad \forall X \in \mathbf{EM}(G) \\ &\iff |\mathscr{C}(U^G(X), A)| = |\mathscr{C}(U^G(X), B)| \\ &\qquad \forall X \in \mathbf{EM}(G) \\ &\iff |\mathscr{C}(C,A)| = |\mathscr{C}(C,B)| \quad \forall C \in \Delta_G \end{aligned}$$

□

Theorem (Dawar–Jakl–Reggio, 2021)
If a category is locally finite, has a proper factorisation system and pushouts, then it is combinatorial.

Proof hint: Use the inclusion–exclusion principle!

Remark: [Pultr, 1973] and [Reggio, 2021] replace pushouts with stronger requirements on the factorisation systems.

Theorem (Dawar–Jakl–Reggio, 2021)
For any comonad G over $\mathbf{Str}(\sigma)$, the category $\mathbf{EM}_{fin}(G)$ of finite coalgebras is combinatorial.

Proof hint: Use the fact that U^G (is comonadic and therefore) creates and preserves colimits and preserves epimorphisms.

Grohe's theorem for $\#\mathrm{FO}_k$ without equality

Observe that $\mathbb{E}_k$ restricts to a comonad on $\mathbf{Str}_{fin}(\sigma)$
$\Rightarrow$ the adjunction $U^{\mathbb{E}_k} \dashv F^{\mathbb{E}_k}$ restricts to $\mathbf{Str}_{fin}(\sigma) \leftrightarrows \mathbf{EM}_{fin}(\mathbb{E}_k)$

Proposition

For finite σ-structures A, B,

$$F^{\mathbb{E}_k}(A) \cong F^{\mathbb{E}_k}(B) \iff |\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)|$$

for every finite C of $\mathrm{td} \leq k$.

For $A \equiv^{\#\mathrm{FO}_k} B$ we need $F^{\mathbb{E}_k}(\mathbf{t}(A)) \cong F^{\mathbb{E}_k}(\mathbf{t}(B))$!

Also, this proves equality elimination for $\#\mathrm{FO}_k$!

Adding equality

Previous proposition also holds for σ^I-structures!

$$\mathbf{Str}(\sigma) \xrightarrow{\mathbf{t}} \mathbf{Str}(\sigma^I) \circlearrowleft \mathbb{E}_k^I \xrightarrow{F^{\mathbb{E}_k^I}} \mathbf{EM}(\mathbb{E}_k^I)$$

Moreover, **t** has a left adjoint:

$$\mathbf{q}\colon \mathbf{Str}(\sigma^I) \to \mathbf{Str}(\sigma), \qquad A \mapsto A/I$$

Last ingredient

Lemma (Combinatorial lemma)

The adjunction

$$\mathbf{Str}(\sigma^I) \underset{\mathbf{t}}{\overset{\mathbf{q}}{\rightleftarrows}} \mathbf{Str}(\sigma), \quad \mathbf{q} \dashv \mathbf{t}$$

restricts to an adjunction between finite σ-structures of $\mathrm{td} \leq k$ and finite σ^I-structures of $\mathrm{td} \leq k$.

Grohe's theorem

Theorem

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \iff |\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{td} \leq k$$

Proof.

$|\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)|$

for every finite σ-structure C of $\mathrm{td} \leq k$

$\iff$ (from the Combinatorial lemma)

$|\,\mathrm{hom}(D, \mathbf{t}(A))| = |\,\mathrm{hom}(D, \mathbf{t}(B))|$

for every finite σ^I-structure D of $\mathrm{td} \leq k$

$\iff$

$F^{\mathbb{E}_k}(\mathbf{t}(A)) \cong F^{\mathbb{E}_k}(\mathbf{t}(B)) \iff A \equiv^{\#\mathrm{FO}_k} B$ □

Adaptations

A minor adaptation needed for $\mathbb{P}_k$, it <u>does not</u> restrict to

$$\mathbf{Str}_{fin}(\sigma) \to \mathbf{Str}_{fin}(\sigma).$$

Combinatorial lemma quite involved.
$\Rightarrow$ Dvořák's theorem.

The abstract result yields new theorems:

- for $\mathbb{M}_k$ (no equality $\Rightarrow$ no combinatorial lemma needed),
- for the $\#\mathrm{FO}_k \cap \#\mathrm{FO}^n$ fragment,
- for the restricted conjunction fragments and path-width.