An Invitation to Game Comonads, day 5, advanced topics: Arboreal Categories

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Towards an axiomatic approach

We studied three different game comonads, $\mathbb{E}_k$, $\mathbb{P}_k$ and $\mathbb{M}_k$.
In each case, we have tight connections with logical fragments, and with combinatorial invariants. These connections involve the categories of coalgebras for the comonads.

Can we capture the significant common elements of these constructions in an axiomatic way?

Potential advantages of an axiomatic framework:

- What properties are needed for various results is clarified
- Generalization - may lead to new *kinds* of examples
- A landscape may emerge, suggesting a structure theory, e.g. tame/wild dichotomies

Towards an axiomatic approach

Recall that, in each of the three cases, the coalgebras have an intrinsic forest structure—preserved by the coalgebra morphisms.

The forest order encodes a process for *generating* (parts of) the relational structure, to which resource notions can be applied.

This allows us to apply resource notions to structures via the adjoint pair of functors

$$\mathbf{Str}(\sigma) \underset{F^G}{\overset{\text{forgetful}}{\leftrightarrows}} \mathbf{EM}(G)$$

Towards an axiomatic approach

We introduce a notion of arboreal category, which captures all essential ingredients of the categories $\mathbf{EM}(G)$, for $G \in \{\mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k\}$ (applies to other comonads as well).

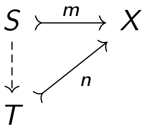
- All the examples of game comonads that we have considered arise from arboreal covers, i.e. adjunctions between extensional categories of relational structures, and arboreal categories.
- Arboreal categories provide a setting for a general notion of bisimulation, which yields a wide range of logical equivalences in the examples.
- Bisimulations in arboreal categories can be equivalently described as certain back-and-forth games.

Embeddings

Fix a category $\mathscr{A}$ with a (proper, stable) factorisation system $(\mathcal{Q}, \mathcal{M})$.

$\mathcal{M}$-morphisms = embeddings, denoted by $\rightarrowtail$.
$\mathcal{Q}$-morphisms = quotients, denoted by $\twoheadrightarrow$.

Given embeddings $m\colon S \rightarrowtail X$ and $n\colon T \rightarrowtail X$, say that $m \leq n$ iff there is $S \to T$ making the following triangle commute.



This is a *preorder* on the set of all embeddings with codomain X.
This canonically yields a poset $\mathbb{S}\, X$, the poset of embeddings of X, by taking the quotient with respect to the equivalence relation

$$m \sim n \iff (m \leq n \text{ and } n \leq m).$$

Definition

An object P of $\mathscr{A}$ is called a path provided the poset $\mathbb{S}\,P$ is a finite chain.

Exercise

In $\mathbf{EM}(\mathbb{E}_k)$, $\mathbf{EM}(\mathbb{P}_k)$ and $\mathbf{EM}(\mathbb{M}_k)$, with the factorisation systems given by (surjective morphisms, embeddings), this coincides with the notion of path previously introduced.

That is, these are precisely those coalgebras (regarded as structures equipped with an appropriate forest order) whose forest order is a finite chain.

Definition

$\mathscr{A}$ is a path category if it satisfies the following conditions:

1. it has all coproducts of small families of paths;
2. every path in $\mathscr{A}$ is connected;
3. for any paths P, Q, R, if the composite $P \to Q \to R$ is a quotient, then so is $P \to Q$.

Example

The following are path categories:

- The category $\mathcal{F}$ of forests and forest morphisms, equipped with the (surjective, injective) factorisation system. The paths are the finite chains. Coproducts are given by disjoint union.
- $\mathbf{EM}(\mathbb{E}_k)$, $\mathbf{EM}(\mathbb{P}_k)$ and $\mathbf{EM}(\mathbb{M}_k)$.

The functor of paths

Since we have axiomatic notions of embeddings and paths, we can generalise a number of notions from the concrete setting to the axiomatic one. E.g., a path embedding in a path category $\mathscr{A}$ is an embedding $P \rightarrowtail X$ whose domain is a path.

Given any object X of $\mathscr{A}$, we let $\mathbb{P}X$ be the sub-poset of $\mathbb{S}X$ consisting of the path embeddings.

Theorem

Let $\mathscr{A}$ be a path category. Then the assignment $X \mapsto \mathbb{P}X$ induces a functor $\mathbb{P}\colon \mathscr{A} \to \mathcal{T}$ into the category of trees.

Similarly, the notions of pathwise embedding and open map make sense in any path category, and so we have bisimulations.
(Recall: these are spans of open pathwise embeddings.)

To fully capture the *dynamic content* of bisimilarity in terms of (abstract) *back-and-forth games*, equivalently *back-and-forth systems*, we need to consider a special class of arboreal categories:

Definition

An arboreal category is a path category in which every object is *path-generated*.

Example

The following are arboreal categories:

- The category $\mathcal{F}$ of forests and forest morphisms; just observe that every forest is the colimit of the diagram given by its branches and the embeddings between them.
- $\mathbf{EM}(\mathbb{E}_k)$, $\mathbf{EM}(\mathbb{P}_k)$ and $\mathbf{EM}(\mathbb{M}_k)$.

Games vs bisimulations: the axiomatic setting

In an arboreal category with *binary products*, the bisimilarity relation can be equivalently described in terms of (the existence of a Duplicator strategy in) an appropriate back-and-forth game.

The latter is a straightforward generalisation of the game $\mathscr{G}$ (in which moves are pairs of path embeddings) defined yesterday.

Similarly, variations of this game capture, in the examples, preservation of various logic fragments, including: existential, positive, existential positive, and counting extensions.

In each case, these correspond to natural relations between objects of the arboreal category. E.g., the preservation of the existential fragment corresponds to the existence of a pathwise embedding between cofree coalgebras.

Arboreal categories can be used e.g. to study homomorphism preservation theorems.

Theorem (Łoś–Tarski–Lyndon, 1955 & 1959)
A first-order sentence $\varphi \in \mathrm{FO}$ is preserved by homomorphisms iff it is equivalent to an existential positive sentence $\psi \in \mathrm{EP}$.

- Admits relativisation to finite structures (Rossman, 2007).
- Equirank HPT: we can replace FO with FO_k, and EP with EP_k (Rossman, 2007).

More generally, we are interested in equi-resource HPTs.

An application: HPTs

Consider an arboreal category $\mathscr{A}$ and an adjunction

$$\mathscr{E} \underset{R}{\overset{L}{\leftrightarrows}} \mathscr{A} \qquad \left(\text{think of: } \ \mathbf{Str}(\sigma) \underset{F^{\mathbb{E}_k}}{\overset{\text{forgetful}}{\leftrightarrows}} \mathbf{EM}(\mathbb{E}_k) \right).$$

This induces relations $\rightarrow^{\mathscr{A}}$ and $\leftrightarrow^{\mathscr{A}}$ on objects of $\mathscr{E}$:

$$A \rightarrow^{\mathscr{A}} B \text{ iff there exists a morphism } RA \rightarrow RB,$$
$$A \leftrightarrow^{\mathscr{A}} B \text{ iff there exists a bisimulation } RA \leftarrow \cdot \rightarrow RB.$$

Example

Let $\mathscr{D}$ be a full subcategory of $\mathbf{Str}(\sigma)$. Then $\mathscr{D}$ is:

- *saturated* under $\leftrightarrow^{\mathbf{EM}(\mathbb{E}_k)}$ iff $\mathscr{D} = \mathbf{Mod}(\varphi)$ for some $\varphi \in \mathrm{FO}_k$;
- *upwards closed* wrt $\rightarrow^{\mathbf{EM}(\mathbb{E}_k)}$ iff $\mathscr{D} = \mathbf{Mod}(\psi)$ for some $\psi \in \mathrm{EP}_k$.

(HP) For any full subcategory $\mathscr{D}$ of $\mathscr{E}$ saturated under $\leftrightarrow^{\mathscr{A}}$, $\mathscr{D}$ is closed under morphisms iff it is upwards closed wrt $\rightarrow^{\mathscr{A}}$.

Theorem

(HP) holds whenever $\mathscr{E} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{A}$ *satisfies either of:*

- *idempotency: the comonad $G := LR$ is idempotent.*
- *bisimilar companion property: $A \leftrightarrow^{\mathscr{A}} GA$ for all $A \in \mathrm{Ob}(\mathscr{E})$.*

Example

$\mathbb{M}_k$ is idempotent, so we get an equidepth HPT for (graded) modal logic. Similarly, using the bisimilar companion property, we obtain an equi-resource HPT for guarded logics.

Forcing the bisimilar companion property

When the bisimilar companion property fails (e.g., for $\mathbb{E}_k$), there are ways to force it. Even though $A \not\leftrightarrow^{\mathscr{A}} GA$, we may attempt to construct extensions $A \hookrightarrow A^*$ and $GA \hookrightarrow (GA)^*$ such that

$$A^* \leftrightarrow^{\mathscr{A}} (GA)^*.$$

These extensions exist whenever the adjunction $\mathscr{E} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{A}$ satisfies appropriate (categorical) properties.

- We recover Rossman's equirank HPT.
- Results can be relativised to certain full subcategories of $\mathscr{E}$.

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