An Invitation to Game Comonads, day 4: Logical Equivalences

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On Day 2, we saw how games provide syntax-free characterisations of various logical equivalences (respectively, preorders) of the form

$$\equiv^{\mathscr{L}} \quad (\text{respectively, } \Rrightarrow^{\mathscr{L}})$$

for an appropriate choice of the logic fragment $\mathscr{L}$.

We then defined comonads

$$\mathbb{E}_k, \ \mathbb{P}_k, \text{ and } \mathbb{M}_k$$

that capture these games (by considering the set of plays in the forth-only games as structures themselves).

Now, we shall look at how to characterise the relations $\equiv^{\mathscr{L}}$ and $\Rrightarrow^{\mathscr{L}}$ using the comonads and their coalgebras.

Existential positive fragments

Capturing the existential positive fragment EP_k of FO_k:
Recall that there exists a Kleisli morphism

$$A \to_{\mathbb{E}_k} B$$

precisely when Duplicator has a winning strategy in the k-round forth-only EF game from A to B. Thus,

Proposition

The following statements are equivalent for all structures A, B:

1. $A \Rrightarrow^{\mathrm{EP}_k} B$. *That is, for all existential positive sentences φ with quantifier rank at most k, $A \vDash \varphi \implies B \vDash \varphi$.*
2. *There exists a Kleisli morphism $A \to_{\mathbb{E}_k} B$.*

Existential positive fragments

Similarly,

- The homomorphism preorder in the Kleisli category $\mathbf{K}(\mathbb{P}_k)$ captures the existential positive fragment of k-variable logic.
- The homomorphism preorder in the Kleisli category $\mathbf{K}(\mathbb{M}_k)$ captures the existential positive fragment of modal logic with modal depth at most k.

Note: The comonads (and their indexed structures) encode the limitation of logical resources.

- Can we capture other logical equivalences using the Kleisli category? For instance, what does isomorphism in the Kleisli category correspond to? (More on this on Day 5.)

From Kleisli to Eilenberg–Moore

To capture e.g. equivalences in the full fragments

$$\mathrm{FO}_k,\ \mathrm{FO}^k \text{ and } \mathrm{ML}_k,$$

corresponding to *back-and-forth* games, we need to move from the Kleisli category to the Eilenberg–Moore category.

Recall that for any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$ there is a functor

$$F^G \colon \mathscr{C} \to \mathbf{EM}(G)$$

sending A to (GA, δ_A), and an arrow $f \colon A \to B$ in $\mathscr{C}$ to Gf.

Idea: Describe logical equivalence of A, B by comparing the corresponding cofree coalgebras $F^G A$ and $F^G B$.

Eilenberg–Moore coalgebras as forest-ordered structures

Yesterday, we saw that the Eilenberg–Moore categories

$$\mathbf{EM}(\mathbb{E}_k),\ \mathbf{EM}(\mathbb{P}_k) \text{ and } \mathbf{EM}(\mathbb{M}_k)$$

can be identified with categories whose objects are structures equipped with an appropriate compatible forest order (and a pebbling function in the case of $\mathbb{P}_k$).

The morphisms are the homomorphisms that preserve the forest orders (and also the pebbling functions in the case of $\mathbb{P}_k$).

E.g., $F^{\mathbb{E}_k}A$ is given by $\mathbb{E}_k A$ equipped with the prefix order $\sqsubseteq$.

Paths and embeddings

We shall now work in the category $\mathbf{EM}(\mathbb{E}_k)$.

An object $(A, \leq)$ of $\mathbf{EM}(\mathbb{E}_k)$ is a path if $\leq$ is a (finite) linear order.
Paths are noted by $P, Q, R \ldots$

For paths, the compatibility condition is trivially satisfied!

An arrow in $f\colon (A, \leq) \to (B, \leq)$ in $\mathbf{EM}(\mathbb{E}_k)$ is an embedding if it is an embedding *qua* σ-homomorphism.
That is, f is injective and for all n-ary $R \in \sigma$

$$(a_1, \ldots, a_n) \in R^A \iff (f(a_1), \ldots, f(a_n)) \in R^B.$$

A path embedding in $\mathbf{EM}(\mathbb{E}_k)$ is an embedding

$$m\colon P \rightarrowtail (A, \leq)$$

whose domain is a path.

Note that the image of m is isomorphic to P.

Example

Consider a path embedding

$$m\colon P \rightarrowtail F^{\mathbb{E}_k} A$$

where P consists of elements $p_1 \prec \cdots \prec p_n$. Because m is a forest morphism, there is a list $[a_1, \ldots, a_n] \in \mathbb{E}_k(A)$ such that

$$\forall i \in \{1, \ldots, n\}, \quad m(p_i) = [a_1, \ldots, a_i].$$

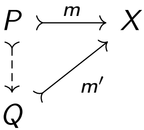
Exercise

Let $m\colon P \rightarrowtail F^{\mathbb{E}_k} A$ and $n\colon P \rightarrowtail F^{\mathbb{E}_k} B$ be path embeddings. Show that, if their images contain only *non-repeating* sequences, m and n induce a partial isomorphism between A and B.

Games via path embeddings

Given any two $X, Y \in \mathbf{EM}(\mathbb{E}_k)$, we shall define a back-and-forth game $\mathscr{G}$ played by Spoiler and Duplicator on the forest-ordered structures X and Y.

To this end, whenever $m \colon P \rightarrowtail X$ and $m' \colon Q \rightarrowtail X$ are path embeddings, let us say that m' covers m, written $m \prec m'$, if $|Q| = |P| + 1$ and there is an embedding $P \rightarrowtail Q$ making the following diagram commute.



The collection of all path embeddings into X is denoted by $\mathbb{P}X$. Similarly for path embeddings into Y.

For all $X, Y \in \mathbf{EM}(\mathbb{E}_k)$, the game $\mathscr{G}$ is defined as follows:

- Positions in the game $\mathscr{G}$ are pairs $(m, n) \in \mathbb{P}X \times \mathbb{P}Y$.
- The winning relation

$$\mathscr{W}(X, Y) \subseteq \mathbb{P}X \times \mathbb{P}Y$$

  consists of the pairs (m, n) such that $\mathrm{dom}(m) = \mathrm{dom}(n)$.
- The initial position is $(\bot_X, \bot_Y)$, where

$$\bot_X : \emptyset \rightarrowtail X \quad \text{and} \quad \bot_Y : \emptyset \rightarrowtail Y$$

  are the unique functions from the empty set.

Games via path embeddings

- At the start of each round, the position is specified by a pair $(m, n) \in \mathbb{P}\,X \times \mathbb{P}\,Y$, and the round proceeds as follows: Either Spoiler chooses some $m' \succ m$ and Duplicator must respond with some $n' \succ n$, or Spoiler chooses some $n'' \succ n$ and Duplicator must respond with $m'' \succ m$.
- Duplicator wins the round if they are able to respond and the new position is in $\mathscr{W}(X, Y)$. Duplicator wins the game if they have a strategy that is winning after j rounds, for all $j \geq 1$.

Since paths in $\mathbf{EM}(\mathbb{E}_k)$ have length at most k, the game terminates after k rounds!

The game $\mathscr{G}$ and its logical counterpart

Assume the game $\mathscr{G}$ is played between cofree coalgebras $X = F^{\mathbb{E}_k}(A)$ and $Y = F^{\mathbb{E}_k}(B)$. After k rounds, let

$$m_1 \prec \cdots \prec m_k \in \mathbb{P}X \quad \text{and} \quad n_1 \prec \cdots \prec n_k \in \mathbb{P}Y$$

be the path embeddings that have been played. Their images yield

$$[a_1] \sqsubseteq \cdots \sqsubseteq [a_1, \ldots, a_k] \in \mathbb{E}_k A \;\text{ and }\; [b_1] \sqsubseteq \cdots \sqsubseteq [b_1, \ldots, b_k] \in \mathbb{E}_k B.$$

Lemma

$(m_i, n_i) \in \mathscr{W}(X, Y)$ for all $i = 1, \ldots, k$ iff $\{(a_i, b_i) \mid i = 1, \ldots, k\}$ is a partial correspondence between A and B.

1. ~~*For all $i, j \in \{1, \ldots, k\}$, $a_i = a_j \iff b_i = b_j$.*~~
2. *For all n-ary relations R and all $i_1, \ldots, i_n \in \{1, \ldots, k\}$,*
   $$(a_{i_1}, \ldots, a_{i_n}) \in R^A \iff (b_{i_1}, \ldots, b_{i_n}) \in R^B.$$

Proposition

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the game $\mathscr{G}$ played between $F^{\mathbb{E}_k}(A)$ and $F^{\mathbb{E}_k}(B)$.*
2. $A \equiv^{\mathrm{FO}_k^-} B$. *I.e., for all first-order sentences φ without equality and with quantifier rank at most k, $A \models \varphi \iff B \models \varphi$.*

- How to recover equivalence with respect to FO_k?
- The previous result relies on a notion of game in the category $\mathbf{EM}(\mathbb{E}_k)$. Can we describe it in a more structural way?

Open morphisms and bisimulations

A morphism $f\colon X \to Y$ in $\mathbf{EM}(\mathbb{E}_k)$ is open if it satisfies the following *path-lifting property*: Given any commutative square

$$\begin{array}{ccc} P & \rightarrowtail & Q \\ \downarrow & \swarrow & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with P, Q paths, there is $Q \to X$ making the triangles commute.

Further, $f\colon X \to Y$ is a pathwise embedding if, for all path embeddings $m\colon P \rightarrowtail X$, the composite $f \circ m$ is a path embedding.

A bisimulation between objects X, Y of $\mathbf{EM}(\mathbb{E}_k)$ is a span of open pathwise embeddings

$$X \leftarrow Z \to Y.$$

If such a bisimulation exists, we say that X and Y are bisimilar.

Games vs bisimulations

Theorem

The following are equivalent for all objects X, Y of $\mathbf{EM}(\mathbb{E}_k)$:

1. *Duplicator has a winning strategy in the game $\mathscr{G}$ played between X and Y.*
2. *X and Y are bisimilar.*

Sketch of proof: See whiteboard.

Corollary

The following statements are equivalent for all structures A, B:

1. $A \equiv^{\mathrm{FO}_k^-} B$.
2. *$F^{\mathbb{E}_k}(A)$ and $F^{\mathbb{E}_k}(B)$ are bisimilar.*

- Similar results hold for the comonads $\mathbb{P}_k$ and $\mathbb{M}_k$.
  (In the former case, we restrict to finite structures.)

- For $\mathbb{M}_k$, since there is no equality symbol in the logic, we get a characterisation of $\equiv^{\mathrm{ML}_k}$ in terms of bisimilarity in **EM**$(\mathbb{M}_k)$.

- We shall now look at how to "add" the equality symbol in the case of $\mathbb{E}_k$ (and $\mathbb{P}_k$). That is, how to go

$$\text{from} \quad \equiv^{\mathrm{FO}_k^-} \quad \text{to} \quad \equiv^{\mathrm{FO}_k} .$$

I-relations

Consider a *fresh* binary relation symbol I and define the signature

$$\sigma^I := \sigma \cup \{I\}.$$

Denote the category of σ^I-structures and their homomorphisms by

$$\mathbf{Str}(\sigma^I).$$

There is a functor

$$\mathbf{t} \colon \mathbf{Str}(\sigma) \to \mathbf{Str}(\sigma^I)$$

that views a σ-structure A as a σ^I-structure where I^A is the identity relation on A.

Since $\mathbb{E}_k$ was defined *uniformly* for all signatures, there is an Ehrenfeucht-Fraïssé comonad $\mathbb{E}_k^I$ on $\mathbf{Str}(\sigma^I)$.

Equivalence in FO_k with equality

$$\mathbb{E}_k \circlearrowright \mathbf{Str}(\sigma) \xrightarrow{\mathbf{t}} \mathbf{Str}(\sigma^I) \circlearrowleft \mathbb{E}^I_k \qquad \mathbf{Str}(\sigma^I) \xrightarrow{F^{\mathbb{E}^I_k}} \mathbf{EM}(\mathbb{E}^I_k)$$

Theorem

The following statements are equivalent for all σ-structures A, B:

1. $F^{\mathbb{E}^I_k}\mathbf{t}(A)$ *and* $F^{\mathbb{E}^I_k}\mathbf{t}(B)$ *are bisimilar.*
2. $A \equiv^{\mathrm{FO}_k(\sigma)} B$.

Proof.

By the characterisation of $\equiv^{\mathrm{FO}^-_k}$, item 1 holds iff

$$\mathbf{t}A \vDash \varphi \iff \mathbf{t}B \vDash \varphi$$

for all $\varphi \in \mathrm{FO}^-_k(\sigma^I)$. But this is equivalent to item 2. □

Outlook

- The same strategy applies to k-variable logic FO^k (provided we restrict ourselves to finite structures).
- In fact, the whole approach can be captured in the axiomatic setting of arboreal categories (Day 5).
- In some cases, there are equality elimination results which, in a sense, tell us that working with σ-structures and σ^I-structures is the same.
- This is especially relevant for counting logics and homomorphism counting theorems (Day 5).

References

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