An Invitation to Game Comonads, day 3: Coalgebras and Combinatorial Parameters

Tomáš Jakl & Luca Reggio

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Summary of Day 2

Was discussed:

- Model comparison games capture relationships in logic.
- Forth-only versions of some games modelled semantically as

$$G(A) \to B$$

- These constructions satisfies axioms of a comonad $(G, \varepsilon, (\cdot)^*)$:

$$\varepsilon_A^* = \mathrm{id}_{G(A)} \qquad \varepsilon_B \circ f^* = f \qquad (g \circ f^*)^* = g^* \circ f^*$$

Obvious questions:

- What can we use from the theory of (co)monads?
- Generic proofs by employing categorical tools?

Functors

Recall, functors are "homomorphisms of categories".

A functor $F\colon \mathscr{C} \to \mathscr{D}$ is given by

- a mapping on objects $F\colon\ \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$
- a mapping on morphisms, for every $A, B \in \mathscr{C}$,

$$F\colon \mathscr{C}(A, B) \to \mathscr{D}(F(A), F(B))$$

which preserves identities and compositions:

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$$
$$F(f \circ g) = F(f) \circ F(g)$$

Example: comonads extend to functors!

Given a comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$, define

$$f\colon A \to B \quad \longmapsto \quad \begin{aligned} &G(f)\colon G(A) \to G(B) \\ &G(f) = (f \circ \varepsilon_A)^* \end{aligned}$$

G is a functor $\mathscr{C} \to \mathscr{C}$ as

$$G(\mathrm{id}_A) = (\mathrm{id}_A \circ \varepsilon_A)^* = (\varepsilon_A)^* = \mathrm{id}_{G(A)}$$

$$\begin{aligned} G(f) \circ G(g) &= (f \circ \varepsilon)^* \circ (g \circ \varepsilon)^* = (f \circ \varepsilon \circ (g \circ \varepsilon)^*)^* \\ &= (f \circ g \circ \varepsilon)^* = G(f \circ g) \end{aligned}$$

Example

For $h\colon A \to B$ in $\mathbf{Str}(\sigma)$, the functor $\mathbb{E}_k(h)\colon \mathbb{E}_k(A) \to \mathbb{E}_k(B)$ maps $[a_1, \ldots, a_n]$ to $[h(a_1), \ldots, h(a_n)]$.

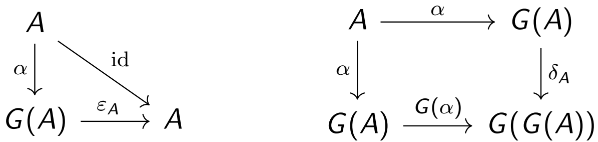
Eilenberg–Moore coalgebras

Given a comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$, for every $A \in \mathrm{Ob}(G)$, define the **comultiplication**

$$\delta_A \colon G(A) \to G(G(A))$$

as the morphism $(\mathrm{id}_{G(A)})^*$.

Then, a morphism $\alpha \colon A \to G(A)$ is a G-**coalgebra on** A if



(i.e. $\varepsilon_A \circ \alpha = \mathrm{id}$ and $\delta_A \circ \alpha = G(\alpha) \circ \alpha$)

Origins – Dual notions

Algebras as functions in **Set**

$$F(A) \to A$$

E.g. for signature $\Sigma = \{\vee, \neg\}$ and $F(A) = (A \times A) \uplus A$

$$\text{functions } (A \times A) \uplus A \to A \quad \approx \quad \Sigma\text{-algebras } (A, \vee, \neg)$$

Correspondence (Σ signature, $\mathcal{E}$ equations) $\longleftrightarrow$ monads T

$$\mathbf{Alg}(\Sigma, \mathcal{E}) \cong \left\{ T(A) \xrightarrow{\alpha} A \;\middle|\; \begin{array}{ccc} A & \xrightarrow{\eta_A} & T(A) \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow{\scriptstyle \alpha} \\ & & A \end{array} \quad \text{and} \quad \begin{array}{ccc} T^2(A) & \xrightarrow{T(\alpha)} & T(A) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \alpha} \\ T(A) & \xrightarrow{\alpha} & A \end{array} \right\}$$

Example: List comonad

Define a comonad on **Set**

$$\texttt{List}\colon \mathrm{Ob}(\textbf{Set}) \to \mathrm{Ob}(\textbf{Set}), \quad A \mapsto \{\, [a_1, \ldots, a_n] \mid a_i \in A\}$$

The counit is

$$\varepsilon_A \colon \texttt{List}(A) \to A, \qquad [a_1, \ldots, a_n] \mapsto a_n$$

and, for a function $f\colon \texttt{List}(A) \to B$, define

$$f^* \colon \texttt{List}(A) \to \texttt{List}(B)$$

by $[a_1, \ldots, a_n] \mapsto [b_1, \ldots, b_n]$ where $b_i = f([a_1, \ldots, a_i])$

Example: `List`-coalgebras, the first axiom

$$
\begin{array}{ccc}
A & & \\
{\scriptstyle \alpha}\downarrow & \searrow^{\mathrm{id}} & \\
\mathtt{List}(A) & \xrightarrow[\varepsilon_A]{} & A
\end{array}
\qquad \text{imposes} \qquad
\begin{array}{ccc}
a & \overset{\mathrm{id}}{\longmapsto} & a \\
{\scriptstyle \alpha}\downarrow & & \| \\
[a_1, \ldots, a_n] & \overset{\varepsilon_A}{\longmapsto} & a_n
\end{array}
$$

Example: List-coalgebras, the second axiom

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathtt{List}(A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta_A} \\ \mathtt{List}(A) & \xrightarrow{\mathtt{List}(\alpha)} & \mathtt{List}(\mathtt{List}(A)) \end{array}$$

imposes

$$\begin{array}{ccc} a & \overset{\alpha}{\longmapsto} & [a_1, \ldots, a_n] \\ & & \downarrow{\scriptstyle\delta_A} \\ {\scriptstyle\alpha}\downarrow & & [[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_n]] \\ & & \shortparallel \\ [a_1, \ldots, a_n] & \overset{\mathtt{List}(\alpha)}{\longmapsto} & [\alpha(a_1), \ldots, \alpha(a_n)] \end{array}$$

Therefore

$$\alpha(a_i) = [a_1, \ldots, a_i]$$

Example: $\texttt{List}$-coalgebras, the second axiom: forest order

For, $w, w' \in \texttt{List}(A)$, write

$$w \sqsubseteq w' \quad \text{for} \quad w \text{ is a prefix of } w'$$

Consequently,

- If $\alpha(a) = [a_1, \ldots, a_n]$ then $\alpha(a_i) \sqsubseteq \alpha(a_j)$ iff $i \leq j$.
- The set $\{a_1, \ldots, a_n\}$ is a chain in the $\leq_\alpha$-order where

$$a \leq_\alpha a' \iff \alpha(a) \sqsubseteq \alpha(a')$$

- $\leq_\alpha$ defines a **forest order**:
  - $(A, \leq_\alpha)$ is a poset
  - $\forall a \in A \quad {\downarrow}a = \{x \in A \mid x \leq_\alpha a\}$ is a finite chain.

Example: List-coalgebras, recovering from forest orders

For a poset $(A, \leq)$ where $\leq$ is a forest order, define

$$\alpha_{\leq} \colon A \to \mathtt{List}(A)$$

by setting

$$\alpha_{\leq}(a) = [a_1, \ldots, a_n]$$

where

$$\downarrow a = \{a_1, \ldots, a_n\} \quad \text{is the chain} \quad a_1 < \cdots < a_n = a$$

Exercise

The mapping $\alpha_{\leq}$ is a List-coalgebra.

Example: `List`-coalgebras, finale

Proposition

For any set $A \in \mathbf{Set}$, there is a bijective correspondence between

- *coalgebras $A \to \mathtt{List}(A)$*
- *forest orders $\leq$ on A*

Proof.

It is enough to observe that $\alpha = \alpha_{\leq_\alpha}$ and $\leq = \leq_{\alpha_\leq}$. □

Morphisms of G-coalgebras

G-coalgebras form a category

$$\mathbf{EM}(G)$$

- Objects: (A, α) where $\alpha \colon A \to G(A)$ is a G-coalgebra
- Morphisms: $(A, \alpha) \to (B, \beta)$ are morphisms $f \colon A \to B$ in $\mathscr{C}$ such that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \alpha \big\downarrow & & \big\downarrow \beta \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

Exercise: Check that $\mathbf{EM}(G)$ is a category.

Example: morphisms of `List`-coalgebras

Proposition

Given `List`*-coalgebras* (A, α), (B, β) *and a function* $f : A \to B$, *the following are equivalent:*

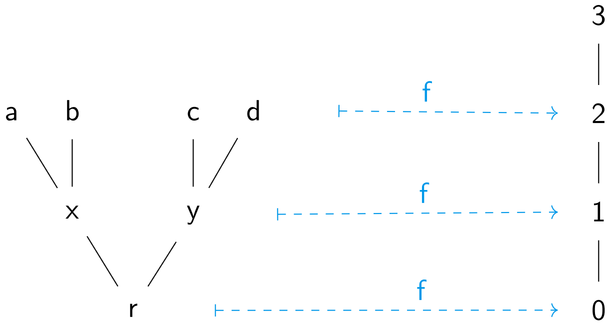
- f *is a coalgebra morphism* $(A, \alpha) \to (B, \beta)$, *i.e.*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \texttt{List}(A) & \xrightarrow{\texttt{List}(f)} & \texttt{List}(B) \end{array}$$

- f *is a* **forest morphism** $(A, \leq_\alpha) \to (B, \leq_\beta)$ *i.e.*
  - f *preserves roots (i.e. minimal elements)*
  - $a \prec a' \implies f(a) \prec f(a')$

  where $a \prec a'$ *iff* $a < a'$ *and* $a \leq z \leq a'$ *implies* $a = z$ *or* $a' = z$.

Example



Theorem

The category $\mathbf{EM}(\mathtt{List})$ *is isomorphic to the category of forest orders and forest morphism.*

Coalgebras of $\mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$

$\mathbb{E}_k$-coalgebras

Proposition

There is a bijection between coalgebras $\alpha\colon A \to \mathbb{E}_k(A)$ and

compatible forest orders $\leq$ on A of depth at most k

that is, relations $\leq$ on A such that

(T1) $\leq$ *is a forest order*

(T2) $\downarrow a$ *has at most* $\leq k$ *elements, for every* $a \in A$

(T3) $(a_1, \ldots, a_n) \in R^A$ *implies* $a_i \leq a_j$ *or* $a_j \leq a_i$ $(\forall i, j)$

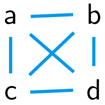
Proof.

$(a_1, \ldots, a_n) \in R^A$ implies $(\alpha(a_1), \ldots, \alpha(a_n)) \in R^{\mathbb{E}_k(A)}$ i.e.

- $\alpha(a_i) \sqsubseteq \alpha(a_j)$ or $\alpha(a_j) \sqsubseteq \alpha(a_i)$ $(\forall i, j)$
- $(\varepsilon(\alpha(a_1)), \ldots, \varepsilon(\alpha(a_n))) = (a_1, \ldots, a_n) \in R^A$ ✓ (always) □

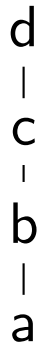
Exercise

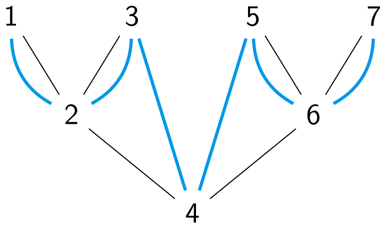
Given graphs



what are the minimal k such that they admit an $\mathbb{E}_k$-coalgebra?

Answer





$\mathbb{P}_k$-coalgebras

Proposition

There is a bijection between coalgebras $\alpha\colon A \to \mathbb{P}_k(A)$ *and*

compatible k**-pebble forest orders** $\leq, p$ **on** A

that is, relations $\leq$ *and pebbling functions* $p\colon A \to \{1, \ldots, k\}$ *satisfying*

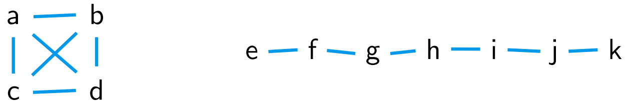
(T1) $\leq$ *is a forest order*

(T3') $(a_1, \ldots, a_n) \in R^A$ *implies*

- $a_i \leq a_j$ *or* $a_j \leq a_i$ $\quad (\forall i, j)$.
- $\forall z \quad a_i < z \leq a_j \implies p(a_i) \neq p(z)$

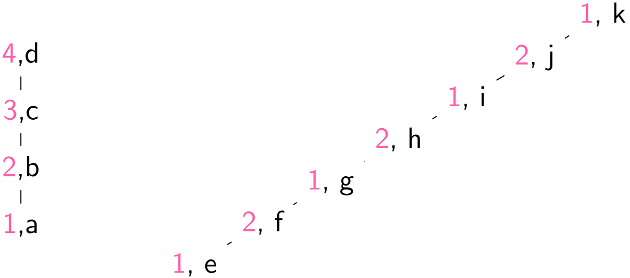
Exercise

Given graphs



what are the minimal k such that they admit an $\mathbb{P}_k$-coalgebra?

Answer



Proposition

There is coalgebra $\alpha\colon (A, a) \to \mathbb{M}_k(A, a)$ *iff*

(A, a) **is a synchronization tree of depth at most** k

i.e., for every $x \in A$*, there is a unique path of length* $\leq k$

$$a \xrightarrow{R_1} a_1 \xrightarrow{R_2} \dots \xrightarrow{R_n} x$$

In fact, synchronization trees are automatically forest ordered:

$$x \prec y \iff (x, y) \in R^A \quad \text{for a (unique) binary } R \in \sigma$$

Theorem (Abramsky–Shah, 2021)

$\mathbf{EM}(\mathbb{E}_k)$ *is isomorphic to the category with*

- *objects:* *σ-structures with a compatible forest order of depth at most k*
- *morphisms:* *homomorphisms of σ-structures that are also forest morphisms.*

Theorem (Abramsky–Shah, 2021)

$\mathbf{EM}(\mathbb{P}_k)$ *is isomorphic to the category with*

- *objects:* *σ-structures with a compatible k-pebble forest order*
- *morphisms:* *homomorphisms of σ-structures that are forest morphisms and preserve the pebbling function.*

Theorem (Abramsky–Shah, 2021)

EM($\mathbb{M}_k$) *is isomorphic to the category with*

- *objects: synchronization trees of depth at most k*
- *morphisms: homomorphisms of σ-structures that are also forest morphisms.*

Cofree coalgebras

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$ and $A \in \mathrm{Ob}(\mathscr{C})$,

$$(G(A),\ G(A) \xrightarrow{\delta_A} G(G(A)))$$

is a G-coalgebra!

Example

For $G = \mathbb{E}_k$ and a σ-structure A, the compatible forest order $\leq$ on $\mathbb{E}_k(A)$ is

$$u \leq w \iff u \text{ is a prefix of } w$$

For $G = \mathbb{P}_k$, the forest order $\leq$ is as above and the pebble function $p\colon \mathbb{P}_k(A) \to \{1, \ldots, k\}$ is defined by

$$p([(p_1, a_1), \ldots, (p_n, a_n)]) = p_n$$

Cofree functors

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$ there is a functor

$$F^G \colon \mathscr{C} \to \mathbf{EM}(G)$$

which sends $A \in \mathrm{Ob}(\mathscr{C})$ to $(G(A), \delta_A)$ and a morphism $f \colon A \to B$ in $\mathscr{C}$ to $G(f)$.

Exercise

Verify that F^G is a functor for $G = \mathbb{E}_k, \mathbb{P}_k$ and/or $\mathbb{M}_k$, from the concrete descriptions of $\mathbf{EM}(G)$.

Combinatorial parameters

Coalgebra numbers

In general, $G = (G_k)_{k \in \mathbb{N}}$ is an **indexed comonad**:

$$G_1, \; G_2, \; G_3, \; G_4, \; \ldots$$

on a category $\mathscr{C}$.

For an object $A \in \mathscr{C}$, define its G-coalgebra number

$$\kappa^G(A) = \min\{k \mid \text{ exists a coalgebra } A \to G_k(A)\}$$

Corollary

- $\kappa^{\mathbb{E}}(A) \leq k \iff \exists$ *compatible forest order on A of depth* $\leq k$
- $\kappa^{\mathbb{P}}(A) \leq k \iff \exists$ *compatible k-pebble forest order on A*
- $\kappa^{\mathbb{M}}(A, a) \leq k \iff (A, a)$ *is a synch. tree of depth* $\leq k$

A **forest cover** of a graph G is a forest $(T, \leq)$ and an injective function $f\colon G \to T$ such that

$$\text{if } (v, w) \in E^G, \text{ then either } f(v) \leq f(w) \text{ or } f(w) \leq f(v).$$

Write

$$\mathrm{td}(G) \leq k$$

if there exists a forest cover $(T, \leq)$ of G such that the size of $\downarrow x$ is at most k, for any $x \in T$.

Theorem (Abramsky–Shah, 2018 & 2021)

$\kappa^{\mathbb{E}}(G) = \mathrm{td}(G)$

Tree-width [Robertson–Seymour, 1986]

A **tree decomposition** of a graph G is a function $f\colon T \to \mathcal{P}(G)$, from a tree $(T, \leq)$ to subsets of G such that

- $\forall v \in G \quad \exists x \in T$ such that $v \in f(x)$,
- $\forall (u, v) \in E^G \quad \exists x \in T$ such that $\{u, v\} \subseteq f(x)$, and
- if $v \in f(x) \cap f(y)$, then $v \in f(z)$ for all z on the unique path between x and y in T.

Write

$$\mathrm{tw}(G) < k,$$

if there exists a tree decomposition $f\colon T \to \mathcal{P}(G)$ of such that $|f(x)| \leq k$ for every $x \in T$.

Theorem (Abramsky–Dawar–Wang, 2017)

$\kappa^{\mathbb{P}}(G) = \mathrm{tw}(G) + 1$

Revisiting the Chandra–Merlin correspondence

Recall the construction

$$\mathbf{M}\colon \mathrm{PP} \to \mathbf{Str}_{fin}(\sigma)$$

transforming φ in steps

1. variable renaming $\Rightarrow$ unique variable usage
2. prenex normal form $\Rightarrow$ $\exists x_1, \ldots, x_n \, (A_1 \wedge \cdots \wedge A_m)$
3. $\mathbf{M}(\varphi)$ on set $\{x_1, \ldots, x_n\}$ with relations as in $A_1, \ldots, A_m$

Theorem

- $\kappa^{\mathbb{E}}(A) \leq k \iff A \cong \mathbf{M}(\varphi)$ *for some* $\varphi \in \mathrm{PP}_k$
- $\kappa^{\mathbb{P}}(A) \leq k \iff A \cong \mathbf{M}(\varphi)$ *for some* $\varphi \in \mathrm{PP}^k$

Proof idea.

Quantifier nesting	$\leftrightarrow$	tree order
Variable usage	$\leftrightarrow$	pebbling function $\square$

Applications

Applications in CSP and logic

Lemma

If $\mathrm{tw}(A) < k$ *and Duplicator has a winning strategy in the* k*-pebble forth-only game from* A *to* B *then there exists a homomorphism* $A \to B$.

Proof.

1. tree-width $< k$ gives a coalgebra $A \to \mathbb{P}_k(A)$
2. a winning strategy gives $\mathbb{P}_k(A) \to B$
3. we compose $A \to \mathbb{P}_k(A) \to B$ □

Observation: Works for arbitrary comonads!

Applications in CSP and logic

Lemma

If $\mathrm{td}(A) \leq k$ *and Duplicator has a winning strategy in the* k*-round Ehrenfeucht-Fraïssé forth-only game from* A *to* B *then there exists a homomorphism* $A \to B$*.*

Lemma

For a synchronisation tree (A, a) *of depth* $\leq k$*, if Duplicator has a winning strategy in the* k*-round simulation game from* (A, a) *to* (B, b) *then there exists a homomorphism* $(A, a) \to (B, b)$*.*

Although, these are not so difficult to prove directly from the definitions.

Applications in combinatorics

There is a "comonad morphism" $\mathbb{E}_k \Rightarrow \mathbb{P}_k$, given by

$$\mathbb{E}_k(A) \xrightarrow{\lambda_A} \mathbb{P}_k(A)$$

$$[a_1, \ldots, a_n] \longmapsto [(1, a_1), (2, a_2), \ldots, (n, a_n)]$$

Lemma

For every σ-structure A, $\mathrm{tw}(A) + 1 \leq \mathrm{td}(A)$.

Proof sketch.

Assume there is a coalgebra $A \xrightarrow{\alpha} \mathbb{E}_k(A)$.

Then, the composition

$$A \xrightarrow{\alpha} \mathbb{E}_k(A) \xrightarrow{\lambda_A} \mathbb{P}_k(A)$$

is a coalgebra too, by the axioms of comonad morphisms. □

Bonus slides:
Different presentations of comonads

Natural transformations

Natural transformations are "morphisms of functors".

Given functors $F\colon \mathscr{C} \to \mathscr{D}$ and $F'\colon \mathscr{C} \to \mathscr{D}$, a **natural transformation**

$$\alpha\colon F \Rightarrow F' \qquad \text{or} \qquad \mathscr{C} \underset{F'}{\overset{F}{\rightrightarrows}} \mathscr{D} \quad (\Downarrow \alpha)$$

is given by a collection of morphisms

$$\{F(A) \xrightarrow{\alpha_A} F'(A) \mid A \in \mathrm{Ob}(\mathscr{C})\}$$

such that, for every $h\colon A \to B$ in $\mathscr{C}$,

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & F'(A) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle F'(h)} \\ F(B) & \xrightarrow{\alpha_B} & F'(B) \end{array} \qquad \text{(i.e. } F'(h) \circ \alpha_A = \alpha_B \circ F(h)\text{)}$$

Example: the identity natural transformations

For any functor $F\colon \mathscr{C} \to \mathscr{D}$, the collection

$$\{\, \mathrm{id}_{F(A)}\colon F(A) \to F(A) \mid A \in \mathrm{Ob}(\mathscr{C}) \,\}$$

is a natural transformation $\mathrm{id}_F\colon F \Rightarrow F$ since

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_{F(A)}} & A \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle F(f)} \\ B & \xrightarrow{\mathrm{id}_{F(B)}} & B \end{array}$$

Example: the counit natural transformation

For a comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$,

$$\{\, \varepsilon_A \colon G(A) \to A \mid A \in \mathrm{Ob}(\mathscr{C}) \,\}$$

is a natural transformation $\varepsilon \colon G \Rightarrow \mathrm{Id}_{\mathscr{C}}$. That is, for any $f \colon A \to B$ in $\mathscr{C}$, we have

$$\begin{array}{ccc} G(A) & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle f} \\ G(B) & \xrightarrow{\varepsilon_B} & B \end{array}$$

Which follows by

$$\varepsilon_B \circ G(f) = \varepsilon_B \circ (f \circ \varepsilon_A)^* = f \circ \varepsilon_A$$

Example: the comultiplication natural transformation

For every comonad $(G, \varepsilon, (\cdot)^*)$ there is a natural transformation

$$\delta\colon G \Rightarrow GG$$

The component δ_A of δ is obtained as the coextension $\mathrm{id}^*_{G(A)}\colon G(A) \to GG(A)$ of $\mathrm{id}_{G(A)}\colon G(A) \to G(A)$.

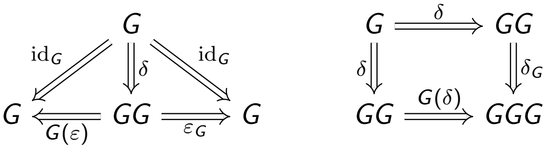
Exercise

Show that δ is a natural transformation.

Two comonad presentations

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$,

- $G : \mathscr{C} \to \mathscr{C}$ is a functor.
- $\varepsilon \colon G \Rightarrow \mathrm{Id}_{\mathscr{C}}$ is a natural transformation.
- $\delta \colon G \Rightarrow GG$ is a natural transformation.
- These satisfy



Fact: The presentation that we use $(G, \varepsilon, (\cdot)^*)$ can be recovered from the data (G, ε, δ), by defining $(\cdot)^*$ as $f^* := G(f) \circ \delta$.