An Invitation to Game Comonads, day 2: Games and Game Comonads

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The What and Why of games

- (Finite) model theory looks at structures up to definable properties.
- Given a logic fragment $\mathscr{L}$, define the equivalence relation

$$A \equiv^{\mathscr{L}} B \ \text{ iff } \ \forall \varphi \in \mathscr{L}. \ (A \models \varphi \iff B \models \varphi).$$

- Games provide semantic characterisations of the syntactic equivalences $\equiv^{\mathscr{L}}$ (and variations thereof).
- Two players: Spoiler aims to show that $A \not\equiv^{\mathscr{L}} B$ and Duplicator that $A \equiv^{\mathscr{L}} B$.
- Logical resources often correspond to natural resource parameters in a game.

Back-and-forth EF games

The (back-and-forth) Ehrenfeucht-Fraïssé game between structures A and B is defined as follows:

- In the i^{th} round, Spoiler chooses an element from A or B;
- Duplicator responds by picking an element in the other structure.
- Duplicator wins after k rounds if $\{(a_i, b_i) \mid i = 1, \ldots, k\}$ is a *partial isomorphism* between A and B.

1. For all $i, j \in \{1, \ldots, k\}$, $a_i = a_j \iff b_i = b_j$.
2. For all relation symbols R of arity n and all $i_1, \ldots, i_n \in \{1, \ldots, k\}$,

$$(a_{i_1}, \ldots, a_{i_n}) \in R^A \iff (b_{i_1}, \ldots, b_{i_n}) \in R^B.$$

Theorem (Ehrenfeucht & Fraïssé, 1954 and 1961)

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round back-and-forth Ehrenfeucht-Fraïssé game between A and B.*
2. $A \equiv^{\mathrm{FO}_k} B$. *That is, for all first-order sentences φ with quantifier rank at most k, $A \models \varphi \iff B \models \varphi$.*

Exercise

Let $A = (\mathbb{N}, <)$ and $B = (\{1, \ldots, 5\}, <)$. Does Duplicator have a winning strategy in the 2-round back-and-forth EF game?

Forth-only EF games

Forth-only variant of the EF game: Spoiler plays always in the same structure, say A, and Duplicator responds in B.

- Duplicator wins after k rounds if $\{(a_i, b_i) \mid i = 1, \ldots, k\}$ is a *partial homomorphism* from A to B.

1. For all $i, j \in \{1, \ldots, k\}$, $a_i = a_j \implies b_i = b_j$.
2. For all relation symbols R of arity n and all $i_1, \ldots, i_n \in \{1, \ldots, k\}$,

$$(a_{i_1}, \ldots, a_{i_n}) \in R^A \implies (b_{i_1}, \ldots, b_{i_n}) \in R^B.$$

Note: Duplicator can win the forth-only game in both directions but still lose the back-and-forth game!

Consider e.g. $A = (\mathbb{N}, \leq)$ and $B = (\{1, \ldots, 5\}, \leq)$.

Forth-only EF games and logic

Theorem

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round forth-only Ehrenfeucht-Fraïssé game played from A to B.*
2. $A \Rrightarrow^{\mathrm{EP}_k} B$. *That is, for all existential positive sentences φ with quantifier rank at most k, $A \models \varphi \implies B \models \varphi$.*

Exercise

Show that Spoiler has a winning strategy in the 3-round forth-only EF game from $A = (\mathbb{N}, <)$ to $B = (\{1, \ldots, 5\}, <)$.

Find an existential positive φ with quantifier rank at most 3 such that $A \models \varphi$ and $B \not\models \varphi$.

The Ehrenfeucht-Fraïssé comonad

Intuition:

- Games as semantic constructions in their own right.
- Make the set of all possible plays (in a given structure) in the forth-only EF game into a structure.

For every structure A, let

Plays in A, at most k rounds

- $\mathbb{E}_k(A)$: set of non-empty lists of length $\leq k$ of elements of A.
- *Last moves*: define $\varepsilon_A \colon \mathbb{E}_k(A) \to A$, $[a_1, \dots, a_j] \mapsto a_j$.
- *Lift relations* from A to $\mathbb{E}_k(A)$: for each relation R of arity n, $R^{\mathbb{E}_k(A)}$ consists of the tuples $(s_1, \dots, s_n) \in \mathbb{E}_k(A)^n$ such that
  1. $s_1, \dots, s_n$ are pairwise comparable in the prefix order, and
  2. $(\varepsilon_A(s_1), \dots, \varepsilon_A(s_n)) \in R^A$.

The Ehrenfeucht-Fraïssé comonad

- The functions $\varepsilon_A \colon \mathbb{E}_k(A) \to A$ become homomorphisms.
- Reconstructing the history of Duplicator's answers:
  Each homomorphism $f \colon \mathbb{E}_k(A) \to B$ induces a homomorphism

$$f^* \colon \mathbb{E}_k(A) \to \mathbb{E}_k(B)$$
$$[a_1, \ldots, a_j] \mapsto [f([a_1]), f([a_1, a_2]), \ldots, f([a_1, \ldots, a_j])].$$

These data define a *comonad*, called Ehrenfeucht-Fraïssé comonad, on the category $\mathbf{Str}(\sigma)$ of σ-structures and their homomorphisms.

Family of comonads, indexed by the *resource parameter* k (number of rounds)

Comonads defined

A comonad (in Kleisli–Manes form) on a category $\mathscr{C}$ is given by:

- an object map $G \colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$,
- a *counit* morphism $\varepsilon_A \colon GA \to A$ for every $A \in \mathrm{Ob}(\mathscr{C})$,
- a *coextension operation* associating with any morphism $f \colon GA \to B$ a morphism $f^* \colon GA \to GB$,

such that for all morphisms $f \colon GA \to B$ and $g \colon GB \to C$:

$$\varepsilon_A^* = \mathrm{id}_{GA}, \quad \varepsilon_B \circ f^* = f, \quad (g \circ f^*)^* = g^* \circ f^*.$$

A Kleisli morphism $A \to_G B$ is a morphism $GA \to B$ in $\mathscr{C}$.
Note: $A \to B$ implies $A \to_G B$, but not vice versa.

Strategies as Kleisli morphisms: the case of $\mathbb{E}_k$

Theorem

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round forth-only EF game played from A to B.*
2. *There exists a Kleisli morphism $A \to_{\mathbb{E}_k} B$.*

Proof.

$1 \Rightarrow 2$. A Duplicator strategy in the k-round forth-only EF game from A to B defines a function $\mathbb{E}_k(A) \to B$. The winning condition ensures that this function is a homomorphism.

$2 \Rightarrow 1$. Fix a homomorphism $f \colon \mathbb{E}_k(A) \to B$ and suppose Spoiler plays $a_1, \ldots, a_k$. Duplicator responds with $b_i = b_j$ if $a_i = a_j$ for some $j < i$, or $b_i = f([a_1, \ldots, a_i])$ otherwise. □

Pebble games

(Back-and-forth) k-pebble game: Each player has k pebbles and the game proceeds as follows.

- In the i^{th} round, Spoiler places some pebble p_i on an element of one of the structures.
- Duplicator places their corresponding pebble p_i on an element of the other structure.
- Duplicator wins after n rounds if the relation determined by the *current placings* of the pebbles is a partial isomorphism, and wins the k-pebble game if they have a strategy which is winning after n rounds, for all $n \geq 0$.

Note: Because pebbles can be moved, this is an infinite game.

Pebble games and logic

Theorem

*The following are equivalent for all *finite* structures A, B:*

1. *Duplicator has a winning strategy in the back-and-forth k-pebble game between A and B.*
2. $A \equiv^{\mathrm{FO}^k} B$. *That is, for all first-order sentences φ with at most k variables, $A \models \varphi \iff B \models \varphi$.*

Similarly, the following are equivalent:

3. *Duplicator has a winning strategy in the forth-only k-pebble game played from A to B.*
4. $A \Rrightarrow^{\mathrm{EP}^k} B$. *That is, for all existential positive sentences φ with at most k variables, $A \models \varphi \implies B \models \varphi$.*

The pebble comonad

For every structure A, let

Plays in A

- $\mathbb{P}_k(A)$: set of non-empty finite lists of elements of $\boldsymbol{k} \times A$, where $\boldsymbol{k} := \{p_1, \ldots, p_k\}$. An element $(p_i, a) \in \boldsymbol{k} \times A$ is a *move* and p_i is the *pebble index* of the move.
- $\varepsilon_A \colon \mathbb{P}_k(A) \to A$, $[(p_1, a_1), \ldots, (p_j, a_j)] \mapsto a_j$.
- Lift relations from A to $\mathbb{P}_k(A)$: for each relation R of arity n, $R^{\mathbb{P}_k(A)}$ consists of the tuples $(s_1, \ldots, s_n) \in \mathbb{P}_k(A)^n$ such that
  1. $s_1, \ldots, s_n$ are pairwise comparable in the prefix order,
  2. for all $i, j \in \{1, \ldots, n\}$, if s_i is a prefix of s_j, the pebble index of the last move of s_i does not appear in the suffix of s_i in s_j,
  3. $(\varepsilon_A(s_1), \ldots, \varepsilon_A(s_n)) \in R^A$.

Extra condition on current placings of the pebbles

The pebble comonad

- The functions $\varepsilon_A \colon \mathbb{P}_k(A) \to A$ become homomorphisms.
- Reconstructing the history of Duplicator's answers:
  Each homomorphism $f \colon \mathbb{P}_k(A) \to B$ induces a homomorphism

$$f^* \colon \mathbb{P}_k(A) \to \mathbb{P}_k(B)$$

$$[(p_1, a_1), \dots, (p_j, a_j)] \mapsto [(p_1, b_1), \dots, (p_j, b_j)]$$

  where $b_i := f([(p_1, a_1), \dots, (p_i, a_i)])$ for all $i = 1, \dots, j$.

These data define a comonad, called pebbling comonad, on the category $\mathbf{Str}(\sigma)$ of σ-structures and their homomorphisms.

Family of comonads, indexed by the *resource parameter k* (number of pebbles)

Bisimulation games

Bisimulation game (for modal logic) between pointed Kripke structures (A, a) and (B, b):

- The initial position is $(a_0, b_0) \coloneqq (a, b)$.
- In the i^{th} round, where the current position is (a_{i-1}, b_{i-1}), Spoiler chooses a binary relation R, one of the two structures, say A, and $a_i \in A$ such that $(a_{i-1}, a_i) \in R^A$.
- Duplicator must respond with an element of the other structure, say $b_i \in B$, such that $(b_{i-1}, b_i) \in R^B$. If there is no such response available, Duplicator loses.
- Duplicator wins after k rounds if, for all unary predicates P, we have $a_i \in P^A \iff b_i \in P^B$ for all $i \in \{0, \ldots, k\}$.

(Bi)simulation games and logic

Theorem

The following statements are equivalent for all pointed Kripke structures $(A, a), (B, b)$:

1. *Duplicator has a winning strategy in the k-round bisimulation game between (A, a) and (B, b).*
2. $A \equiv^{\mathrm{ML}_k} B$. *That is, for all modal formulas φ of modal depth at most k, $A, a \models \varphi \iff B, b \models \varphi$.*

Similarly, the following are equivalent:

3. *Duplicator has a winning strategy in the k-round simulation game played from (A, a) to (B, b).*
4. *For all existential positive modal formulas φ of modal depth at most k, $A, a \models \varphi \implies B, b \models \varphi$.*

The modal comonad

Plays in $\mathbf{A}$, at most k rounds

For every pointed Kripke structure $\mathbf{A} = (A, a)$,

- $\mathbb{M}_k(\mathbf{A})$: set of paths of length $\leq k$ starting from a:

$$a \xrightarrow{R_1} a_1 \xrightarrow{R_2} a_2 \to \cdots \xrightarrow{R_n} a_n$$

  where $R_1, \ldots, R_n$ are binary relations.
- $\varepsilon_{\mathbf{A}} \colon \mathbb{M}_k(\mathbf{A}) \to A$ sends a path to its last element.
- Lift relations from A to $\mathbb{M}_k(\mathbf{A})$: for each unary relation P, $P^{\mathbb{M}_k(\mathbf{A})}$ consists of the paths s such that $\varepsilon_{\mathbf{A}}(s) \in P^A$. For each binary relation R, $R^{\mathbb{M}_k(\mathbf{A})}$ consists of the pairs of paths (s, t) such that t is obtained by extending s by one step along R.
- The distinguished element of $\mathbb{M}_k(\mathbf{A})$ is the trivial path (a).

The modal comonad

- The functions $\varepsilon_{\mathbf{A}} \colon \mathbb{M}_k(\mathbf{A}) \to \mathbf{A}$ become homomorphisms of pointed Kripke structures.
- Each homomorphism $f \colon \mathbb{M}_k(\mathbf{A}) \to \mathbf{B}$ yields a homomorphism

$$f^* \colon \mathbb{M}_k(\mathbf{A}) \to \mathbb{M}_k(\mathbf{B})$$

$$(a \xrightarrow{R_1} a_1 \cdots \xrightarrow{R_n} a_n) \mapsto (b \xrightarrow{R_1} b_1 \cdots \xrightarrow{R_n} b_n)$$

where $b_i \coloneqq f(a \xrightarrow{R_1} a_1 \cdots \xrightarrow{R_i} a_i)$.

These data define a comonad, called modal comonad, on the category $\mathbf{Str}_*(\sigma)$ of pointed Kripke structures and their homomorphisms.

Family of comonads, indexed by the *resource parameter* k (number of rounds)

Strategies as Kleisli morphisms: the case of $\mathbb{P}_k$ and $\mathbb{M}_k$

Theorem

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the forth-only k-pebble game played from A to B.*
2. *There exists a Kleisli morphism $A \to_{\mathbb{P}_k} B$.*

Theorem

The following statements are equivalent for all pointed Kripke structures $\mathbf{A}, \mathbf{B}$:

1. *Duplicator has a winning strategy in the k-round simulation game played from $\mathbf{A}$ to $\mathbf{B}$.*
2. *There exists a Kleisli morphism $\mathbf{A} \to_{\mathbb{M}_k} \mathbf{B}$.*

The Kleisli category of a comonad

Let G be a comonad on a category $\mathscr{C}$.

- Kleisli morphisms compose: given Kleisli morphisms $f\colon A \to_G B$ and $g\colon B \to_G C$, their composition is

$$GA \xrightarrow{f^*} GB \xrightarrow{g} C.$$

- The identity $A \to_G A$ is the counit $\varepsilon_A\colon GA \to A$.

The Kleisli category of G is the category $\mathbf{K}(G)$ such that

- $\mathrm{Ob}(\mathbf{K}(G)) = \mathrm{Ob}(\mathscr{C})$
- $\mathbf{K}(G)(A, B)$ consists of the Kleisli morphisms $A \to_G B$.

Note: In the case of $\mathbb{E}_k, \mathbb{P}_k$ and $\mathbb{M}_k$, composition of Kleisli morphisms corresponds to *composition of winning strategies*.

Outlook

The Kleisli category $\mathbf{K}(G)$ arises naturally by considering winning strategies in various forth-only games.

- From a logical viewpoint $\mathbf{K}(G)$ captures preservation of existential positive fragments, in the sense that

  $$\rightarrow_G \; = \; \Rrightarrow^{\mathscr{L}}$$

  for appropriate choices of G and $\mathscr{L}$.
  E.g., if $G = \mathbb{E}_k$ then $\mathscr{L}$ consists of all existential positive sentences with quantifier rank $\leq k$.
- $\mathbf{K}(G)$ sits in a larger category of *coalgebras* for G that capture combinatorial parameters of structures.
  This is the topic of tomorrow's lecture.

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