DISCRETE DENSITY COMONADS AND GRAPH PARAMETERS

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Abstract. ...

1. Introduction

An important feature of the emerging theory of game comonads is that game comonads classify a number of important classes of finite relational structures. We say that a comonad $\mathbb{C}$ *classifies a class* Δ if a finite structure A is in the class Δ precisely when A admits a $\mathbb{C}$-coalgebra. For example, the Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$ classifies the structures of tree-depth $\leq k$ and, similarly, the pebbling comonad $\mathbb{P}_k$ classifies tree-width $<k$.

In this paper we take a look at the theoretical limits of the comonadic approach. In particular, we would like to identify classes of structures which can be classified by comonads. We can readily predict two necessary requirements. Namely, since the problem is stated in the language of category theory, we know that the classes of structures classified by comonads need to be closed under isomorphisms and, moreover, since finite coalgebras are closed under binary coprododucts $+$ (i.e. disjoint unions), so need to be the classes classifiable by comonads.

In fact, we show that only one further natural requirement suffices in order to be able to classify a class of structures by a comonad. In particular, it suffices to assume that the class contains all connected substructures of the structures already in the class. In summary, the first of our main results is the following theorem.

Theorem 1.1. *Any* component-based *class* Δ *of finite relational structures or graphs, i.e. a class closed under*

- *isomorphism,*
- *finite coproducts, and*
- *subcomponents*
  (i.e. if $A + B$ *is in* Δ *then so are* A *and* B*),*

can be classified by a comonad.

The theorem applies to a wide variety of classes of structures studied in the literature. In particular, these assumptions hold for all classes of finite

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structures classified by our game comonads and, moreover, by a number of typical examples of classes of structures for which a given graph parameter is bounded by a constant. For example, we obtain comonads for planar graphs, bipartite graphs, or graphs of the max degree or clique-width bounded by a constant. Moreover, we show that the constructed comonad $\mathbb{C}$ is weakly initial among the comonads classifying Δ, meaning that for any comonad $\mathbb{D}$ classifying Δ, there is a comonad morphism $\mathbb{C} \Rightarrow \mathbb{D}$. This initiality allows us to obtain a characterisation of comonads that classify monotone nowhere dense classes.

Another important aspect of game comonads is that they classify various well-known binary relations between relational structures. We say that a comonad $\mathbb{C}$ *classifies relation* $\asymp$ whenever $A \asymp B$ holds precisely whenever the cofree $\mathbb{C}$-coalgebras on A and B are isomorphic. For example, the comonad $\mathbb{E}_k$ classifies the relation that expresses that Duplicator has a winning strategy in the bijective k-round variant of the Ehrenfeucht–Fraïssé game and, similarly, the $\mathbb{P}_k$ classifies the existence of a winning strategy in the bijective k-pebble game. Furthermore, it was recently shown that the relation classified by $\mathbb{E}_k$ admits a Lovász-type theorem. In particular, finite structures A, B have isomorphic $\mathbb{E}_k$-coalgebras if, and only if, they admit the same homomorphism counts from finite structures of tree-depth $\leq k$, i.e. when there is a bijection between $\hom(C, A)$ and $\hom(C, B)$ for every finite C of tree-depth $\leq k$. Similar Lovász-type theorems have been also shown for $\mathbb{P}_k$ and the pebble-relation comonads [13, 23].

We show that the comonad constructed in the proof of Theorem 1.1 automatically admits a Lovász-type theorem for the class of structures it classifies. In fact, we show that such comonad always has *finite rank*[1], which ensures that the category of coalgebras for the comonad is locally finitely presentable (cf. [14, Proposition 1.12.1], see also [25, Appendix B]) and therefore, by a recent result of Luca Reggio [25, Corollary 5.15], admits the following Lovász-type result.

Corollary 1.2. *Let Δ be a component-based class of finite structures and let $\asymp$ be a binary relation on finite structures satisfying that, for any two finite structures A, B,*

$$A \asymp B \iff \hom(C, A) \cong \hom(C, B) \quad \text{for all } C \in \Delta$$

Then, we may assume that the comonad classifying Δ, by Theorem 1.1, also classifies $\asymp$.

As an example, we obtain that the comonad obtained by Theorem 1.1 which classifies planar graphs also classifies quantum isomorphism (cf. [22]), and similarly, the comonad for coproducts of cycles classifies co-spectrality and the comonad for bipartite graphs classifies isomorphic bipartite double covers.

[1] Do not confuse comonads of finite rank with finitary comonads, which is a weaker notion.

The density comonad construction is the main technical tool of this paper. In fact, we develop most of our theory by means of discrete density comonads, that is, density comonads of functors with discrete domain. A general overview of the necessary categorical terminology and results is given in Section 2. Discrete density comonads are introduced in Section 3 and Theorem 1.1 is proved in Section 4. In Section 5 we take a look at how graph parameters correspond to coalgebra numbers of graded comonads, compare discrete density comonads with game comonads and characterise comonads classifying monotone nowhere dense classes of graphs. Lastly, in Section 6 we prove Corollary 1.2 by showing that, under mild conditions, discrete density comonads have finite rank.

2. Preliminaries

In this section we fix notation and us recall some basic facts about comonads and the density construction. We assume the reader has familiarity with elementary category theory notions such as functors, natural transformations, adjunctions, limits and colimits (see e.g. [5] or [9]).

Throughout the paper we use the following notation. Given a natural transformation $\lambda\colon E \Rightarrow F$ between functors $E, F\colon \mathcal{A} \to \mathcal{B}$ and functors $G\colon \mathcal{B} \to \mathcal{B}'$ and $H\colon \mathcal{A}' \to \mathcal{A}$, we denote by $G\lambda$ and λH the obvious natural transformations of type $GE \Rightarrow GF$ and $EH \Rightarrow FH$, respectively.

2.1. **Comonads and coalgebras.**

A *comonad* (on category $\mathcal{A}$) is a triple $(\mathbb{C}, \varepsilon, \delta)$ where $\mathbb{C}\colon \mathcal{A} \to \mathcal{A}$ is an endofunctor, and $\varepsilon\colon \mathbb{C} \Rightarrow \mathrm{Id}$ and $\delta\colon \mathbb{C} \Rightarrow \mathbb{C}^2$ are natural transformations such that the following diagrams commute.

$$
\begin{array}{ccc}
\mathbb{C} & \xRightarrow{\ \delta\ } & \mathbb{C}^2 \\
{\scriptstyle\delta}\Downarrow & & \Downarrow{\scriptstyle\delta\mathbb{C}} \\
\mathbb{C}^2 & \xRightarrow[\mathbb{C}\delta]{} & \mathbb{C}^3
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C} & \xRightarrow{\ \delta\ } & \mathbb{C}^2 \\
 & {\scriptstyle\mathrm{id}}\searrow & \Downarrow{\scriptstyle\varepsilon\mathbb{C}} \\
 & & \mathbb{C}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C} & \xRightarrow{\ \delta\ } & \mathbb{C}^2 \\
 & {\scriptstyle\mathrm{id}}\searrow & \Downarrow{\scriptstyle\mathbb{C}\varepsilon} \\
 & & \mathbb{C}
\end{array}
$$

A morphism $\alpha\colon a \to \mathbb{C}(a)$ is an *(Eilenberg–Moore)* $\mathbb{C}$*-coalgebra*[2] if the following diagrams commute.

$$
\begin{array}{ccc}
a & & \\
{\scriptstyle\alpha}\downarrow & \searrow{\scriptstyle\mathrm{id}} & \\
\mathbb{C}(a) & \xrightarrow[\varepsilon_a]{} & a
\end{array}
\qquad
\begin{array}{ccc}
a & \xrightarrow{\ \alpha\ } & \mathbb{C}^2(a) \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta_a} \\
\mathbb{C}(a) & \xrightarrow[\mathbb{C}\alpha]{} & \mathbb{C}^2(a)
\end{array}
\tag{1}
$$

We say that an object of a *admits a coalgebra* if there exists a morphism $a \to \mathbb{C}(a)$ which is a $\mathbb{C}$-coalgebra.

Coalgebras form a category $\mathtt{EM}(\mathbb{C})$ where morphisms between coalgebras $(a, \alpha) \to (b, \beta)$ are morphisms $h\colon a \to b$ such that $\beta \circ h = \mathbb{C}(h) \circ \alpha$. Moreover,

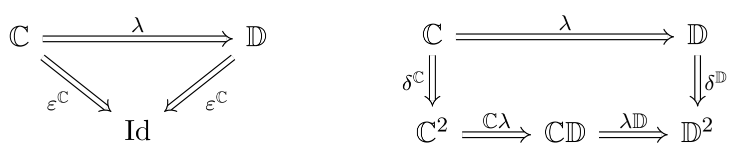
[2] All coalgebras in this text are Eilenberg–Moore coalgebras. We do not work with functor coalgebras at any point.

there is a pair of adjoint functors

$$U^{\mathbb{C}}\colon \mathtt{EM}(\mathbb{C}) \to \mathcal{A} \quad \text{and} \quad F^{\mathbb{C}}\colon \mathcal{A} \to \mathtt{EM}(\mathbb{C})$$

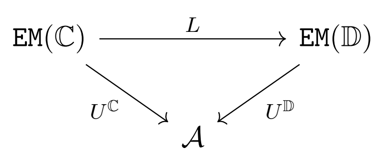
between $\mathtt{EM}(\mathbb{C})$ and the underlying category $\mathcal{A}$. The left adjoint is just a forgetful functor, it sends a coalgebra $\alpha\colon a \to \mathbb{C}(a)$ to its underlining object $U^{\mathbb{C}}(a, \alpha) = a$. The right adjoint returns the *cofree coalgebra* $F^{\mathbb{C}}(a)$ *on* a, represented by the morphism $\delta_a\colon \mathbb{C}(a) \to \mathbb{C}^2(a)$.

2.2. **Comonad morphisms.** Given two comonads $(\mathbb{C}, \varepsilon^{\mathbb{C}}, \delta^{\mathbb{C}})$ and $(\mathbb{D}, \varepsilon^{\mathbb{D}}, \delta^{\mathbb{D}})$ on $\mathcal{A}$, a natural transformation $\lambda\colon \mathbb{C} \Rightarrow \mathbb{D}$ is a *comonad morphism* if the following two diagrams of natural transformations commute.



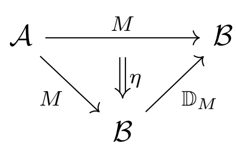
(2)

Note that comonad morphisms can be equivalently presented as functors $L\colon \mathtt{EM}(\mathbb{C}) \to \mathtt{EM}(\mathbb{D})$ such that the following diagram of functors commutes.

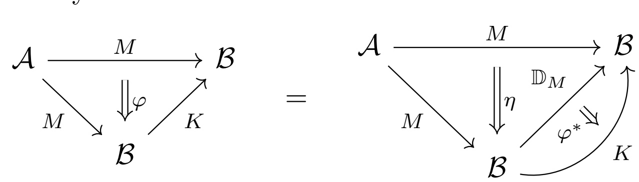


The functor L is constructed from the comonad morphism given as a natural transformation λ by sending $a \xrightarrow{\alpha} \mathbb{C}(a)$ to $a \xrightarrow{\alpha} \mathbb{C}(a) \xrightarrow{\lambda_a} \mathbb{D}(a)$. For details, see e.g. [26].

2.3. **Density comonads.** The *density comonad* of a functor $M\colon \mathcal{A} \to \mathcal{B}$ is a functor $\mathbb{D}_M\colon \mathcal{B} \to \mathcal{B}$ with a natural transformation $\eta\colon M \Rightarrow \mathbb{D}_M M$.



Moreover, η is required to be the initial natural transformation with this property. In other words, for any functor $K\colon \mathcal{B} \to \mathcal{B}$ and a natural transformation $\varphi\colon M \Rightarrow KM$ there is a *unique* $\varphi^*\colon \mathbb{D}_M \Rightarrow K$ such that $\varphi = \varphi^* M \circ \eta$, i.e. diagramatically



Density comonads are special types of left Kan extensions. They do not exist for all functors. However, when $\mathcal{A}$ is a small category and $\mathcal{B}$ is cocomplete then $\mathbb{D}_M$ exists for every functor $M\colon \mathcal{A} \to \mathcal{B}$. In such case, $\mathbb{D}_M(b)$ is computed as the colimit of the diagram:

$$M \downarrow b \xrightarrow{V} \mathcal{A} \xrightarrow{M} \mathcal{B}$$

where V is the forgetful functor from the comma category $M \downarrow b$, which consists of pairs (a, f) where $f\colon M(a) \to b$ is a morphism in $\mathcal{B}$, and morphisms $(a, f) \to (a', f')$ between such pairs are morphisms $g\colon a \to a'$ in $\mathcal{A}$ making the following triangle commute.

$$\begin{array}{ccc} M(a) & \xrightarrow{\quad M(g) \quad} & M(a') \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & b & \end{array}$$

We may express the same fact by the formula:

$$\mathbb{D}_M(b) \;=\; \operatorname*{colim}_{a \in \mathcal{A},\ M(a) \to b} M(a). \tag{3}$$

Note that $\mathcal{A}$ does not have to be small nor $\mathcal{B}$ cocomplete in general. It is enough that the colimit above exists. In such case we speak of *pointwise* density comonads. Denote by

$$\iota_f\colon M(a) \to \mathbb{D}_M(b)$$

the inclusion morphism of the copy of $M(a)$, corresponding to the morphism $f\colon M(a) \to b$, into the colimit. Then, the component $\eta_a\colon M(a) \to \mathbb{D}_M(M(a))$ of the natural transformation $\eta\colon M \Rightarrow \mathbb{D}_M M$ is given as ι_f for f equal to the identity morphism $\mathrm{id}\colon M(a) \to M(a)$.

2.4. **The comonad structure.** The initiality of $\eta\colon M \Rightarrow \mathbb{D}_M M$ ensures that we can equip $\mathbb{D}_M$ with a comonad structure. In particular, the identity natural transformation $M \Rightarrow \mathrm{Id} \circ M$ uniquely factors as the composition of η with the counit $\varepsilon\colon \mathbb{D}_M \Rightarrow \mathrm{Id}$ and, similarly, $\mathbb{D}_M(\eta) \circ \eta\colon M \Rightarrow \mathbb{D}_M \circ \mathbb{D}_M \circ M$ factors through the comultiplication $\delta\colon \mathbb{D}_M \Rightarrow \mathbb{D}_M^2$. In other words, the counit and the comultiplication are uniquely determined by the equations

$$\varepsilon M \circ \eta = \mathrm{id} \quad \text{and} \quad \delta M \circ \eta = \mathbb{D}_M(\eta) \circ \eta. \tag{4}$$

Moreover, these two equations guarantee that the functor

$$M^\dagger\colon \mathcal{A} \to \mathtt{EM}(\mathbb{D}_M) \tag{5}$$

which sends $a \in \mathcal{A}$ to the coalgebra $\eta_a\colon M(a) \to \mathbb{D}_M(M(a))$ is well-defined.

Lastly, we recall three equations of density comonads, which we use extensively throughout the paper. For $f\colon M(a) \to b$ and $h\colon b \to c$, the following three triangles commute.

$$\begin{array}{ccc} M(a) & & \\ {\scriptstyle \iota_f}\downarrow & \searrow{\scriptstyle \iota_{h \circ f}} & \\ \mathbb{D}_M(b) & \xrightarrow[\mathbb{D}_M(h)]{} & \mathbb{D}_M(c) \end{array} \qquad \begin{array}{ccc} M(a) & & \\ {\scriptstyle \iota_f}\downarrow & \searrow{\scriptstyle f} & \\ \mathbb{D}_M(b) & \xrightarrow[\varepsilon_b]{} & b \end{array} \qquad \begin{array}{ccc} M(a) & & \\ {\scriptstyle \iota_f}\downarrow & \searrow{\scriptstyle \iota_{\iota_f}} & \\ \mathbb{D}_M(b) & \xrightarrow[\delta_b]{} & \mathbb{D}_M(\mathbb{D}_M(b)) \end{array}$$

Equivalently, in the equational form we have:

(DC1) $\mathbb{D}_M(h) \circ \iota_f = \iota_{h \circ f}$
(DC2) $\varepsilon_a \circ \iota_f = f$
(DC3) $\delta_a \circ \iota_f = \iota_{\iota_f}$

All these constructions and equations above appeared already in [8].

2.5. **Comonad morphisms from composites.** Let M be the composite of functors

$$\mathcal{A}_0 \xrightarrow{M_0} \mathcal{A} \xrightarrow{M_1} \mathcal{B}$$

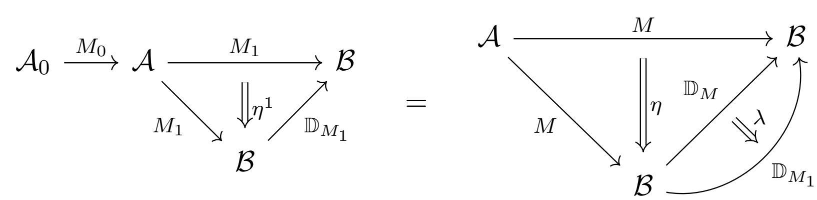
such that the density comonads $\mathbb{D}_M$ and $\mathbb{D}_{M_1}$ exist. Let

$$\eta\colon M \Rightarrow \mathbb{D}_M M \qquad \eta^1\colon M_1 \Rightarrow \mathbb{D}_{M_1} M_1$$

be the corresponding initial natural transformations. By initiality of η, there is a unique natural transformation

$$\lambda\colon \mathbb{D}_M \Rightarrow \mathbb{D}_{M_1}$$

such that:



Lemma 2.1. $\lambda\colon \mathbb{D}_M \Rightarrow \mathbb{D}_{M_1}$ *is a comonad morphism.*

In fact, $\mathbb{D}_{(-)}$ is a functor from the category of functors $X \to \mathcal{A}$, which admit density comonads, into the category of comonads and comonad morphisms. See e.g. [19].

3. Discrete density comonads

Recall that a category is *discrete* whenever it only has identity morphisms. By *discrete density comonads* we mean density comonads for functors whose domain is discrete. An important feature of discrete density comonads is that the formula in (3) simplifies dramatically. Indeed, assume that

$$M\colon \mathcal{A} \to \mathcal{B}$$

is a fixed functor from a small and discrete category $\mathcal{A}$. Then, since $\mathcal{A}$ is discrete, there are no morphisms between objects $M(a)$, given by morphisms $M(a) \to b$, in the colimit formula (3). Therefore, the density comonad $\mathbb{D}_M\colon \mathcal{B} \to \mathcal{B}$ is computed as a coproduct, that is, the colimit of a discrete diagram. Concretely, for $b \in \mathcal{B}$,

$$\mathbb{D}_M(b) \;=\; \coprod_{a \in \mathcal{A}} \; \coprod_{f\colon M(a) \to b} M(a). \tag{6}$$

Note that $\mathbb{D}_M$ exists whenever the above coproduct exists in $\mathcal{B}$, for every object $b \in \mathcal{B}$. In particular, $\mathbb{D}_M$ exists whenever $\mathcal{B}$ is cocomplete or has all coproducts.

As with general density comonads, we have inclusion morphisms

$$\iota_f \colon M(a) \to \mathbb{D}_M(b)$$

for every $f \colon M(a) \to b$, which satisfy axioms (DC1)–(DC3) from Section 2.4.

When proving Theorem 1.1, the category $\mathcal{B}$ will be either the category $\mathcal{R}(\sigma)$ of σ-structures and homomorphisms (where σ is a fixed relational signature) or the category **Graph** of graphs and their homomorphisms (where by graphs we mean undirected loopless graphs). For example, the former case, we may describe the comonad $\mathbb{D}_M$ even more explicitly. For a σ-structure B, the universe of $\mathbb{D}_M(B)$ consists of tuples

$$(M(a), h, x)$$

where $h \colon M(a) \to B$ is a homomorphism of relational structures and x is an element of $M(a)$. Further, an n-ary relation R in σ is interpreted as the set of all tuples

$$(M(a), h, x_1), \quad \dots\ , \quad (M(a), h, x_n)$$

such that $R(x_1,\ \dots,\ x_n)$ in $M(a)$.

The axiom (DC1) determines the action of $\mathbb{D}_M$ on morphisms. Given a homomorphism $g \colon B \to B'$, we have a homomorphism $\mathbb{D}_M(g) \colon \mathbb{D}_M(B) \to \mathbb{D}_M(B')$ defined by

$$(M(a), h, x) \mapsto (M(a),\ g \circ h,\ x).$$

Moreover, by (DC2), the counit $\varepsilon_B \colon \mathbb{D}_M(B) \to B$ is the following map

$$(M(a), h, x) \mapsto h(x)$$

and the comultiplication $\delta_B \colon \mathbb{D}_M(B) \to \mathbb{D}_M^2(B)$ is given by

$$(M(a), h, x) \mapsto (M(a), \iota_h, x),$$

as dictated by (DC3).

Example 3.1. Let $\mathcal{A}$ be the discrete subcategory of graphs, consisting of only the triangle graph, and let $M \colon \mathcal{A} \to$ **Graph** be the inclusion of $\mathcal{A}$ into the category of graphs. Then, given an arbitrary graph G, the graph computed as $\mathbb{D}_M(G)$ is the disjoint union of $k \times l$ copies of triangles, where k is the number of triangles in G and l is the number of automorphisms of the triangle graph.

Note that the extra copies arising from the automorphisms can be eliminated by adding all automorphisms of the triangle graph into $\mathcal{A}$. However, then $\mathcal{A}$ will be a groupoid and not just a discrete category. In such case the proofs that we present later should still go through, albeit with the added burden of keeping track of the added automorphisms.

4. The abstract classification theorem

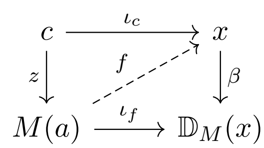
In this section we prove Theorem 1.1. Since the entire reasoning can be carried abstractly in the language of category theory, we actually prove a general categorical statement that can be applied in other scenarios too. In the following we fix a functor

$$M \colon \mathcal{A} \to \mathcal{B}$$

from a discrete category $\mathcal{A}$ into $\mathcal{B}$. We further assume that the pointwise density comonad $\mathbb{D}_M$ exists on $\mathcal{B}$ (i.e. it is given by the formula (6)).

We start with a useful observation. Recall that an object c is *connected* iff, for every morphism $f \colon c \to \coprod_i a_i$ into a coproduct, the morphism f factors uniquely through one of the inclusions morphisms $\iota_i \colon a_i \to \coprod_i a_i$.[3] Whenever a connected c is such that $a \cong c + x$ for some a, x, we say that c is a *(sub)component* of a.

Lemma 4.1. *Let $\beta \colon x \to \mathbb{D}_M(x)$ be a $\mathbb{D}_M$-coalgebra and let $\iota_c \colon c \to x$ be a subcomponent inclusion. Further, let $f \colon M(a) \to x$ be the morphism for which $\beta \circ \iota_c$ decomposes through ι_f, as shown below.*



(7)

Then, also the two triangles in the diagram above commute, that is, $\iota_c = f \circ z$ and $\iota_f = \alpha \circ f$.

Proof. The first equality is obtained immediately from the triangle law of coalgebras (cf. (1)) together with (DC2) from Section 2.4 as

$$\iota_c = \varepsilon_c \circ \beta \circ \iota_c = \varepsilon_c \circ \iota_f \circ z = f \circ z.$$

To show that also $\iota_f = \alpha \circ f$ we apply the square law of coalgebras (cf. (1)). Observe that

- $\delta_c \circ \beta \circ \iota_c = \delta_c \circ \iota_f \circ z = \iota_{\iota_f} \circ z$ by (DC3), and
- $\mathbb{D}_M(\beta) \circ \beta \circ \iota_c = \mathbb{D}_M(\beta) \circ \iota_f \circ z = \iota_{\beta \circ f} \circ z$ by (DC1).

Because c is connected, the factorisation $c \to M(a) \to \mathbb{D}_M(\mathbb{D}_M(c))$ into the coproduct must be unique, hence $\iota_f = \beta \circ f$. □

In the following we need to assume that $\mathcal{B}$ is a *constituental category*, i.e. that

- every element in $\mathcal{B}$ is (isomorphic to) a coproduct of connected elements, and
- the inclusion morphisms $\iota_i \colon a_i \to \coprod_i a_i$ are monomorphisms for any coproduct $\coprod_i a_i$.

[3] Equivalently, c is connected iff $\hom(c, -)$ distributes over coproducts. Observe that, by this definition, the initial object (if it exists in the category) is connected only if all objects in the category are isomorphic to one another.

Further, we say that an object of $\mathcal{B}$ is *essentially in* $\mathcal{A}$ if it is isomorphic to $M(a)$, for some a in $\mathcal{A}$.

We can now state and prove a version of Theorem 1.1 for connected objects.

Lemma 4.2. *If $\mathcal{B}$ is a constituental category, then a connected object c of $\mathcal{B}$ is essentially in $\mathcal{A}$ iff it admits a $\mathbb{D}_M$-coalgebra.*

Proof. The left-to-right implication follows immediately from the fact that, $M^\dagger(a)$ is a coalgebra on $M(a)$, for every $a \in \mathcal{A}$ (cf. (5)). For the right-to-left implication, let $\beta\colon c \to \mathbb{D}_M(c)$ be a coalgebra. Since c is connected,

$$\beta\colon c \to \coprod_{a\in\mathcal{A}} \; \coprod_{f\colon M(a)\to c} M(a)$$

uniquely factors as $\beta = \iota_f \circ \beta_0$, for some $\beta_0\colon c \to M(a)$ and $f\colon M(a) \to c$. By Lemma 4.1, the following diagram commutes.

$$\begin{array}{ccc} c & \xrightarrow{\mathrm{id}} & c \\ {\scriptstyle\beta_0}\big\downarrow & \overset{f}{\nearrow} & \big\downarrow{\scriptstyle\beta} \\ M(a) & \xrightarrow{\iota_f} & \mathbb{D}_M(c) \end{array}$$

One triangle immediately gives $f \circ \beta_0 = \mathrm{id}$ and the other triangle together with $\beta = \iota_f \circ \beta_0$ and the fact that inclusions are monomorphisms (in constituental categories) entails $\beta_0 \circ f = \mathrm{id}$. □

To make progress, we need to assume that $\mathcal{A}$ is *component-based.* This means that, whenever a is equal to a coproduct of connected objects $\coprod_i c_i$ and a is essentially $\mathcal{A}$, then so is c_i, for every i. Note that this condition mirrors the third item in the list of assumptions in Theorem 1.1.

With this we show the main technical lemma of this section.

Lemma 4.3. *If $\mathcal{A}$ is component-based and $\mathcal{B}$ is a constituental category, then any subcomponent c of an object x of $\mathcal{B}$ which admits a $\mathbb{D}_M$-coalgebra $\alpha\colon x \to \mathbb{D}_M(x)$ is essentially in $\mathcal{A}$.*

Proof. Let $\iota_c\colon c \to x$ be the inclusion morphism of c into one of the subcomponents of x. Recall that $\mathbb{D}_M(x)$ is (isomorphic to) a coproduct of elements of the form $M(a)$. Therefore, because c is connected, the morphism $c \xrightarrow{\iota_c} x \xrightarrow{\alpha} \mathbb{D}_M(x)$ factors uniquely through $M(a)$ of some $f\colon M(a) \to x$ and, by Lemma 4.1, the following diagram commutes.

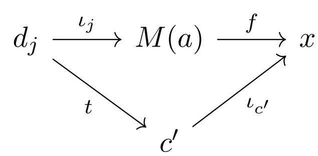
$$\begin{array}{ccc} c & \xrightarrow{\iota_c} & x \\ {\scriptstyle z}\big\downarrow & \overset{f}{\nearrow} & \big\downarrow{\scriptstyle\alpha} \\ M(a) & \xrightarrow{\iota_f} & \mathbb{D}_M(x) \end{array} \tag{8}$$

Further, since $\mathcal{B}$ is a constituental category, $M(a) \cong \coprod_j d_j$ for some connected objects d_j from $\mathcal{B}$. Again, since c is connected, there is a j such that

z factors as the composite

$$c \xrightarrow{z_0} d_j \xrightarrow{\iota_j} M(a)$$

Since $\mathcal{A}$ is component-based, d_j is essentially in $\mathcal{A}$. Therefore, in order to show that also c is essentially in $\mathcal{A}$, it is enough to show that c is isomorphic to d_j. To this end, since d_j is connected and x is a coproduct of connected elements in $\mathcal{B}$, the composite $f \circ \iota_j$ must factor through one of the subcomponents c' of x, as shown in the following diagram.



Consequently, $\iota_c = \iota_{c'} \circ t \circ z_0$ and so $c = c'$ and $t \circ z_0 = \mathrm{id}$ since c is connected. Further, observe that since $\iota_f = \alpha \circ f$ we have that

$$\iota_f \circ \iota_j = \alpha \circ f \circ \iota_j = \alpha \circ \iota_c \circ t = \iota_f \circ z \circ t = \iota_f \circ \iota_j \circ z_0 \circ t.$$

Therefore, $z_0 \circ t = \mathrm{id}$ since inclusions are monomorphisms in constituental categories. We have shown that z_0 is an isomorphism, hence c is essentially in $\mathcal{A}$. □

The main classification theorem, which we state in full, is obtained as a consequence of the previous two lemmas.

Theorem 4.4. *Let $M\colon \mathcal{A} \to \mathcal{B}$ be a functor from a discrete component-based category $\mathcal{A}$ into a constituental category $\mathcal{B}$, such that the pointwise density comonad $\mathbb{D}_M$ exists.*

Then an element $b \in \mathcal{B}$ is isomorphic to a coproduct of objects essentially in $\mathcal{A}$ if and only if b admits a $\mathbb{D}_M$-coalgebra.

Proof. The left-to-right implication follows from Lemma 4.2 and the fact that coalgebras are closed under coproducts that exist in $\mathcal{B}$. Conversely, if $\alpha\colon x \to \mathbb{D}_M(x)$ is a coalgebra then, by our assumptions, x is isomorphic to a coproduct $\coprod_i c_i$ of connected elements by Lemma 4.3 and all those subcomponents are essentially in $\mathcal{A}$. □

Observe that both $\mathcal{R}(\sigma)$ and **Graph** are constituental categories. Therefore, the previous theorem immediately yields Theorem 1.1. Given a component-based class Δ of relational structures or graphs, let Δ_c be the subclass of Δ consisting of connected structures only. We then set $\mathcal{A}$ to be a discrete subcategory of $\mathcal{R}(\sigma)$ or **Graph**, consisting of one representative from every isomorphism class in Δ_c. Since we picked only one representative from every equivalence class, the class Δ_c is just a set and, therefore, the density comonad $\mathbb{D}_M$, for the inclusion functor $M\colon \mathcal{A} \to \mathcal{R}(\sigma)$, exists because both $\mathcal{R}(\sigma)$ and **Graph** have all (small) coproducts. Observe that the comonad $\mathbb{D}_M$ classifies Δ. Indeed, by Theorem 4.4, a finite relational structure B has a $\mathbb{D}_M$-coalgebra if and only if there exist $C_1, \ldots, C_n$ in Δ_c such that

$B \cong C_1 + \cdots + C_n$. In turn, this is equivalent to B being in Δ, which follows from being component-based as then $C_1 + \cdots + C_n$ is in Δ iff all the individual structures $C_1, \ldots, C_n$ are.

Remark 4.5. In the proof of Theorem 1.1, in the previous paragraph, we made sure that all objects in the image of M are connected. This is stronger than assuming that $\mathcal{A}$ is component-based. By carefully inspecting the proof of Theorem 4.4 and the preceding lemmas one can check that this extra assumption allows us to drop the requirement that $\mathcal{B}$ is constituental. See Lemma 6.3 below for more details.

Remark 4.6. Theorem 1.1 says that, for a class Δ of structures closed under isomorphisms and finite coproducts, if Δ is also closed under subcomponents then it can be classified by a comonad. However, the converse does not hold. Let C and D be two graphs with no homomorphism $C \to D$ nor any homomorphisms $D \to C$. For example, if C is the triangle and D is the cycle on 5 vertices. Take $\mathcal{A}$ to be the discrete subcategory of **Graph** consisting of $C + D$ only and let $M \colon \mathcal{A} \to \mathbf{Graph}$ be the subcategory inclusion. Then, despite $C + D$ admitting a $\mathbb{D}_M$-coalgebra, no connected graph admits a $\mathbb{D}_M$-coalgebra (by Lemma 4.2). It is easy to see that the class of finite structures classified by $\mathbb{D}_M$ is the class consisting of graphs isomorphic to

$$C + \cdots + C + D + \cdots + D$$

where the number of copies of C has to be at least 1 and similarly for D. Consequently, the class of structures classified by $\mathbb{D}_M$ is not closed under subcomponents.

4.1. **Examples.** The category $\mathcal{R}(\sigma)$ of σ-structures is a constituental category. Therefore, in our applications we only need to check that a class Δ of σ-structures is closed under finite coproducts and taking subcomponents of its objects. However, these are fairly weak conditions satisfied by many known examples of classes from the literature. In particular, this includes classes of finite structures closed under finite coproducts which are

1. monotone, i.e. class closed under substructures,
2. hereditary, i.e. class closed under taking induced substructures, or
3. closed under taking graph minors.

Further examples include

4. Fraïssé classes closed under free amalgamations, or
5. classes of coproducts of connected cores.

Recall that core is a structure with the property that all of its endomorphims are automorphisms. An example of a class from (5) is the class of coproducts of cycles. Note that the discrete density comonad for this class captures co-spectrality, see Section 6.

As an example of a non-example, take the class of graphs that can be drawn on a surface of genus n, for $n > 1$. This class it is specified by a finite set of forbidden minors. However, it is not closed under taking coproducts and hence is not a component-based class. On the other hand, any minor-closed class can be completed under finite coproducts. The resulting class is then still minor closed [11, Lemma 5] and hence is classified by a comonad.[4]

Remark 4.7. The proof of Theorem 1.1 is carried abstractly, in the language of category theory, and thus can be dualised. In the dual statement we have monads instead of comonads and instead of component-based classes speak about classes closed under isomorphisms, direct (categorical) products, and *factors*, i.e. with the property that if $A \times B$ is in the class then so are A and B. For such classes there is a monad which classifies the class, i.e. a finite structure is in the class iff it admits an algebra for the monad. An example of a class of graphs which can be classified this way is the class of connected non-bipartite graphs, cf. Chapter 8 in [17].

5. Graph parameters

A *graph parameter* is a mapping $\mu\colon \mathbf{Graph}_{fin} \to \overline{\mathbb{R}}$, from the class of finite graphs $\mathbf{Graph}_{fin}$ to the class of extended real numbers $\overline{\mathbb{R}} = [-\infty, +\infty]$, which gives the same value to any two isomorphic graphs. Moreover, we say that it is *standard*[5] if $\mu(G_1 + G_2) = \max\{\mu(G_1), \mu(G_2)\}$.

Standard graph parameters cover many well-known examples of graph parameters from the literature, such as

- clique number, chromatic number, max-degree,
- tree-depth, tree-width, path-width, clique-width, etc.

In this section we show that every standard graph parameter μ gives rise to a *graded comonad* $(\mathbb{C}_k)_k$, that is, a sequence of comonads $(\mathbb{C}_k)_k$ indexed by extended real numbers and comonad morphisms $g_{k,l}\colon \mathbb{C}_k \Rightarrow \mathbb{C}_l$, for every $k \leq l$ in $\overline{\mathbb{R}}$, such that $g_{k,l} = g_{j,l} \circ g_{k,j}$ for any $k \leq j \leq l$ in $\overline{\mathbb{R}}$.[6] Given a graded comonad $(\mathbb{C}_k)_k$ we define the *coalgebra number* $\kappa^{\mathbb{C}}(G)$, of a graph G, to be the infimum of $k \in \overline{\mathbb{R}}$ such that G admits a $\mathbb{C}_k$-coalgebra. In fact, we show that the coalgebra number, for the constructed graded comonad, agrees with the standard graph parameter μ we started with. In other words, we have that $\mu(G) \leq k$ iff G admits a $\mathbb{C}_k$-coalgebra.

Note that every graded comonad $(\mathbb{C}_k)_k$ trivially determines a graph parameter, by setting $\mu(G) := \kappa^{\mathbb{C}}(G)$. We have already mentioned graded comonads characterising graph parameters this way. For example, the (graded) Ehrenfeucht–Fraïssé comonad $(\mathbb{E}_k)_k$ characterises tree-depth [3, 4], the pebbling comonad $(\mathbb{P}_k)_k$ characterises tree-width [1] and the pebble-relation comonad $(\mathbb{PR}_k)_k$ classify path-width [23].

---

[4] We are grateful to Anuj Dawar for pointing out these facts.

[5] Also known as *maxing*, e.g. in [21]. Note that this does not include a large class of parameters such as *additive* or *multiplicative* graph parameters where $\mu(G_1 + G_2)$ is either $\mu(G_1) + \mu(G_2)$ or $\mu(G_1) \cdot \mu(G_2)$, respectively.

[6] In fact, this is a special type of graded comonads, with the grading being over the fixed monoid $(\overline{\mathbb{R}}, \min, +\infty)$.

To start with, observe that there is a one-to-one correspondence between *graph properties*, i.e. graph parameters valued in $\{0,1\}$, and classes of finite graphs which are closed under isomorphisms. Furthermore, it is easy to see that the correspondence restricts to that of standard graph properties and component-based classes of graphs:

Lemma 5.1. *Given a standard graph property* $\mu\colon \mathbf{Graph}_{fin} \to \{0,1\}$*, the class of graphs G such that $\mu(G) = 0$ is closed under isomorphisms, finite coproducts, and subcomponents. In fact, every such class is obtained from a standard graph property this way.* □

Therefore, by Theorem 1.1, there is a comonad $\mathbb{C}^\mu$ which classifies the class Δ of finite graphs G such that $\mu(G) = 0$, for every standard graph property μ. We construct $\mathbb{C}^\mu$ explicitly, as the density comonad for the inclusion functor

$$\mathcal{A} \to \mathbf{Graph}, \tag{9}$$

where $\mathcal{A}$ is a discrete subcategory of finite connected graphs consisting precisely of one graph from every isomorphism class in Δ. Then, by Theorem 4.4, $\mathbb{C}^\mu$ classifies Δ.

5.1. **Grading graph parameters.** We use this to construct a sequence of comonads for a given standard graph parameter μ. For every extended real number k, we turn μ into a graph property

$$\mu_{\le k}\colon \mathbf{Graph} \to \{0,1\}$$

by setting $\mu_{\le k}(G) = 0$ iff $\mu(G) \le k$. Then, the density comonad $\mathbb{C}^\mu_k$, defined as $\mathbb{C}^\mu$ for $\mu := \mu_{\le k}$, classifies finite graphs G such that $\mu(G) \le k$.

Moreover, we can make sure that there is a linearly ordered chain of embeddings of discrete categories

$$\mathcal{A}_{-\infty} \hookrightarrow \cdots \hookrightarrow \mathcal{A}_k \hookrightarrow \mathcal{A}_l \hookrightarrow \cdots \hookrightarrow \mathcal{A}_{+\infty} \qquad \text{(with } k \le l\text{)}$$

where each $\mathcal{A}_k$ is a category as in (9), for the class of graphs G such that $\mu(G) \le k$. Then, by Lemma 2.1 in Section 2.5, the composite

$$\mathcal{A}_k \hookrightarrow \mathcal{A}_l \to \mathbf{Graph},$$

for $k \le l$, gives rise to a comonad morphism $g_{k,l}\colon \mathbb{C}^\mu_k \Rightarrow \mathbb{C}^\mu_l$. In fact, we have $g_{k,l} = g_{j,l} \circ g_{k,j}$ for every $k \le j \le l$, by functoriality of $\mathbb{D}_{(-)}$. Hence, $(\mathbb{C}^\mu_k)_k$ is a graded comonad with the property that $\kappa^{\mathbb{C}^\mu}(G) = \mu(G)$ for every finite graph G.

Remark 5.2. The procedure to produce sequences of comonads for standard graph parameters can be defined dually for graph parameters μ with the property that $-\mu$ is standard, i.e. graph parameters such that $\mu(G_1 + G_2) = \min\{\mu(G_1), \mu(G_2)\}$. This is done by constructing a sequence of standard graph properties

$$\mu_{\ge k}\colon \mathbf{Graph} \to \{0,1\}$$

and inducing comonads classifying the classes of graphs such that $\mu(G) \geq k$ in a similar spirit as before. This then covers examples of graph parameters such as min-degree and girth.

5.2. **Comparison with game comonads.** For some graph parameters and classes of structures we already knew how to construct comonads that classify them. For example, for the classes of structures classified by comonads $\mathbb{P}_k$, $\mathbb{E}_k$, and $\mathbb{PR}_k$. In this section, we explain the relationship between those comonads and discrete density comonads constructed directly for given classes.

In fact, we will show that discrete density comonads are weakly initial in the category of comonads that classify the same class. To this end, denote by $\mathbb{D}_\Delta$ the discrete density comonad constructed as in (9) above, for a component-based class Δ.

Proposition 5.3. *Let Δ be a component-based class of relational structures or graphs and let $\mathbb{C}$ be a comonad that classifies a class Γ. Then, $\Delta \subseteq \Gamma$ if, and only if, there exists a comonad morphism $\mathbb{D}_\Delta \Rightarrow \mathbb{C}$.*

Observe that the right-to-left direction is immediate as a comonad morphism $\mathbb{D}_\Delta \Rightarrow \mathbb{C}$ lifts to a functor $L\colon \mathtt{EM}(\mathbb{D}_\Delta) \to \mathtt{EM}(\mathbb{C})$ making the following diagram commute (cf. Section 2.2).

$$\begin{array}{ccc} \mathtt{EM}(\mathbb{D}_\Delta) & \xrightarrow{\quad L \quad} & \mathtt{EM}(\mathbb{C}) \\ & {\scriptstyle U^{\mathbb{D}_\Delta}} \searrow \quad \swarrow {\scriptstyle U^{\mathbb{C}}} & \\ & \mathcal{B} & \end{array}$$

(Here $\mathcal{B}$ is either the category of relational structures or graphs.) For a structure B in Δ, let $\beta\colon B \to \mathbb{D}_\Delta(B)$ be a $\mathbb{D}_\Delta$-coalgebra, which exists because $\mathbb{D}_\Delta$ classifies Δ. Then, by the commutativity of the above triangle $L(\beta)$ is a $\mathbb{C}$-coalgebra $B \to \mathbb{C}(B)$ making $B \in \Gamma$ because $\mathbb{C}$ classifies Γ.

We carry out the left-to-right direction of the proof abstractly, for arbitrary categories rather than just relational structures or graphs. Let $\mathbb{D} := \mathbb{D}_M$ be the density comonad of a functor $M\colon \mathcal{A} \to \mathcal{B}$ from a discrete category $\mathcal{A}$. Further, assume that $\mathbb{C}$ is a comonad on $\mathcal{B}$ such that for every $a \in \mathcal{A}$, there exists a coalgebra

$$\varphi_a\colon M(a) \to \mathbb{C}(M(a)).$$

Observe that, since $\mathcal{A}$ is a discrete category, the collection of morphisms $\{\varphi_a \mid a \in \mathcal{A}\}$ trivially forms a natural transformation $\varphi\colon M \Rightarrow \mathbb{C}M$. Since $\mathbb{D}$ is a density comonad of M, there is a natural transformation $\varphi^*\colon \mathbb{D} \Rightarrow \mathbb{C}$ such that

$$\varphi = \varphi^* M \circ \eta \tag{10}$$

where $\eta\colon M \Rightarrow \mathbb{D}M$ is the initial natural transformation determining $\mathbb{D}$ (cf. Section 2.3).

To prove Proposition 5.3, it is enough to show the following.

Lemma 5.4. *Natural transformation $\varphi^*\colon \mathbb{D} \Rightarrow \mathbb{C}$ is a morphism of comonads.*

Proof. First, let us check commutativity of the triangle diagram in (2). By (4), $\varepsilon^{\mathbb{D}} M \circ \eta = \mathrm{id}$. Similarly, by (10), $\varepsilon^{\mathbb{C}} \circ \varphi^* \circ \eta = \varepsilon^{\mathbb{C}} \circ \varphi = \mathrm{id}$ where the last equality follows from commutativity of the triangle law in (1) for every individual coalgebra $\varphi_a\colon M(a) \to \mathbb{C}(M(a))$. We see that $\varepsilon^{\mathbb{D}} M \circ \eta = (\varepsilon^{\mathbb{C}} \circ \varphi^*) M \circ \eta$ from which it follows that $\varepsilon^{C} = \varepsilon^{\mathbb{C}} \circ \varphi^*$ by the universal property of η.

Next, we check the oblong law in (2). The composition $\delta^{\mathbb{C}} M \circ \varphi^* M \circ \eta$ is equal to $\delta^{\mathbb{C}} M \circ \varphi$, by (10), which is equal to $\mathbb{C}\varphi \circ \varphi$ by the square law of coalgebras (1). Similarly,

$$
\begin{aligned}
\varphi^* \mathbb{C} M \circ \mathbb{D}\varphi^* M \circ \delta^{\mathbb{D}} M \circ \eta &= \varphi^* \mathbb{C} M \circ \mathbb{D}\varphi^* M \circ \mathbb{D}\eta \circ \eta \\
&= \varphi^* \mathbb{C} M \circ \mathbb{D}\varphi \circ \eta \\
&= \mathbb{C}\varphi \circ \varphi^* M \circ \eta \\
&= \mathbb{C}\varphi \circ \varphi
\end{aligned}
$$

The first equality follows from (4), the second and fourth from (10) and the third is naturality of φ^*. We have shown that $(\varphi^* \mathbb{C} \circ \mathbb{D}\varphi^* \circ \delta^{\mathbb{D}}) M \circ \eta$ is equal to $(\delta^{\mathbb{C}} \circ \varphi^*) M \circ \eta$, which by the universal property of η implies that $\varphi^* \mathbb{C} \circ \mathbb{D}\varphi^* \circ \delta^{\mathbb{D}} = \delta^{\mathbb{C}} \circ \varphi^*$. □

Example 5.5. Proposition 5.3 gives us that that for our running examples of comonads $\mathbb{E}_k$, $\mathbb{P}_k$, $\mathbb{PR}_k$, there exist comonad morphisms

$$
\mathbb{D}_{\mathcal{TD}_k} \Rightarrow \mathbb{E}_k, \quad \mathbb{D}_{\mathcal{TW}_k} \Rightarrow \mathbb{P}_k, \quad \text{and} \quad \mathbb{D}_{\mathcal{PW}_k} \Rightarrow \mathbb{PR}_k,
$$

where $\mathcal{TD}_k$, $\mathcal{TW}_k$, and $\mathcal{PW}_k$ are the classes of structures of tree-depth, tree-width, and path-width $\leq k$, respective.

Note that unlike with the game comonads $\mathbb{E}_k$, $\mathbb{P}_k$, and $\mathbb{PR}_k$, the discrete density comonads $\mathbb{D}_{\mathcal{TD}_k}$, $\mathbb{D}_{\mathcal{TW}_k}$, and $\mathbb{D}_{\mathcal{PW}_k}$ do not classify infinite structures of said properties.

5.3. **Nowhere dense comonads.** A direct consequence of Proposition 5.3 is that we can characterise comonads that classify monotone nowhere dense classes of graphs in terms of non-existence of certain comonad morphisms. Recall that a class Δ is *somewhere dense* if there exists a natural number p such that, for every n, the p-th subdivision K_n^p of all edges in the clique graph K_n on n vertices is a subgraph of some graph in Δ. Then, a class is *nowhere dense* if it is not somewhere dense.

It is immediate that, a monotone class of graphs Δ (i.e. a class closed under substructures) is somewhere dense if and only if $\mathbf{Cli}_p \subseteq \Delta$, for some p, where

$$
\mathbf{Cli}_p = \{K_n^p \mid n \in \mathbb{N}\}
$$

is the class of p-th subdivisions of all cliques. Finally, let Cli_p be the density comonad for the inclusion functor $\mathbf{Cli}_p \to \mathbf{Graph}$, where $\mathbf{Cli}_p$ is viewed as a discrete subcategory of $\mathbf{Graph}$. We can now state the characterisation.

Proposition 5.6. *Assume that* $\mathbb{C}$ *classifies a class of graphs* Δ*. Then,* Δ *is nowhere dense if, and only if, there is no comonad morphism* $\mathrm{Cli}_p \Rightarrow \mathbb{C}$ *for any* $p \in \mathbb{N}$*.*

Proof. Define $\overline{\mathbf{Cli}_p}$ to be the closure of $\mathbf{Cli}_p$ under finite coproducts. Observe that $\overline{\mathbf{Cli}_p}$ is component-based and, since the connected objects in $\overline{\mathbf{Cli}_p}$ are precisely the objects in $\mathbf{Cli}_p$, the comonad Cli_p classifies $\overline{\mathbf{Cli}_p}$. Moreover, since any class classified by a comonad needs to be closed under finite coproducts, $\mathbf{Cli}_p \subseteq \Delta$ iff $\overline{\mathbf{Cli}_p} \subseteq \Delta$. The result follows by monotonicity of Δ and by Proposition 5.3. □

6. Lovász-type theorems for free

A classic result of Lovász [20] says that two finite structures are isomorphic if and only if they admit the same number of homomorphisms from all finite structures. This result has been extended in many different ways. In one type of generalisations, isomorphism are replaced by a selected equivalence relations $\asymp$ on finite structures, and the class of all finite structures by a class of selected finite structures Δ. Then a typical Lovász-type theorems expresses that, for finite structures A, B,

$$A \asymp B \iff \hom(C, A) \cong \hom(C, B) \quad \text{for every } C \in \Delta.$$

A number of well-known equivalence relations on finite structures have been characterised this way. See Figure 1 for an overview of some Lovász-type results.

Δ	$\asymp$	reference
cycles	co-spectrality	(folklore)
trees	fractional isomorphism	[24]
bipartite graphs	isomorphic bipartite double covers	[10, 21] [7]
planar graphs	quantum isomorphism	[22]
tree-depth $\leq k$	Duplicator wins the bijective k-round Ehrenfeucht–Fraïssé game	[16]
tree-width $< k$	Duplicator wins the bijective k-pebble game	[15]
path-width $< k$	Duplicator wins the bijective k-pebble relation game	[23]
with k-pebble tree cover of height $\leq n$	Duplicator wins the bijective n-round k-pebble E.F. game	[13]
synchronization trees of height $\leq k$	equivalence in graded modal logic of modal depth $\leq k$	[13]

Figure 1. Examples of Lovász-type theorems

Note that the equivalence relations $\asymp$ in Figure 1 corresponding to winning strategies of Duplicator can be equivalently described as logical equivalences with respect to a fragment of first-order logic with counting quantifiers. A comonadic proof of the first two Lovász-type theorems that identify a logic fragment was established in [13] and later adapted in [23] to obtain the new result for path-width. For the comonadic proof to work it is necessary that the comonad $\mathbb{C}$ classifies the said relation $\asymp$, i.e. that $A \asymp B$ holds precisely whenever the cofree coalgebras $F^{\mathbb{C}}(A)$ and $F^{\mathbb{C}}(B)$ are isomorphic.

Anuj Dawar has asked (in private communication) whether there are comonads covering the other listed cases as well. In our terminology, this means finding comonads that classify both the class Δ as well as the corresponding $\asymp$ relation, in the same row of the table. We answer this question in the positive for any component-based class Δ. We show that the discrete density comonad $\mathbb{C}$ that classifies such class also classifies the relation $\asymp$, with $A \asymp B$ whenever $\hom(C, A) \cong \hom(C, B)$ for every $C \in \Delta$, thereby proving Corollary 1.2.

The main ingredient of our proof is the following recent result due to Luca Reggio, proved abstractly for locally finitely presentable categories.

Theorem 6.1 (Corollary 5.15 in [25]). *Let $\mathbb{C}$ be a comonad of finite rank on* **Graph** *or $\mathcal{R}(\sigma)$. Then, for two finite structures A, B,*

$$F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B) \quad \textit{iff} \quad \hom(C, A) \cong \hom(C, B),$$

for every finite C which admits a $\mathbb{C}$-coalgebra.

We see that in order to prove Corollary 1.2, it is enough to show that the comonad $\mathbb{C}$ constructed in the proof of Theorem 1.1 has finite rank. In the next section we define finite rank comonads and show that the discrete density comonad constructed in the proof of Theorem 1.1 does have finite rank.

6.1. **Finite rank comonads.** We work in the general setting of categories, rather than the more concrete setting of relational structures or graphs. Recall that an object c of a category $\mathcal{B}$ is *finitely presentable* if any morphism from c into a colimit d of a filtered diagram $\{d_i\}_i$ factors through one of the inclusion morphisms $d_i \to d$. Equivalently, the functor $\hom(c, -)$ preserves filtered colimits.

Let $\mathbb{C}$ be a comonad over a category $\mathcal{B}$ and let $U \colon \mathtt{EM}(\mathbb{C}) \to \mathcal{B}$ be the usual forgetful functor. We say that a comonad $\mathbb{C}$ has *finite rank* if

(1) $\mathbb{C}$ is finitary, i.e. its underlying functor preserves filtered colimits,

---

[7] The *bipartite double cover* of a graph G is the product graph $G \times K_2$ where K_2 is the clique on two vertices.

The fact that isomorphic bipartite double covers correspond to counting homomorphisms from bipartite graphs was worked out by Böker in his master thesis [10]. He later observed (in private communication) that the same result already follows from Section 5.4.2 in [21].

(2) if every morphism of the form $f\colon c \to U(x)$, from a finitely presentable c, admits a factorisation

$$f = c \xrightarrow{f_0} U(y) \xrightarrow{U(\gamma)} U(x)$$

for some $\gamma\colon y \to x$ such that $U(y)$ is finitely presentable, and

(3) this factorisation is essentially unique, i.e. if $g\colon c \to U(y)$ satisfies $f = U(\gamma) \circ g$ then for some factorisation of γ into $\lambda\colon y \to y'$ and $\gamma'\colon y' \to x$ such that $U(y')$ is finitely presentable, $U(\lambda)\circ f_0 = U(\lambda)\circ g$.

Observe that if $U(\gamma)$ in the second item is a monomorphism, then essential uniqueness is automatic. In fact, this will be the case in our construction.

In the following we fix a functor

$$M\colon \mathcal{A} \to \mathcal{B}$$

from a discrete category $\mathcal{A}$ and assume that the density comonad $\mathbb{D}_M$ for M exists.

Proposition 6.2. *If all objects in the image of M are finitely presentable, then $\mathbb{D}_M$ is finitary.*

Proof. Let $D\colon I \to \mathcal{B}$ be a (small) directed diagram with a colimit $\mathrm{colim}(D)$. We show that the diagram $\mathbb{D}_M \circ D$ has a colimit and it coincides with $\mathbb{D}_M(\mathrm{colim}(D))$. Let $f\colon M(c) \to \mathrm{colim}(D)$ be a morphism, for some $c \in \mathcal{A}$. Since $M(c)$ is finitely presentable, there is essentially unique factorisation

$$f = M(c) \xrightarrow{g} D(i) \xrightarrow{d_i} \mathrm{colim}(D)$$

where $d_i\colon D(i) \to \mathrm{colim}(D)$ is the cocone morphism for $D(i)$. Being essentially unique means that for any other $g'\colon M(c) \to D(i)$ with $f = d_i \circ g'$ there is a morphism $t\colon i \to j$ such that $D(t) \circ g = D(t) \circ g'$ and, consequently, also $f = d_i \circ g = d_j \circ D(t) \circ g$. Furthermore, if $f = d_k \circ h$ for some $h\colon M(c) \to D(k)$ then, because the diagram is directed, there are morphisms $z_i\colon i \to l$ and $z_k\colon k \to l$ such that $d_i = d_l \circ D(z_i)$ and $d_k = d_l \circ D(z_k)$. Then, $f = d_l \circ D(z_i) \circ g = d_l \circ D(z_k) \circ h$.

In other words, we just showed that morphisms $M(c) \to \mathrm{colim}(D)$ are in bijection with the set

$$S_c \;:=\; \left(\coprod_{i\in I} \hom(M(c), D(i))\right) / \sim$$

where $\sim$ is the least equivalence relation containing

$$(M(c) \xrightarrow{f} D(i)) \;\sim\; (M(c) \xrightarrow{f} D(i) \xrightarrow{D(t)} D(j))$$

for any $t\colon i \to j$ in I.

On the other hand, given $t\colon i \to j$ in I, the morphism

$$\mathbb{D}_M(D(t))\colon\; \mathbb{D}_M(D(i)) \to \mathbb{D}_M(D(j))$$

maps the $M(c)$ subcomponent corresponding to $f\colon M(c) \to D(i)$ to the subcomponent corresponding to $D(t) \circ f\colon M(c) \to D(j)$. In other words, precisely the subcomponents related by $\sim$ are identified. ☐

Next, we prove a technical lemma needed to show that $\mathbb{D}_M$ has finite rank. It is essentially a strengthening of Lemma 4.3.

Lemma 6.3. *Assume that all objects in the image of M are connected. Let c be a subcomponent of x and let $\beta\colon x \to \mathbb{D}_M(x)$ be a $\mathbb{D}_M$-coalgebra. Then, c can be equipped with a $\mathbb{D}_M$-coalgebra $\alpha\colon c \to \mathbb{D}_M(c)$ such that the inclusion $\iota_c\colon c \to x$ extends to a coalgebra homomorphism:*

$$\begin{array}{ccc} c & \xrightarrow{\iota_c} & x \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \mathbb{D}_M(c) & \xrightarrow{\mathbb{D}_M(\iota_c)} & \mathbb{D}_M(x) \end{array}$$

Proof. By Lemma 4.1, $\beta \circ \iota_c$ has a decomposition through $M(a)$ of some $f\colon M(a) \to x$ such that the following diagram commutes.

$$\begin{array}{ccc} c & \xrightarrow{\iota_c} & x \\ {\scriptstyle z}\big\downarrow & \overset{f}{\nearrow} & \big\downarrow{\scriptstyle\beta} \\ M(a) & \xrightarrow{\iota_f} & \mathbb{D}_M(x) \end{array} \tag{11}$$

This time, we have that $M(a)$ is connected and so $f\colon M(a) \to x$ decomposes as

$$M(a) \xrightarrow{f_0} c' \xrightarrow{\iota_{c'}} x.$$

Again, since c is connected, the decomposition

$$\iota_c = \varepsilon_x \circ \alpha \circ \iota_c = \varepsilon_x \circ \iota_f \circ z = f \circ z = \iota_{c'} \circ f_0 \circ z$$

is unique and so $c = c'$ and $f_0 \circ z = \mathrm{id}$. Then, $z \circ h_0 = \mathrm{id}$ since ι_h is a monomorphism and since $\alpha \circ h = \alpha \circ \iota_c \circ h_0 = \iota_h \circ z \circ h_0$.

With this, it is routine to check that the following morphism

$$\alpha = \iota_{h_0} \circ z\colon c \to \mathbb{D}_M(c)$$

is a $\mathbb{D}_M$-coalgebra. Lastly, we check that ι_c is a coalgebra homomorphism $(c, \alpha) \to (x, \beta)$, that is, that $\beta \circ \iota_c = \mathbb{D}_M(\iota_c) \circ \alpha$. By (11), $\beta \circ \iota_c = \iota_h \circ z$. On the other hand, by (DC1), $\mathbb{D}_M(\iota_c) \circ \alpha = \mathbb{D}_M(\iota_c) \circ \iota_{h_0} \circ z = \iota_{\iota_c \circ h_0} \circ z = \iota_h \circ z$. ☐

Lastly, we show that also conditions (2) and (3) are satisfied for discrete density comonads.

Proposition 6.4. *Assume that all objects in the image of M are connected and finitely presentable, and that $\mathcal{B}$ is a constituental category with finite coproducts. Then, every morphism $f\colon a \to U(\beta)$ in $\mathcal{B}$, for finitely presentable a in $\mathcal{B}$, admits a unique factorisation (up to isomorphism)*

$$f \;=\; a \xrightarrow{g} U(\alpha) \xrightarrow{U(\gamma)} U(\beta)$$

where $U(\alpha)$ is finitely presentable.

Proof. In the following, we denote by U the forgetful functor $U^{\mathbb{C}}\colon \mathtt{EM}(\mathbb{D}_M) \to \mathcal{B}$. By Theorem 4.4, we may assume that the underlying object x of $\beta\colon x \to \mathbb{D}_M(x)$ is a coproduct $\coprod_{i\in I} c_i$ of a collection of connected, finitely presentable objects c_i essentially in $\mathcal{A}$.

Recall that $\coprod_{i\in I} c_i$ is isomorphic to the directed colimit of the following directed diagram

$$\{\coprod_{i\in F} c_i \mid F \text{ is a finite subset of } I\}$$

with the obvious morphisms between these finite coproducts. Since a is finitely presentable, $f\colon a \to x$ decomposes as

$$f = a \xrightarrow{g} \coprod_{i\in F} c_i \xrightarrow{\iota_F} x$$

for some finite $F \subseteq I$. By Lemma 6.3, for each $i \in F$, the inclusion morphism $\iota_i\colon c_i \to x$ is a coalgebra morphism $(c_i, \alpha_i) \to (x, \beta)$, for some comonad coalgebra $c_i \xrightarrow{\alpha_i} \mathbb{D}_M(c_i)$.

Lastly, because the forgetful functor $U\colon \mathtt{EM}(\mathbb{D}_M) \to \mathcal{B}$ creates colimits (see e.g. Proposition 20.12 in [6]), $\coprod_{i\in F} c_i$ can be equipped with the coalgebra structure of the coproduct of the coalgebras $\beta_i\colon c_i \to \mathbb{D}_M(c_j)$. Moreover, $\iota_F\colon \coprod_{i\in F} c_i \to x$ is a coalgebra morphism because each of its subcomponents is. Also, $\coprod_{i\in F} c_j$ is finitely presentable because it is a finite coproduct of finitely presentable objects (see e.g. Proposition 1.3 in [7]). Finally, g is unique because ι_F is the inclusion morphism of $\coprod_F c_i$ into the coproduct $\coprod_I c_i$ and hence a monomorphism, because $\mathcal{B}$ is a constituental category. □

As a corollary of Propositions 6.2 and 6.4 we obtain the main theorem of this section.

Theorem 6.5. *Let $M\colon \mathcal{A} \to \mathcal{B}$ be a functor from a discrete category $\mathcal{A}$ to a constituental category $\mathcal{B}$ with finite coproducts and assume that the density comonad $\mathbb{D}_M$ for M exists. If all objects in the image of M are connected and finitely presentable then $\mathbb{D}_M$ has finite rank.*

Observe that the constructed functor $M\colon \mathcal{A} \to \mathcal{B}$ in the proof of Theorem 1.1 (cf. the paragraph following Theorem 4.4) automatically satisfies the assumptions of Theorem 6.5. Indeed, M is an inclusion of a class of finite connected structures and finite structures are precisely the finitely presentable objects in the category of relational structures or the category of graphs. Therefore, Theorem 6.5 together with Theorems 6.1 concludes the proof Corollary 1.2.

7. Conclusion

In this paper we have showed that classes of structures closed under isomorphism, disjoint unions which also contain subcomponents of its structures can be always classified by a comonad and this comonad admits a

Lovász-type theorem. We also show that standard graph parameters give rise to graded comonads, i.e. sequences of comonads indexed by natural numbers, such that the graph parameter is captured by the coalgebra number of a given structure. Both results cover a huge class of examples of classes or graph parameters from the literature.

The comonads we construct are, in some sense, the minimal solutions to this problem in that they are weakly initial among the comonads that classify the same class of structures. What our proofs show is that the classifying comonads (or graded comonads) can be very simple and do not need to be specifically tailored for the studied concept.

However, the power of game comonads is that they shed light on some previously known constructs and explain what makes them related. We do not experience the same eureka moment with discrete density comonads. The philosophical outcomes of our investigations are two-fold.

For one, we learned that it is not so interesting to ask whether there is a comonad classifying a given class or an equivalence relation given by homomorphism-counts. Some comonad usually exists, as we showed, but the proof of this might not explain what makes the particular Lovász-type result work. The more interesting question is whether there is a comonadic reformulation of a known proof. When such proof is present, a new result can be obtained by replacing a comonad in question by a comonad expressing a different combinatorial property, as was demonstrated [18] and also in [23], by leveraging the comonadic proof present in [1].

On the other hand, if we start from a model-comparison game classifying a particular logic fragment then we typically obtain a close connection with a particular combinatorial property, as demonstrated, for example, in [1, 2, 3, 12]. Another potential source of useful comonads is when a particular class of structures is given by a *construction*, similar to the inductive definition of clique-width or the algebraic definition of planar graphs found e.g. in [22]. We hope to explore comonads arising from inductive constructions in the future work.

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