Algebraic and topological methods in computer science

(report 3, thesis proposal)

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Contents

Chapter 1

Introduction

Coalgebras provide a unifying framework for all kinds of state based systems such as labeled processes or nondeterministic automata. In *“Universal coalgebra: a theory of systems”* [Rut00] Rutten showed not only that but also that the standard constructions such as bisimilarity, coinduction, taking subsystems, their coproduct and many more can be easily expressed in the language of coalgebras. An important part of the theory of coalgebras is the study of coalgebraic logics, where various kinds of generalisations of modal logics has proved to be the most successful over the years [KP11].

A long time before Rutten’s discovery, Belnap gave a philosophical motivation to study 4-valued logics for analysis of behaviour of systems [Bel77]. The idea is to, in addition to the standard boolean values, also add a value for diverging computations and for computations giving contradicting results. Since then, many formalisations of Belnap’s logic have been introduced [AA96].

Both lines of research are (in short) presented in the text. The author’s main goal is to try to combine the best of both worlds and hopefully obtain a four-valued coalgebraic modal logic. In order to be able to do that, a lot of small steps need to be taken first; we outline some of them in the final chapter.

Preliminaries

We assume that the reader has a very basic knowledge of category theory, logic, topology and universal algebra i.e. familiarity with concepts such as category, functor, adjunction, natural transformation, topology, continuous map, compactness of space, separation axioms, (algebraic) variety, signature, free algebra, etc. Everything needed can be found in [Awo06], [BDRV02], [Kel75] and [SB81].

Throughout the whole text, we denote the category of sets and functions, bounded distributive lattices and homomorphisms and Boolean algebras and homomorphisms by **Set**, **DLat** and **Bool** respectively. Also, $\mathscr{P}$ denotes the *covariant* powerset functor over **Set** and $\mathscr{P}_f$ the finite covariant powerset functor.

Chapter 2

Coalgebras as Systems

In this chapter we give a brief overview of the theory of coalgebras and, unlike in the later chapters, we will be working in the category of sets. The main source of inspiration was Rutten [Rut00], Kurz [Kur01], Pattinson [Pat03a], Venema [Ven06a], and Jung and Chen [CJ14a].

2.1 Examples of coalgebras

In this section we will argue that we can describe most of the transition systems as coalgebras. Abstractly, for an endofunctor T, a coalgebra $\xi : X \to T(X)$ represents a transition system where X represents set of *states*, the *transition map* ξ describes how a state changes after one step of computation and T describes the *shape* or *type* of the resulting set of states.

2.1.1 Basic systems

The examples are given inductively. Starting with the simplest systems and then describing how all the other systems can be obtained as coalgebras of certain functors/shapes obtained as a combination of shapes of simpler systems.

The most basic system is a system which only produces an output in the first step and then halts. Such systems are described as functions from a set of states to a set given by a constant functor:

$$\xi : X \to A.$$

For every state $x \in X$ we designate an output $\xi(x) \in A$.

The second simplest kind of behaviour is a system which for a fixed set of states describes how the states change as the computation goes on. We can imagine such systems as functions from a set of states to the same set, that is the set given by the identity functor:

$$\xi : X \to X.$$

Again, for a state $x \in X$, after one step of computation, we end up in a state given by ξ such that $\xi(x) \in X$.

Notice that in the first case, for the constant functors, the systems always halt after one step and produce an output whereas in the second example the systems never produce any output and also never stop.

2.1.2 Combined systems

2.1.1 Parallel composition. Now, given two functors T_1 and T_2 representing shapes of systems, we can create functor which represents a parallel composition of both systems:

$$\xi : X \to T_1(X) \times T_2(X).$$

Then, a value of a state after one step of computation is a pair of values and we can think of the system ξ as of a system with two operations: $\pi_1 \circ \xi$ and $\pi_2 \circ \xi$.

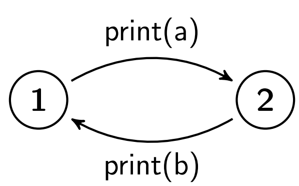
A nice example of a system obtained as a composition of two systems is the following labeled transition system of shape $T(X) = \{\mathsf{a}, \mathsf{b}\} \times X$:

$$\xi_{\text{alternate}} : \{1, 2\} \to \{\mathsf{a}, \mathsf{b}\} \times \{1, 2\}$$

such that

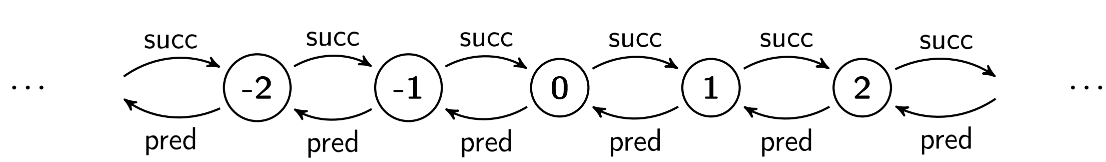
$$\xi_{\text{alternate}} : 1 \mapsto (\mathsf{a}, 2), \quad 2 \mapsto (\mathsf{b}, 1).$$

We can imagine $\xi_{\text{alternate}}$ as the transition system depicted in the following schema:



For the state 1, after one step, it produces output a and changes to the state 2, and when the system is in state 2 it produces output b and changes to the state 1. We can see that if we start in the state 1 and compute several steps, $\xi_{\text{alternate}}$ produces the following sequence of outputs: abababababababa. . . .

Similarly, for the system $\langle \mathsf{pred}, \mathsf{succ} \rangle : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $z \mapsto (z - 1, z + 1)$ we get the following transition system:

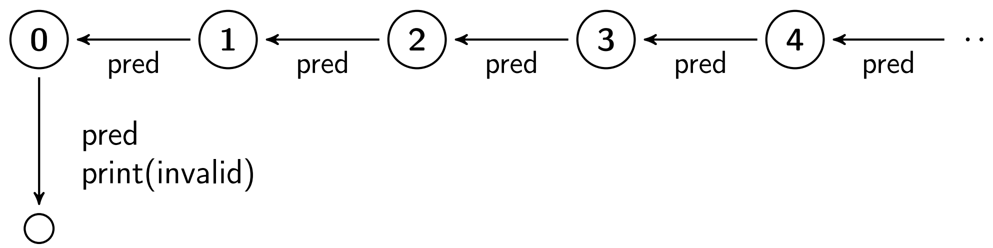


2.1.2 Choice. An another way how we can create systems of more complicated shapes is adding a choice and this can be done by taking a coproduct:

$$\xi: X \to T_1(X) + T_2(X).$$

This way, when starting from a state $x \in X$, in the next step the system will be in a $T_1(X)$ state or a $T_2(X)$ state; that is $\xi(x) \in T_1(X)$ or $\xi(x) \in T_2(X)$.

As an example, take natural numbers and subtracting by one. The coalgebra is a function $\mathsf{pred}: \mathbb{N} \to \{\mathsf{invalid}\} + \mathbb{N}$, $n \in \mathbb{N} \setminus \{0\} \mapsto n-1$ and $0 \mapsto \mathsf{invalid}$ and its diagram looks as follows:

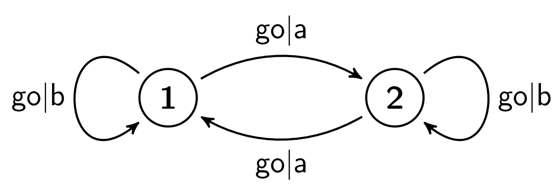


2.1.3 Input. For transition systems, it is very important to be able to make decisions based on inputs. In order to be able to reproduce this behaviour for coalgebras, we can use exponentiations:

$$\xi: X \to T(X)^A.$$

Then for a state $x \in X$ and a character $a \in A$ on input, we get to the state $\xi(x)(a) \in T(X)$ in the next step.

As an example, take a system $\mathsf{go}: \{1,2\} \to \{1,2\}^{\{\mathsf{a},\mathsf{b}\}}$ which changes a state only if it reads a, otherwise it keeps it. The diagrams looks as follows:

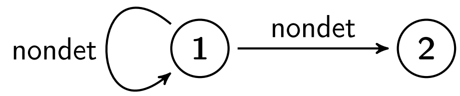


2.1.4 Nondeterminism. Finally, the last very important concept in the theory of systems is nondeterminism. We can achieve nondeterminism by using the (covariant) powerset functor (or the finite powerset functor). Say,

$$\xi: X \to \mathscr{P}(T(X))$$

sends a state s, after one step, to a one of the states in $\xi(x) \subseteq T(X)$ and the actual state where x goes after one step is picked nondeterministically.

And as an example take a system $\mathsf{nondet}: \{1,2\} \to \mathscr{P}(\{1,2\})$ such that $\mathsf{nondet}: 1 \mapsto \{1,2\}$ and $\mathsf{nondet}: 2 \mapsto \emptyset$. Then nondet represents a system which in state 1 nondeterministically chooses whether to stay in the state 1 or to go to the state 2 where the computation stops:



2.1.3 Classical examples

The examples of systems given in the previous sections are just very simple. We should show that we can also describe the fundamental systems of computer science as certain coalgebras.

Automatons. For a fixed alphabet Σ, an automaton is usually described as a quadruple (S,δ,s_0,F), where S is a set of states, $\delta: S\times\Sigma\to S$ is a transition function, s_0 is an initial state and $F\subseteq S$ is a set of accepting states.

This representation can be modified in order to get a coalgebra. The transition function is equivalent (by currying) to some $\widetilde{\delta}: S\to S^\Sigma$ and the set F can be represented as a function $f: S\to 2$ (such that $f^{-1}(1)=F$). Then an automaton is just a coalgebra

$$\left\langle\widetilde{\delta},f\right\rangle : S\to S^\Sigma\times 2$$

with a designated (starting) state s_0. (Coalgebras with a designated state are sometimes called *pointed systems* [Ven06a].)

To get a nondeterministic automaton it is enough to replace the set of states in the previous example by the set of all possible subsets of states: $S\to\mathscr{P}(S)^\Sigma\times 2$.

Kripke frames. An another very important structure in computer science are graphs or Kripke frames. There is a one-to-one correspondence between graphs and $\mathscr{P}$-coalgebras: For a graph (V,E), set $\eta_E(v)$ to be the set of neighbours of v; and for a coalgebra $\eta: V\to\mathscr{P}(V)$ define the relation E_η such that $(u,v)\in E_\eta$ iff $v\in\eta(u)$.

Having multiple kinds of edges (or binary relations in the language of Kripke frames) can be also represented as coalgebras: Graphs of the following signature $(V,E_1,E_2,\ldots,E_n)$ correspond to coalgebras of shape $V\to\mathscr{P}(V)^n$.

However, instead of graph homomorphisms modal logicians use *p-morphisms* as morphisms of Kripke frames. A p-morphism $f:(V,E)\to(V',E')$ between two Kripke frames is a function satisfying the following conditions:

$$\text{(forth condition)}\qquad xEy \implies f(x)E'f(y)$$
$$\text{(back condition)}\qquad f(x)E'y' \implies \exists y\in V \text{ s.t. } f(y)=y' \text{ and } xEy.$$

Or, as stated in terms of neighbour maps $\eta: W\to\mathscr{P}(W)$ and $\eta': W'\to\mathscr{P}(W')$ associated with E and E' respectively:

$$\text{(forth condition')}\qquad y\in\eta(x) \implies f(y)\in\eta'(f(x))$$
$$\text{(back condition')}\qquad y'\in\eta'(f(x)) \implies \exists y\in f^{-1}(y') \text{ s.t. } y\in\eta(x).$$

Then forth condition just says that $f[\eta(x)] \subseteq \eta'(f(x))$ and back condition $f[\eta(x)] \supseteq \eta'(f(x))$. We can sum up the previous discussion in the following Theorem

2.1.5 Theorem. *The category of $\mathscr{P}$-coalgebras is equivalent to the category of Kripke frames and p-morphisms.*

2.1.4 The syntax of shape functors

From the discussion above we can see that *Kripke polynomial functors*, that is the functors of the syntax

$$F ::= \mathsf{Id} \mid A \mid F_1 + F_2 \mid F_1 \times F_2 \mid F^B \mid \mathscr{P}F \qquad \text{(for sets } A \text{ and } B\text{)},$$

should allow us to represent any kind of state based system as a coalgebra. Sometimes, we may want to restrict our attention to finitely branching systems only, that is the coalgebras of *finite Kripke polynomial functors*:

$$F ::= \mathsf{Id} \mid A \mid F_1 + F_2 \mid F_1 \times F_2 \mid F^B \mid \mathscr{P}_f F \qquad \text{(for sets } A \text{ and } B\text{)}.$$

2.2 Coalgebra homomorphisms

Now, we will focus on similarity of behaviours and how this concept can be expressed coalgebraically – by coalgebra homomorphisms.

Take, for example, the coalgebra $\xi_{\text{alternate}} \colon \{1,2\} \to \{\mathsf{a},\mathsf{b}\} \times \{1,2\}$ which was defined above and the coalgebra $\mathsf{isOdd} \colon \mathbb{N} \to \{\mathsf{a},\mathsf{b}\} \times \mathbb{N}$ defined as:

$$\mathsf{isOdd} \colon n \mapsto \begin{cases} (\mathsf{a}, n+1) & \text{if } n \text{ is odd, and} \\ (\mathsf{b}, n+1) & \text{if } n \text{ is even.} \end{cases}$$

Now, we can see that the function $i \colon \mathbb{N} \longrightarrow \{1,2\}$, sending odd numbers to 1 and even numbers to 2, connects states that behave the same. Namely, 1 and odd states respectively 2 and even states print the same character after one step of computation and they both change to the state that will print again the same character in the next step. Moreover, this goes on and on, after three, four, five, ... steps they always print the same character.

We can describe this behaviour equationally:

$$\pi_1 \circ (\pi_2 \circ \xi_{\text{alternate}})^n \circ i = \pi_1 \circ (\pi_2 \circ \mathsf{isOdd})^n$$

is true for all n (where π_1 and π_2 are the projections).

Similarly, go from above and $\mathsf{go}' \colon \{\star\} \to \{\star\}^{\{\mathsf{a},\mathsf{b}\}}$ can be connected by a constant function $c \colon 1,2 \mapsto \star$. This is because the system never prints anything and never stops, both states are indistinguishable from each other.

An another example is $\mathsf{nondet}\colon \{1,2\} \to \mathscr{P}(\{1,2\})$ and $\mathsf{nondet}'\colon \mathbb{N} \cup \{\bot\} \to \mathscr{P}(\mathbb{N} \cup \{\bot\})$ where $n \mapsto \{n+1, \bot\}$ and $\bot \mapsto \emptyset$. Then, for the function f defined as $f : n \in \mathbb{N} \mapsto 1$ and $f : \bot \mapsto 2$; again, f maps states to the states of the same behaviour. Intuitively, 1 nondeterministically either stays 1 or goes to 2 which terminates; and, for nondet', $\bot$ always halts and any other state nondeterministically changes to $\bot$ or goes to a state with the same behaviour.

Notice that all of these examples share the same pattern. For two coalgebras $\xi\colon X \to T(X)$ and $\xi'\colon X' \to T(X')$, and a mapping $f : X \to X'$ connecting states of the same behaviour, it always holds that

$$\xi' \circ f = Tf \circ \xi.$$

Or in other words, f is a T-coalgebra homomorphisms. Note that being coalgebra homomorphism implies that the following equation holds for all n:

$$(T^n\xi' \circ T^{n-1}\xi' \circ \cdots \circ T\xi' \circ \xi') \circ f = T^n f \circ (T^n\xi \circ T^{n-1}\xi \circ \cdots \circ T\xi \circ \xi).$$

We can think of coalgebra homomorphisms as the behaviour preserving maps and by behaviour we only mean the observable behaviour such as printing an output and halting.

2.2.1 Definition. The category $\mathsf{Coalg}(T)$ is the category of all T-coalgebras and T-coalgebra homomorphisms.

Similarly, we denote the category of all T-algebras and T-algebra homomorphisms as $\mathsf{Alg}(T)$.

The resulting category $\mathsf{Coalg}(T)$ plays the role of a category of systems and behaviour preserving maps.

One may ask, why is the description of systems as coalgebras useful at all and whether it gives us any insights about the actual systems. As we will see in the following sections, the translation has proved to be very useful. Just by thinking about systems using the language of category theory gives us a lot of powerful tools and techniques.

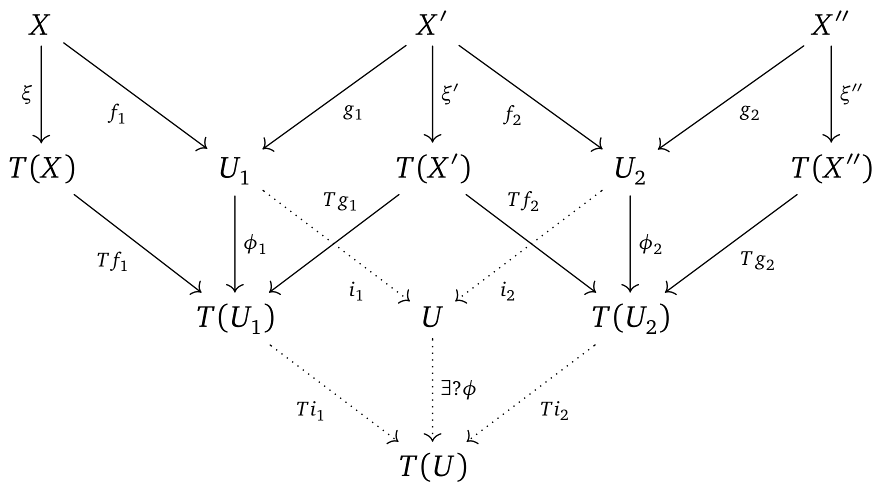
2.3 Behavioural Equivalence and Bisimulation

2.3.1 Behavioural Equivalence

Once we convinced ourselves that the coalgebra homomorphisms map elements only to elements of the same behaviour we can define what does it mean that elements have the same behaviour.

2.3.1 Definition. For two coalgebras $\xi\colon X \to T(X)$ and $\xi'\colon X' \to T(X')$, we say that two states $x \in X$ and $x' \in X'$ are *behaviourally equivalent* and write $x \approx x'$ if there exists a coalgebra $\phi : U \to T(U)$ and two coalgebra homomorphisms $f : X \to U$ and $g : X' \to U$ such that $f(x) = g(x')$.

From the definition we can see that behaviour equivalence is reflexive and symmetric. We can also show that it is transitive. Take three coalgebras $\xi : X \to T(X)$, $\xi' : X' \to T(X')$ and $\xi'' : X'' \to T(X'')$ and $x \in X$, $x' \in X'$ and $x'' \in X''$ such that $x \Leftrightarrow x'$ and $x' \Leftrightarrow x''$. Then, there exist two coalgebras $\phi_1 : U_1 \to T(U_1)$ and $\phi_2 : U_2 \to T(U_2)$ and four coalgebra homomorphisms $f_1 : X \to U_1$, $g_1 : X' \to U_1$, $f_2 : X' \to U_2$ and $g_2 : X'' \to U_2$ such that $f_1(x) = g_1(x')$ and $f_2(x') = g_2(x'')$ as is depicted in the following diagram:



Define $(U, i_1 : U_1 \to U, i_2 : U_2 \to U)$ to be the pushout of g_1 and f_2. Then, by diagram chasing, we can see that $Ti_1 \circ \phi_1 \circ g_1 = Ti_2 \circ \phi_2 \circ f_2$ and, therefore, there exists a factorising map $\phi : U \to T(U)$. Moreover, i_1 and i_2 are coalgebra homomorphisms and $i_1(f_1(x)) = i_1(g_1(x')) = i_2(f_2(x')) = i_2(g_2(x''))$. Therefore, we can conclude that behavioural equivalence is indeed an equivalence relation.

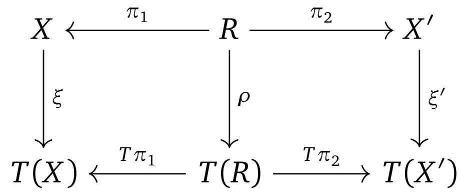
Observation. *The definition and the proof of transitivity of behavioural equivalence would work for any concrete category with pushouts.*

2.3.2 Example. Any coalgebra homomorphism $f : X \to X'$ already points to a whole collection of pairs of elements that are behaviourally equivalent. Namely, for any $x \in X$ we have that $x \Leftrightarrow f(x)$ simply by taking g (as requested by the definition) to be the constant coalgebra homomorphism on X'.

2.3.2 Bisimulation

Bisimulation or *bisimilarity* is an another concept how to say that two states behave the same. Historically, it was prior to behavioural equivalence. Intuitively, a bisimulation is a relation that is preserved by computations. Formally:

2.3.3 Definition. For two coalgebras $\xi : X \to T(X)$ and $\xi' : X' \to T(X')$, we say that two states $x \in X$ and $x' \in X'$ are *bisimilar* and write $x \sim x'$ if there exists a relation $R \subseteq X \times X'$ and a coalgebra $\rho : R \to T(R)$ such that $(x, x') \in R$ and the following diagram commutes

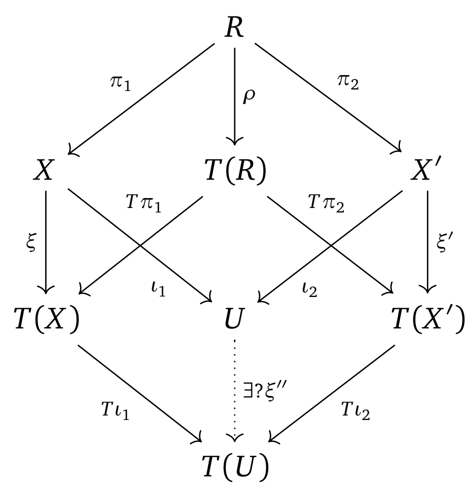


Where, π_1 and π_2 in the diagram are projections. The following theorem shows how is bisimilarity related to behavioural equivalence.

2.3.4 Theorem. *Let $\xi: X \to T(X)$ and $\xi': X' \to T(X')$ be two coalgebras and $x \in X$ and $x' \in X'$ two states. Then,*

1. *$x \sim x'$ implies $x \leftrightarroweq x'$, and*
2. *if T preserves weak pullbacks, then $x \leftrightarroweq x'$ implies $x \sim x'$.*

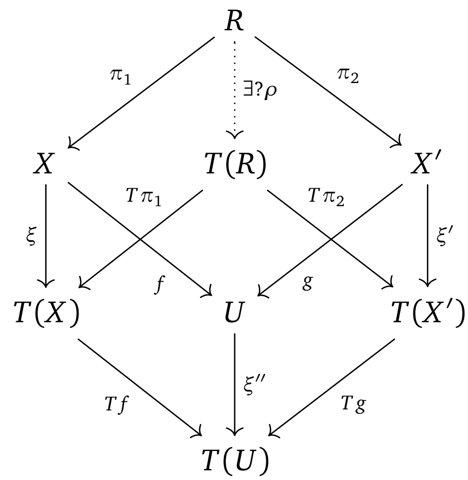
Proof. For 1., let $R \subseteq X \times X'$ be a relation and $\rho: R \to T(R)$ be a coalgebra witnessing that $x \sim x'$. Then, let U together with the maps $\iota_1: X \to U$ and $\iota_2: X' \to U$ be a pushout of π_1 and π_2. The situation looks as follows:



Because the bottom square commutes and π_1 and π_2 are coalgebra homomorphisms, we get that $T\iota_1 \circ \xi \circ \pi_1 = T\iota_2 \circ \xi' \circ \pi_2$ and, because U is a pushout, there exists a $\xi'': U \to T(U)$ such that it makes the front squares of the diagrams to commute. This just means that ι_1 and ι_2 are coalgebra homomorphisms and, since $(x, x') \in R$ and the top square commutes, $\iota_1(x) = \iota_2(x')$.

To prove 2., take a coalgebra $\xi'': U \to T(U)$ and two coalgebra homomorphisms $f: X \to U$ and $g: X' \to U$ witnessing that $x \leftrightarroweq x'$, that is $f(x) = g(x')$. Take R with $\pi_1: R \to X$ and $\pi_2: R \to T$ to be a pullback of f and g. Notice that as a pullback in the category of sets, R is isomorphic to the set of pairs of elements that are identified by the functions f and g. Therefore, we can think of π_1 and π_2 as projections and we have that $(x, x') \in R$.

Similarly to previous we have the following



Also, $Tf \circ \rho \circ \pi_1 = Tg \circ \xi' \circ \pi_2$ because f and g are coalgebra homomorphisms and the top square commutes. Therefore, there exists a $\rho : R \to T(R)$ such that it makes the back squares to commute (that is $\rho \circ T\pi_1 = \xi \circ \pi_1$ and $\xi' \circ \pi_2 = T\pi_2 \circ \rho$) since T makes the bottom square to be a weak pullback.

In conclusion, there exists a relation $R \subseteq X \times X'$ containing the pair (x, x') and π_1 and π_2 are coalgebra homomorphisms. □

Observation. *The definition of bisimilarity and the proof of the previous theorem would work for any concrete category with pushouts and pullbacks.*

Note that to (directly) show that bisimilarity is indeed an equivalence relation, one needs to use Axiom of Choice [Rut00, Theorem 5.4]. However, from previous it seems to the author that one can show transitivity simply from the transitivity of behavioural equivalence.

2.4 Final coalgebras

Let us think for a while what would it mean for a functor to have a final coalgebra; that is a final object in the category Coalg(T). If a functor T has a final coalgebra $(\nu T, \nu_T : \nu T \to T(\nu T))$, it is unique up to an isomorphism. By $!_X : X \to \nu T$ we will denote the unique homomorphisms from a coalgebra (X, ρ). We will also sometimes drop the lower index of $!_X$ if there is no danger in confusion.

Because coalgebra homomorphisms map states only to the states of the same behaviour, the final coalgebra must contain a state of every possible kind of behaviour. Also, a final coalgebra can not contain two different states of the same behaviour. Take two behaviourally equivalent states $x, x' \in \nu T$. There exists a coalgebra $\phi : U \to T(U)$ an two homomorphisms $f, g : \nu T \to U$ such that $f(x) = g(x')$. But, because νT is final, $! \circ f = ! \circ g = 1_P$. Therefore, $x = x'$. As a sum up of the previous discussion we have

2.4.1 Proposition. *Behavioural equivalence is equality on final coalgebras.*

For functors with a final coalgebra we can define behavioural equivalence in the following equivalent form:

2.4.2 Corollary. *If a functor T has a final coalgebra, then $x \in X$ and $x' \in X'$ are behaviourally equivalent iff $!_X x = !_{X'} x'$.*

As is shown above, it can be very useful for a functor to have a final coalgebra. Later, when we talk about coinduction, we will see more examples of why we would like our functor to have a final coalgebra.

Notice that not every functor has a final coalgebra. Take, for example, the powerset functor. By the Cantor diagonal argument, there can not be a bijection between $\mathscr{P}(X)$ and X for any set X and, by the following theorem, the final coalgebra map is always an isomorphism (which is in the category of sets a bijection):

2.4.3 Theorem (Lambek)**.** *Transition morphism of a final coalgebra is an isomorphisms.*

Proof. For a final coalgebra $(\nu T, \nu_T)$, take the following situation:

$$\begin{array}{ccccc} T(\nu T) & \xrightarrow{\;!\;} & \nu T & \xrightarrow{\;\nu_T\;} & T(\nu T) \\ \downarrow{\scriptstyle T\nu_T} & & \downarrow{\scriptstyle \nu_T} & & \downarrow{\scriptstyle T\nu_T} \\ T^2(\nu T) & \xrightarrow{\;T!\;} & T(\nu T) & \xrightarrow{\;T\nu_T\;} & T^2(\nu T) \end{array}$$

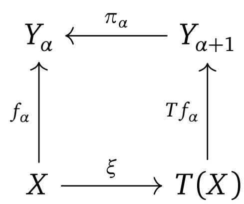
We can immediately see that $! \circ \nu_T = 1$ because composition of homomorphisms is a homomorphism, 1 is a homomorphism and it must be unique. On the other hand, because ! is a homomorphisms, we know that $\nu_T \circ ! = T! \circ T\nu_T = T(! \circ \nu_T) = T1 = 1$. Therefore, ν_T is an isomorphism. □

2.4.1 Final coalgebra construction

Let us describe a construction of a final coalgebra. For a category with all limits and an endofunctor T, an *terminal sequence for T* is a sequence of objects and morphisms $(\pi_\kappa \colon Y_\kappa \longleftarrow Y_{\kappa+1} : \kappa \in \mathbf{Ord})$, where

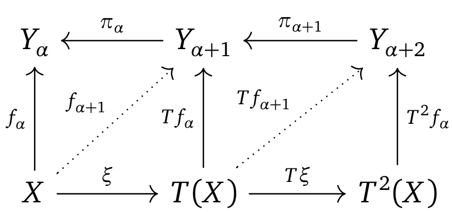
1. Y_0 is a final object (obtained as a limit of empty diagram),
2. $Y_\kappa = T(Y_{\kappa-1})$, for a successor ordinal κ,
3. $\pi_1 \colon Y_1 \to Y_0$ is the unique morphism to the final object,
4. $\pi_{\kappa+1} = T(\pi_\kappa)$, for a successor ordinal κ,
5. $Y_\kappa = \lim(\pi_\alpha : \alpha < \kappa)$, for a limit ordinal κ,
6. $\pi_{\kappa+1} \colon Y_{\kappa+1} \to Y_\kappa$, for a limit ordinal κ, is the factorising morphism of the cone $(\pi_\alpha \circ Tp^\kappa_\alpha : \alpha < \kappa)$ where $(p^\kappa_\alpha \colon Y_\kappa \to Y_\alpha : \alpha < \kappa)$ is the limit cone for Y_κ.

For a coalgebra $\xi : X \to T(X)$, we say that a morphism $f_\alpha : X \to Y_\alpha$ *makes the diagram for α commute* if the following diagram commutes



We will prove that given a coalgebra $\xi : X \to T(X)$ such f_α always exists and is uniquely determined.

Let us proceed by transitive induction. First, assume that there exists such uniquely determined f_α and we will show that $f_{\alpha+1}$ also exists and it is also uniquely determined. The situation is as follows



Observe that $f_{\alpha+1}$ defined as $Tf_\alpha \circ \xi$ makes the diagram for $\alpha + 1$ commute. Indeed, the right square of the diagram commutes simply from the fact that the diagram for α commutes and then, from the definition of $f_{\alpha+1}$, we also know that $Tf_{\alpha+1} = T^2 f_\alpha \circ T\xi$ and we get what we wanted to prove:

$$f_{\alpha+1} = Tf_\alpha \circ \xi = \pi_{\alpha+1} \circ T^2 f_\alpha \circ T\xi \circ \xi = \pi_{\alpha+1} \circ Tf_{\alpha+1} \circ \xi.$$

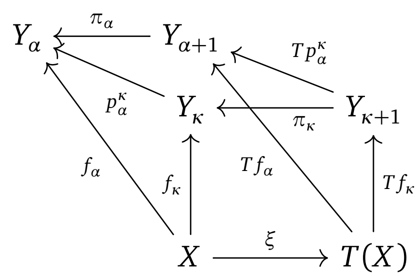
Now, assume that there exists a morphism $f : X \to Y_{\alpha+1}$ also making the diagram for $\alpha + 1$ commute. Since, from induction hypothesis, f_α is uniquely defined and $\pi_\alpha \circ f$ makes also the diagram for α commute, f_α has to be equal to $\pi_\alpha \circ f$. But then

$$f = \pi_{\alpha+1} \circ Tf \circ \xi = T(\pi_\alpha \circ f) \circ \xi = Tf_\alpha \circ \xi.$$

Before we continue, let us also define p_α^β for β's iterating over all successor ordinals (and still assuming that $\alpha < \beta$):

$$p_\alpha^\beta = \begin{cases} \pi_\alpha \circ \pi_{\alpha+1} \circ \cdots \circ \pi_{\alpha+n-1} & \text{for } \beta = \alpha + n; \text{ for some } n \in \omega \\ p_\alpha^\kappa \circ \pi_\kappa \circ \cdots \circ \pi_{\kappa+n-1} & \text{for } \beta = \kappa + n; \text{ for some limit } \kappa \text{ and some } n \in \omega. \end{cases}$$

To prove the limit step, take a limit ordinal $\kappa < \lambda$ and assume that, for all $\alpha < \kappa$, f_α makes the diagram for α commute and is uniquely determined. Observe that this also means that for any two ordinals $\alpha < \beta$, $f_\alpha = p_\alpha^\beta \circ f_\beta$ (the $\beta = \alpha + n$ steps are proven inductively from $f_\alpha = \pi_\alpha \circ f_{\alpha+1}$ and the limit steps are just a property of limits). Hence, there exists a $f_\kappa : X \to Y_\kappa$ such that $p_\alpha^\kappa \circ f_\kappa = f_\alpha$, for all $\alpha < \kappa$, and we have the following situation:



In the diagram $Tp^\kappa_\alpha \circ Tf_\kappa = Tf_\alpha$ from the definition of f_κ, $f_\alpha = \pi_\alpha \circ Tf_\alpha \circ \xi$ is just the diagram for α, and $p^\kappa_\alpha \circ \pi_\kappa = \pi_\alpha \circ Tp^\kappa_\alpha$ from the definition of π_κ. Therefore, by a diagram chasing,

$$f_\alpha = \pi_\alpha \circ (Tf_\alpha) \circ \xi = (\pi_\alpha \circ Tp^\kappa_\alpha) \circ Tf_\kappa \circ \xi = p^\kappa_\alpha \circ (\pi_\kappa \circ Tf_\kappa \circ \xi).$$

But, $f_\kappa = \pi_\kappa \circ Tf_\kappa \circ \xi$ because we know that factorising morphism f_κ is uniquely determined. Uniqueness of a f making the diagram for κ commute follows from the universal property of the limit construction and from the fact that all the f_α's are also uniquely determined.

If the *construction stops*, that is π_κ is an isomorphism for some κ, then $\nu_T = (\pi_\kappa)^{-1} \colon Y_\kappa \to Y_{\kappa+1}$ is a final coalgebra and the diagram for κ is exactly the diagram for f_κ to be a coalgebra homomorphism. Also notice that if π_κ is an isomorphism, then π_λ is also an isomorphism, for all $\lambda > \kappa$ (meaning that the construction has really stopped).

Moreover, assume that there exists another coalgebra homomorphism $f : X \to Y_\kappa$. Then, for all $\alpha < \kappa$, $p^\kappa_\alpha \circ f$ makes the diagram for α commute; therefore, it has to be equal to f_α and $f = f_\kappa$.

We can see that, if the construction stops, the resulting coalgebra is indeed a final coalgebra.

2.4.2 Further notes on the construction

An old result of Adámek and Koubek [AK95] shows that if a set functor has a final coalgebra then it can be obtained from the terminal sequence construction as described above.

Notice also that the same construction, but reversed, would also work for initial algebra construction for categories with all colimits. It is a relatively recent result of Adámek and Trnková that such construction for set functors, if it stops, it stops either in 3 steps or at an infinite regular cardinal steps [AT11].

As we mentioned earlier, Kripke polynomial functors do not have a final coalgebras in general. On the other hand, one can prove that Finite Kripke polynomial functors always have final coalgebras and the construction always terminates in $\omega + \omega$ steps (as a special case of a Theorem in [Wor99]).

2.4.3 Examples of final coalgebras

The simplest example of a final coalgebra is the final coalgebra for $T(X) = A \times X$. The terminal sequence for T is

$$\{\star\} \longleftarrow A \longleftarrow A^2 \longleftarrow A^3 \longleftarrow A^4 \longleftarrow \ \dots$$

and the final coalgebra is then $\langle \mathsf{head}_1, \mathsf{tail}_1 \rangle : A^\omega \to A \times A^\omega$ (as the least fixed point). We can think of elements of A^ω as if they were the infinite sequences of elements of A. The function head_1 then just returns the first element of the sequence and tail_1 returns the same sequence after dropping the first element.

As the next example, let us just change the previous functor a bit. Take the functor $T'(X) = A \times X + \{\star\}$. Then the terminal sequence becomes

$$\{\star\} \longleftarrow A + \{\star\} \longleftarrow A^2 + A + \{\star\} \longleftarrow A^3 + A^2 + A + \{\star\} \longleftarrow \ \dots .$$

As a result, the final coalgebra is the coalgebra $\langle \mathsf{head}, \mathsf{tail} \rangle : A^{\leq\omega} \to A \times A^{\leq\omega} + 1$ of empty, all finite and infinite sequences. The maps head and tail return first element of the sequence and the rest of the sequence respectively if the list is not empty, or $\star$ if it is.

2.5 Coinduction

2.5.1 Coinductive definitions

Once we have a final coalgebra we can define operations on it using its universal property. For example, take the final coalgebra $\langle \mathsf{head}, \mathsf{tail} \rangle : A^{\leq\omega} \to A \times A^{\leq\omega} + \{\star\}$ for the functor T' from the previous example and define a function $\alpha \colon A^{\leq\omega} \times A^{\leq\omega} \to A \times A^{\leq\omega} \times A^{\leq\omega} + \{\star\}$:

$$\alpha((a_i)_i, (b_j)_j) = \begin{cases} (a_0,\ (a_i)_{i\geq 1},\ (b_j)_j) & \text{for } (a_i)_i \text{ nonempty,} \\ (b_0,\ (),\ (b_j)_{j\geq 1}) & \text{for } (a_i)_i \text{ empty and } (b_j)_j \text{ nonempty,} \\ \star & \text{otherwise;} \end{cases}$$

(where () denotes empty sequence) or in the terms of head and tail:

$$\alpha((a_i)_i,\ (b_j)_j) = \begin{cases} (\mathsf{head}((a_i)_i),\ \mathsf{tail}((a_i)_i),\ (b_j)_j) & \text{for } (a_i)_i \text{ nonempty,} \\ (\mathsf{head}((b_j)_j),\ (),\ \mathsf{tail}((b_j)_j)) & \text{for } (a_i)_i \text{ empty and } (b_j)_j \text{ nonempty,} \\ \star & \text{otherwise.} \end{cases}$$

Now, α defines a T'-coalgebra and, therefore, there exists a unique coalgebra homomorphism $(+\!\!+)\colon\ A^{\leq\omega} \times A^{\leq\omega} \to A^{\leq\omega}$ satisfying

$$\langle \mathsf{head}, \mathsf{tail} \rangle \circ (+\!\!+) = T'(+\!\!+) \circ \alpha. \qquad \text{(def-}+\!\!+\text{)}$$

We say that that $+\!\!+$ is defined by a *coinduction*.

2.5.2 Proofs by coinduction

Proving equational properties for an operation defined on a final coalgebra (i.e. like above) is usually shown by defining an appropriate bisimulation. For example, to show that $() \mathbin{+\!\!+} (b_j)_j = (b_j)_j$, for all $(b_j)_j$, we can define a binary relation $R_1 = \{(() \mathbin{+\!\!+} (b_j)_j,\ (b_j)_j) : (b_j)_j \in A^{\leq\omega}\}$ on $A^{\leq\omega}$ and prove that it is a bisimulation.

First, observe that, for a pair $(() \mathbin{+\!\!+} (b_j)_j,\ (b_j)_j)$, after one step of computation (taken pointwise) we get again two elements related by R_1:

$$\alpha((),\ (b_j)_j) = (\mathsf{head}((b_j)_j),\ (),\ \mathsf{tail}((b_j)_j)), \quad \nu_{T'}((b_j)_j) = (\mathsf{head}((b_j)_j),\ \mathsf{tail}((b_j)_j))$$

They both print $\mathsf{head}((b_j)_j)$ and $(() \mathbin{+\!\!+} \mathsf{tail}((b_j)_j),\ \mathsf{tail}((b_j)_j)) \in R_1$.

To show that R_1 is a bisimulation, we need to define a transition structure $\rho : R_1 \to T'(R_1)$. But, there is only one reasonable way to do it

$$\rho : (() \mathbin{+\!\!+} (b_j)_j,\ (b_j)_j) \mapsto \begin{cases} \star & \text{for } (b_j)_j \text{ empty,} \\ (\mathsf{head}((b_j)_j),\ (() \mathbin{+\!\!+} \mathsf{tail}((b_j)_j),\ \mathsf{tail}((b_j)_j))) & \text{otherwise} \end{cases}$$

Since α was defined in terms of head and tail, we can easily see that projections π_1 and π_2 are coalgebra homomorphisms.

What we have just shown is that all pairs $() \mathbin{+\!\!+} (b_j)_j$ and $(b_j)_j$ are bisimilar and by Theorem 2.3.4 we know that they are also behaviourally equal. But, this just means that they must be equal as shown in Proposition 2.4.1.

Similarly, we can show something a bit more complicated; for example, that $\mathbin{+\!\!+}$ is associative. Take $R_2 = \{((a_i)_i \mathbin{+\!\!+} ((b_j)_j \mathbin{+\!\!+} (c_k)_k),\ ((a_i)_i \mathbin{+\!\!+} (b_j)_j) \mathbin{+\!\!+} (c_k)_k) : (a_i)_i,\ (b_j)_j,\ (c_k)_k\}$. We have that both $() \mathbin{+\!\!+} (() \mathbin{+\!\!+} (c_k)_k)$ and $(() \mathbin{+\!\!+} ()) \mathbin{+\!\!+} (c_k)_k$ print $\mathsf{head}((c_k)_k)$ and continue to the state $() \mathbin{+\!\!+} (() \mathbin{+\!\!+} \mathsf{tail}((c_k)_k))$ and $(() \mathbin{+\!\!+} ()) \mathbin{+\!\!+} \mathsf{tail}((c_k)_k)$ (notice that we know that $() \mathbin{+\!\!+} () = ()$ from (def-$\mathbin{+\!\!+}$)). We can continue similarly for $() \mathbin{+\!\!+} ((b_j)_j \mathbin{+\!\!+} (c_k)_k)$ and $(() \mathbin{+\!\!+} (b_j)_j) \mathbin{+\!\!+} (c_k)_k$, where $(b_j)_j$ is nonempty, or $(a_i)_i \mathbin{+\!\!+} ((b_j)_j \mathbin{+\!\!+} (c_k)_k)$ and $((a_i)_i \mathbin{+\!\!+} (b_j)_j) \mathbin{+\!\!+} (c_k)_k$, where $(a_i)_i$ is nonempty.

Chapter 3

Logics of Coalgebras

In the previous chapter we saw that the most interesting structures arising in computer science can be represented as coalgebras. One particular structure among the presented played an important role in the theory of coalgebras – Kripke frames – the most natural models of Hennessy-Milner logic. Realisation of this started a whole new area of research. The first real attempt to generalise modal logics for coalgebras was made by Lawrence S. Moss with his theory of coalgebraic logics via relation liftings in [Mos99] and shortly after was followed by many others [Pat03b; BK05; Kli07; CJ14b].

Coalgebraic logics have proved to provide a general framework for logics for systems over the years. Moreover, the universality of the framework allows us to prove important properties of logics such as soundness, completeness or Hennessy-Milner property once for all. Thanks to the flexibility and generality of the logics the slogan that

"Modal logics are coalgebraic"

is now widely accepted [Cîr+11].

In this chapter we present gradually more and more abstract logics for coalgebras; starting with Hennessy-Milner logic and finishing with categories of logics by Chen. We are sometimes a bit vague when introducing proof theories for the logics because our main interests are in analysing the structures and not the proof systems.

3.1 Hennessy-Milner logic

A variant of Hennessy-Milner logic [HM80] is obtained from propositional logic by adding one modal operation. The syntax is given by the following grammar:

$$\varphi ::= \mathsf{t} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi.$$

The other propositional connectives such as $\vee$, $\rightarrow$, $\leftrightarrow$ and f can be obtained from $\neg$, $\wedge$ and t. Also, the modal operator representing *possibility* can be introduced in similar fashion from the operator of *necessity* (or *certainty*): $\Diamond\varphi \stackrel{\text{def}}{\equiv} \neg(\Box\neg\varphi)$. The following axioms are added in addition to the axioms of propositional logic:

$$\Box \mathrm{t} \longleftrightarrow \mathrm{t}, \quad \text{and} \quad \Box(\varphi_1 \wedge \varphi_2) \longleftrightarrow \Box\varphi_1 \wedge \Box\varphi_2. \tag{Ax-$\Box$}$$

Intuitively, when interpreted in the theory of systems, $\Box\varphi$ means that the formula φ holds in any future state accessible in one step and $\Diamond\varphi$ that there exist a state accessible in one step where φ holds.

The models of the above defined modal logic are *Kripke frames*, that is the $\mathscr{P}$-coalgebras. The *truth relation* is defined inductively. For a Kripke frame (X,ξ) and an $x \in X$:

1. $(X,\xi), x \models \mathrm{t}$,
2. $(X,\xi), x \models \neg\varphi$ if not $(X,\xi), x \models \varphi$,
3. $(X,\xi), x \models \varphi \wedge \psi$ if $(X,\xi), x \models \varphi$ and $(X,\xi), x \models \psi$,
4. $(X,\xi), x \models \Box\varphi$ if, for all $y \in \xi(x)$, $(X,\xi), y \models \varphi$.

For a Kripke frame (X,ξ) we say that $(X,\xi) \models \varphi$ if $(X,\xi), x \models \varphi$ for all $x \in X$. It is well known fact that

3.1.1 Theorem ([BDRV02])**.** *Hennessy-Milner logic (as defined above) is sound and complete with respect to the class of Kripke frames.*

Moreover, finite branching Kripke frames have the *Hennessy-Milner property*, that is the behavioural equivalence and the logical equivalence coincide:

3.1.2 Theorem ([BDRV02])**.** *Let* $(X,\ \xi\colon X \to \mathscr{P}_f(X))$ *and* $(X',\ \xi'\colon X' \to \mathscr{P}_f(X'))$ *be two finitely branching Kripke frames. Then,*

$$\forall x \in X, x' \in X'. \quad x \leftrightarrows x' \quad \textit{iff} \quad (\forall\varphi.\ (X,\xi), x \models \varphi \iff (X',\xi'), x' \models \varphi).$$

We can define a semantics of a formula to be the set of states where the formula is true

$$[\![\varphi]\!]_\xi = \{\, x \in X \mid (X,\xi), x \models \varphi \}$$

and then the standard equivalence between the truth relation and semantics holds

$$(X,\xi), x \models \varphi \iff x \in [\![\varphi]\!]_\xi.$$

3.1.3 Remark. Note that in the same fashion we can define modal logics with multiple modalities. The Kripke frames are then of the shape $\mathscr{P}(-)^n$ (if we have n modalities) and axioms (Ax-$\Box$) are added for every modality independently.

3.2 1st tier logics – Modalities via predicate liftings

An obvious disadvantage of Hennessy-Milner logic is that it is suitable for the powerset coalgebras only. On the other hand, the logic itself is very simple and straightforward. One can try to take advantage of that and try to somehow embed general T-coalgebras into powerset coalgebras and use variants of Hennessy-Milner logics for T-coalgebras.

Intuitively, for a T-coalgebra (X,ξ), any natural transformation $\mu\colon T \Longrightarrow \mathscr{P}$ tells us how to extract a set of accessible states from T and, therefore, introduces a new modality. For example, for $T(X) = A + B \times X$, define $\mu_X : a \in A \mapsto \emptyset,\ (b,x) \in B \times X \mapsto \{x\}$. Then, we can define a semantics of $[\mu]$ by $[\![[\mu]\varphi]\!] = \xi^{-1} \circ (\mu^{-1})_X([\![\varphi]\!])$. Or, take $T(X) = A \times X^I$ and define $\mu^i_X : (a, f) \mapsto \{f(i)\}$, for all $i \in I$. Again, we have a set of modalities $\{[\mu^i] : i \in I\}$ via $[\![[\mu^i]\varphi]\!] = \xi^{-1} \circ (\mu^i_X)^{-1}([\![\varphi]\!])$, for all $i \in I$.

Notice that, since we only needed the preimages of μ, we can just focus on natural transformations $2^{(-)} \Longrightarrow 2^{T(-)}$ or $(2^{(-)})^n \Longrightarrow 2^{T(-)}$ for n-ary modalities. Then, a *modal logic via predicate liftings* for T-coalgebras is a set of axioms $\mathbb{A}$ and a set of modalities Λ. Where, for each (n-ary) $\triangle \in \Lambda$, there is a natural transformation

$$[\![\triangle]\!] : (2^{(-)})^n \Longrightarrow 2^{T(-)}.$$

The semantics of formulas is introduced inductively. For propositional symbols it is defined as one would expect and for modalities it is defined similarly to the examples above

$$[\![\varphi \wedge \psi]\!]_\xi = [\![\varphi]\!]_\xi \cap [\![\psi]\!]_\xi, \quad [\![\neg\varphi]\!]_\xi = X \setminus [\![\varphi]\!]_\xi,$$
$$[\![\triangle(\varphi_1, \varphi_2, \dots, \varphi_n)]\!]_\xi = \xi^{-1} \circ [\![\triangle]\!]([\![\varphi_1]\!]_\xi, [\![\varphi_2]\!]_\xi, \dots, [\![\varphi_n]\!]_\xi).$$

Having defined semantics first, we can now recover the truth relation. For every state $x \in X$

$$(X,\xi), x \models \varphi \ \stackrel{\text{def}}{\equiv}\ x \in [\![\varphi]\!]_\xi.$$

3.2.1 Example. We can reconstruct the classical modal logic defined above as a modal logic via predicate liftings. Set $\Lambda = \{\Box\}$, and define $[\![\Box]\!](Z) = \{Y \in \mathscr{P}(X) \mid Y \subseteq Z\}$. One can check that $[\![\Box]\!] : 2^{(-)} \Longrightarrow 2^{\mathscr{P}(-)}$ defined this way is natural and that the resulting semantics is the same as before. For example, we have that

$$[\![\Box\varphi]\!]_\xi = \{x \in X \mid \forall y \in \xi(x).\ (X,\xi), y \models \varphi\}.$$

Note. The requirement that the $[\![\triangle]\!]$'s are natural guarantees that the resulting logic is invariant under behavioural equivalence [KL12]. For a detailed overview of the logics via predicate liftings see [KP11], [Pat03b] or [Kur01].

3.3 2nd tier logics – Alexander Kurz's framework

3.3.1 Basic ingredients

Let $\mathscr{X}$ be a concrete category; let $\mathscr{A}$ be a category of algebras forming a variety; and, finally, let $Q\colon \mathscr{X} \to \mathscr{A}$ and $X : \mathscr{A} \to \mathscr{X}$ be contravariant dually adjoint functors. Intuitively, the signature of $\mathscr{A}$ gives us the syntax and the identities that hold for $\mathscr{A}$ correspond to the tautologies of our logic. The objects of the category $\mathscr{X}$ are going to be carriers of our transition systems.

We have the following picture

$$\mathscr{X} \underset{S}{\overset{Q}{\rightleftarrows}} \mathscr{A}$$

For example, if $\mathscr{A} = \mathbf{Bool}$, the syntax is formed of t, f, $\wedge$, $\vee$ and $\neg$, and the tautologies are just the standard tautologies of propositional logic. More formally, a formula φ holds in propositional logic iff $\varphi = 1$ is true in **Bool** (where φ is a term now). Similarly, if $\mathscr{A}$ is the category of Heyting algebras[1] or distributive lattices, then the logic is the intuitionistic propositional logic or the positive logic respectively[2].

When $\mathscr{X}$ is the category of sets, we expect carriers of our transition systems to be mere sets with no additional structure (as we saw in the previous chapter). Ideally, the category $\mathscr{X}$ has products and finite coproducts and one can define some kind of powerset functor for $\mathscr{X}$. If this is the case, $\mathscr{X}$ is suitable for representing transition systems because we then have everything we need for Kripke polynomial functors as before.

3.3.2 The logics

Fix a shape functor $T : \mathscr{X} \to \mathscr{X}$. An *abstract logic* for T is an endofunctor $L\colon \mathscr{A} \to \mathscr{A}$ and a natural transformation $\delta\colon LQ \Longrightarrow QT$. The functor L is called a *logic* functor and δ is called an *interpretation of L in T*.

Observe that any such δ defines a contravariant functor $\widetilde{Q}\colon \mathsf{Coalg}(T) \to \mathsf{Alg}(L)$:

$$\widetilde{Q}\colon (X,\ \xi : X \to T(X)) \longmapsto (Q(X),\ Q\xi \circ \delta_X : LQ(X) \to Q(X))$$

Note that the naturality of δ is crucial for $\widetilde{Q}$ to be a functor. For a coalgebra homomorphism $f : (X, \xi) \to (X', \xi')$ we have a commutative diagram

[1] *Heyting algebras* are bounded distributive lattices with exponentiation $\to$ satisfying: $a \wedge b \leq c$ if and only if $a \leq b \to c$.

[2] The process when we associate a certain derivation system with an equational logic of a class of algebras is called *algebraization* [FJP03].

$$\begin{array}{ccccc} LQ(X) & \xrightarrow{\delta_X} & QT(X) & \xrightarrow{Q\xi} & Q(X) \\ {\scriptstyle LQf}\uparrow & & {\scriptstyle QTf}\uparrow & & {\scriptstyle Qf}\uparrow \\ LQ(X') & \xrightarrow{\delta_{X'}} & QT(X') & \xrightarrow{Q\xi'} & Q(X') \end{array}$$

Assume that the category $\mathsf{Alg}(L)$ is still a variety and has an initial object (I,Φ). Then, for any T-coalgebra (X,ξ), we have a unique L-algebra homomorphism $[\![-]\!]_\xi\colon (I,\Phi) \to \widetilde{Q}(X,\xi)$ making the following diagram to commute

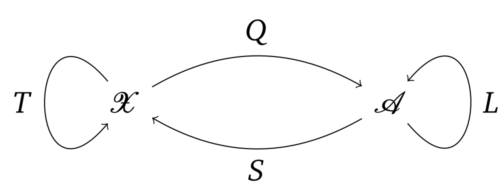
$$\begin{array}{ccccc} L(I) & & \xrightarrow{\quad\Phi\quad} & & I \\ \downarrow{\scriptstyle L[\![-]\!]_\xi} & & & & \downarrow{\scriptstyle [\![-]\!]_\xi} \\ LQ(X) & \xrightarrow{\delta_X} & QT(X) & \xrightarrow{Q\xi} & Q(X) \end{array}$$

We can think of the initial object (I,Φ) of $\mathsf{Alg}(L)$, as if it was a Lindenbaum-Tarski[3] algebra of some logic, then elements of I correspond to formulas (or, to be more precise, to equivalence classes of formulas) in the language of the logic corresponding to $\mathsf{Alg}(L)$. Therefore, given a coalgebra (X,ξ), we can define the truth relation as follows

$$(X,\xi), x \models \varphi \ \stackrel{\text{def}}{\equiv}\ x \in [\![\varphi]\!]_\xi,$$

for $x \in X$ and $\varphi \in I$. Note that the previous definition can only make sense if $U_{\mathscr{A}} \circ Q(X) \subseteq \mathscr{P} \circ U_{\mathscr{X}}(X)$ where $U_{\mathscr{A}}$ and $U_{\mathscr{X}}$ are the forgetful functors for $\mathscr{A}$ and $\mathscr{X}$ respectively.

We have the following picture



3.3.1 Remark. The assumption that $\mathsf{Alg}(L)$ is a variety with an initial object can be made more precise. For example, if $\mathscr{A}$ has free algebras, it is enough to assume that the functor L has a *finite presentation*.

Formally, for a variety $\mathscr{A} \cong \mathtt{Alg}\langle\Sigma, E\rangle$, a functor $L\colon \mathscr{A} \to \mathscr{A}$ is *equationally presented* by $\langle \Sigma_L \mid E_L \rangle$ if, for an algebra A with a presentation $\langle A, E_A \rangle$, is $L(A)$ obtained as a term

[3] Lindenbaum-Tarski algebra for a logic is a term algebra of all well-formed terms of that logic factored by logical equivalence. The elements of the Lindenbaum-Tarski algebra are congruence classes of formulas where the factoring relation is interderivability in the logic.

algebra over the set of variables $\{s(\alpha) : s \in \Sigma_L,\ \alpha \in A^{\mathsf{ar}(s)}\}$ and over the signature $\Sigma \cup \Sigma_L$ and factorised by the smallest congruence containing all instances of the equations in $E \cup E_L \cup E_A$.

Finally, we say that a functor $L\colon \mathscr{A} \to \mathscr{A}$ has a *finite presentation* if it is equationally presentable, the set of operations Σ_L is finite, with all the operations of a finite arity and the equations in E_L are of rank-1 (i.e. without nesting of the operations from Σ_L). For more details see [Kur06] and [CJ14b].

3.3.2 Example. The definition of modal logics via predicate liftings is just a special case of an abstract logic. Take $\mathscr{X} = \mathbf{Set}$ and $\mathscr{A} = \mathbf{Bool}$, and Q to be the contravariant powerset functor assigning to a set the Boolean algebra of all its subsets and S the functor assigning to a Boolean algebra the set of all its ultrafilters. Q and S are dually adjoint.

Let $(\Lambda, \{[\![\triangle]\!]\}_{\triangle\in\Lambda})$ be a modal logic defined via predicate liftings. For a $\triangle \in \Lambda$, let $\mathsf{ar}(\triangle)$ be the arity of $\triangle$. Take the functor $L(X) = \coprod_{\triangle\in\Lambda}\{\triangle\} \times X^{\mathsf{ar}(\triangle)}$ and the natural transformation $\delta\colon L \circ 2^{(-)} \Longrightarrow 2^{T(-)}$ defined pointwise:

$$(\triangle, X_1, X_2, \ldots, X_{\mathsf{ar}(\triangle)}) \in \{\triangle\} \times X^{\mathsf{ar}(\triangle)} \quad \longmapsto \quad [\![\triangle]\!](X_1, X_2, \ldots, X_{\mathsf{ar}(\triangle)}).$$

Observe that, since $Q\xi = \xi^{-1}$, the set $[\![\triangle(\varphi_1, \varphi_2, \ldots, \varphi_{\mathsf{ar}(\triangle)})]\!]_\xi$ is exactly the same as before. In order to also reflect the axioms that come with the logic, one has to factorise $L(X)$ by an appropriate congruence relation.

3.3.3 The framework

Now we present a list of conditions that should not be too restrictive and at the same time allows us to prove important properties we would like our abstract logics to have relatively easily. The conditions were inspired by Alexander Kurz's original conditions in [BK05; Kur06]. Let us assume that

(Ax-1) $\mathscr{X}$ is a concrete category and $\mathscr{A}$ is a variety with an initial algebra (for example, if it is a finitary variety);

(Ax-2) $Q\colon \mathscr{X} \to \mathscr{A}$ and $S\colon \mathscr{A} \to \mathscr{X}$ are contravariant functors forming a dual equivalence of categories, that is the units of adjunction $\eta\colon \mathrm{Id} \Longrightarrow QS$ and $\varepsilon\colon \mathrm{Id} \Longrightarrow SQ$ are natural isomorphisms;

(Ax-3) $U_{\mathscr{A}} \circ Q(X) \subseteq \mathscr{P} \circ U_{\mathscr{X}}(X)$, $Q(f) = f^{-1}$ and, for every $x \neq x' \in X$, there exists a $P \in Q(X)$ such that either $x \in P \not\ni x'$ or $x \notin P \ni x'$;

(Ax-4) $T\colon \mathscr{X} \to \mathscr{X}$ and $L\colon \mathscr{A} \to \mathscr{A}$ are functors and L has an initial L-algebra (I, Φ) (for example, if L has a finite presentation); and finally

(Ax-5) $\delta\colon LQ \Longrightarrow QT$ is an interpretation of L in T, moreover, δ is a natural isomorphism.

3.3.3 Remark. The axiom (Ax-3) just basically says that the algebra $Q(X)$ corresponding to X is formed of predicates of X and that any two points can be distinguished by some predicate of that algebra. This condition is easy to satisfy in practice. For example, the distributive lattice/frame of all open sets of a T_0 space has this property.

Having said that, now we see that the name of δ is justified. L applied to $Q(X)$ enriches the algebra of predicates $Q(X)$ by new predicates and δ shows how to *interpret* them as predicates of $T(X)$. The very same ideas were already used by Abramsky [Abr91].

Since Q and S form a dual equivalence and δ is a natural isomorphism, we can define a functor $\widetilde{S}\colon \mathsf{Alg}(L) \to \mathsf{Coalg}(T)$ by

$$\widetilde{S}\colon (A,\, \alpha\colon L(A) \to A) \quad \longmapsto \quad (S(A),\, \rotatebox[origin=c]{180}{δ}^{-1} \circ S\alpha\colon S(A) \to TS(A))$$

where $\rotatebox[origin=c]{180}{δ}\colon TS \Longrightarrow SL$ is the natural isomorphism given by

$$TS(A) \xrightarrow{\varepsilon_{TS(A)}} SQTS(A) \xrightarrow{S\delta_{S(A)}} SLQS(A) \xrightarrow{SL\eta_A} SL(A).$$

We extend the notation from the previous section and denote the unique algebra homomorphisms from (I,Φ) to an algebra (A,α) by $[\![-]\!]_\alpha\colon (I,\Phi) \to (A,\alpha)$. Let (X,ξ) be a T-coalgebra, we say that

$$(X,\xi) \models \varphi = \psi \ \stackrel{\text{def}}{\equiv}\ \widetilde{Q}(X,\xi) \models [\![\varphi]\!]_\xi = [\![\psi]\!]_\xi \qquad \text{for (formulas) } \varphi,\psi \in I.$$

We say that $\mathsf{Alg}(L) \models \varphi = \psi$ holds if $(A,\alpha) \models [\![\varphi]\!]_\alpha = [\![\psi]\!]_\alpha$ holds for any L-algebra (A,α) and, similarly, $\mathsf{Coalg}(T) \models \varphi = \psi$ if $(X,\xi) \models [\![\varphi]\!]_\xi = [\![\psi]\!]_\xi$ holds for any T-coalgebra (X,ξ).

3.3.4 Theorem (Invariance under $\leftrightarrows$)**.** *Let $f : (X,\xi) \to (X',\xi')$ be a coalgebra homomorphism. Then, for a $x \in X$,*

$$(X,\xi), x \models \varphi \quad \Longleftrightarrow \quad (X',\xi'), f(x) \models \varphi.$$

Proof. Since (I,Φ) is an initial algebra, the following diagram commutes

$$\begin{array}{ccc} (I,\Phi) & \xrightarrow{[\![-]\!]_\xi} & \widetilde{Q}(X,\xi) \\ & \searrow^{[\![-]\!]_{\xi'}} & \uparrow{\scriptstyle \widetilde{Q}f = f^{-1}} \\ & & \widetilde{Q}(X',\xi') \end{array}$$

Therefore, $x \in [\![\varphi]\!]_\xi$ if and only if $f(x) \in [\![\varphi]\!]_{\xi'}$. □

Next, we show soundness and completeness of the equational theory of L-algebras with respect to the T-coalgebras:

3.3.5 Theorem (Soundness and Completeness)**.**

$$\mathsf{Alg}(L) \models \varphi = \psi \quad \textit{if and only if} \quad \mathsf{Coalg}(T) \models \varphi = \psi.$$

Proof. The "⇒" (soundness) follows from the fact that equality is preserved by all algebra homomorphisms, i.e. by all $[\![-]\!]_\xi : (I,\Phi) \to \widetilde{Q}(X,\xi)$. For "⇐" (completeness), first observe that $\eta_I : I \to QS(I)$ is the unique L-algebra homomorphism $[\![-]\!] : (I,\Phi) \to \widetilde{Q}\widetilde{S}(I,\Phi)$ because all the morphisms in the following diagram are isomorphisms:

$$
\begin{array}{ccccc}
L(I) & & \xrightarrow{\quad\quad\quad\Phi\quad\quad\quad} & & I \\
\big\downarrow{\scriptstyle L\eta_I} & & & & \big\downarrow{\scriptstyle \eta_I} \\
LQS(I) & \xrightarrow{\delta_{S(I)}} & QTS(I) & \xrightarrow{Q(\mathfrak{z}^{-1}\circ S\Phi)} & QS(I)
\end{array}
$$

Now, assume that there exists an L-algebra where $\varphi \neq \psi$. Then, in the initial algebra $(I,\Phi) \not\models \varphi = \psi$ and, therefore, the elements corresponding to φ and ψ are not equal in I. But, since η_I is one-one, $\widetilde{Q}\widetilde{S}(I,\Phi) \models [\![\varphi]\!]_\xi \neq [\![\psi]\!]_\xi$ and $\widetilde{S}(I,\Phi) \models \varphi \neq \psi$. □

Note that, if (I,Φ) indeed represents a Lindenbaum-Tarski algebra of some logic, then an equation holds for all algebras if and only if the corresponding formulas are interderivable. On the other hand, $\varphi = \psi$ for all T-coalgebras if and only if φ and ψ correspond to the same predicates or, in other words, the states satisfying φ are exactly the same states that satisfy ψ. Then, the theorem just says that two formulas are interderivable if and only if they are semantically equivalent.

3.3.6 Theorem (Hennessy-Milner property)**.** *Let* $(X,\ \xi : X \to T(X))$ *and* $(X',\ \xi' : X \to T(X'))$ *be two coalgebras. Then,*

$$\forall x \in X, x' \in X'. \quad x \leftrightarrows x' \quad \textit{iff} \quad (\forall \varphi.\ (X,\xi), x \models \varphi \iff (X',\xi'), x' \models \varphi).$$

Proof. The Theorem 3.3.4 implies that it is enough to check the statement just for the states of the final coalgebra $\widetilde{S}(I,\Phi)$. In a final coalgebra, two states are behaviourally equivalent if they are equal. This proves the left-to-right implication (adequacy). For converse (expressiveness), assume $x \neq x'$ in F. Then, x and x' have to be distinguished by some element P in $\widetilde{S}(I,\Phi)$ by (Ax-3). Finally, since η_I is onto, $[\![-]\!] : (I,\Phi) \to \widetilde{Q}\widetilde{S}(I,\Phi)$ is also onto (see the previous proof), and therefore the exists a $\phi \in I$ such that $[\![\phi]\!] = P$ distinguishing x and x'. □

3.3.4 Generalisations

In some applications the list of axioms we assumed may be too restrictive. One can still obtain the same results assuming weaker conditions. One does not have to assume that an initial L-algebra exists; or that Q and S does not form an equivalence (only I-th component of η needs to be iso to prove soundness and expressiveness) [BK05]. One can also omit the assumption that δ is a natural isomorphism but this requires more assumptions for the category $\mathscr{X}$ (such as having a factorisation system) and for the functor T [Kli07; JS10].

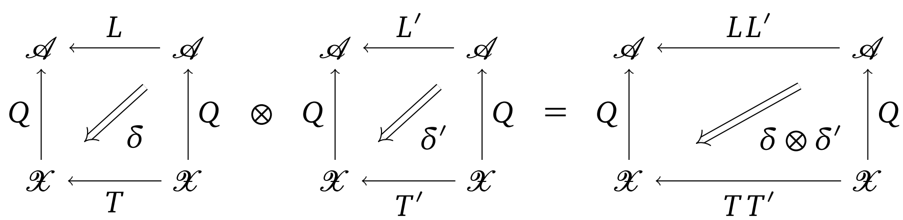
3.4 3rd tier logics – the categories CoLog

Coalgebraic logics, when viewed from Chen's perspective, are even a bit more abstract. First, we fix a contravariant functor $Q\colon \mathscr{X} \to \mathscr{A}$ where $\mathscr{X}$ is a concrete category and $\mathscr{A}$ a variety. The triples (L, T, δ) define a category CoLog^Q of *one-step semantics* where T is an endofunctor on $\mathscr{X}$, L is an endofunctor on $\mathscr{A}$ and $\delta\colon LQ \Longrightarrow QT$ is a natural transformation. Morphisms of CoLog^Q are pairs of natural transformations $(\tau\colon L \Longrightarrow L',\ \gamma\colon T' \Longrightarrow T)$ making the following diagram (in the category $[\mathscr{X}, \mathscr{A}]$ of functors from $\mathscr{X}$ to $\mathscr{A}$) commute:

$$\begin{array}{ccc} LQ & \overset{\delta}{\Longrightarrow} & QT \\ {\scriptstyle \tau Q}\Downarrow & & \Downarrow{\scriptstyle Q\gamma} \\ L'Q & \underset{}{\overset{\delta'}{\Longrightarrow}} & QT' \end{array}$$

3.4.1 Remark. The category CoLog^Q can be alternatively defined as the comma category $(Q^* \downarrow Q_*)$, where $Q^*\colon [\mathscr{A}, \mathscr{A}] \to [\mathscr{X} \to \mathscr{A}]$ and $Q_*\colon [\mathscr{X}, \mathscr{X}] \to [\mathscr{X} \to \mathscr{A}]$ are the post– and pre–compositions with Q.

There is a strict monoidal structure in CoLog^Q: Two objects (L, T, δ) and (L', T', δ') can be composed the following way:



Where $\delta \otimes \delta'$ is defined as $\delta T' \circ L\delta'$. The unit with respect to $\otimes$ is $\mathbb{I} = (\mathrm{Id}_{\mathscr{X}}, \mathrm{Id}_{\mathscr{A}}, \mathrm{id}_P)$. We can think of $\otimes$ as a composition of the languages and the transition systems.

One can define soundness an completeness and the Hennessy-Milner property for one-step semantics in a similar fashion as we described in the previous section.

When inspecting logics for coalgebras of a certain functor T, it useful to study the subcategory CoLog^Q_T of the category CoLog^Q containing only the objects of the form (L, T, δ) and the morphisms $(\tau,\ \mathrm{id}_T)$.

3.4.2 Theorem. *Let $Q\colon \mathscr{X} \to \mathscr{A}$ has a (dual) left adjoint $S\colon \mathscr{A} \to \mathscr{X}$ and let $\epsilon\colon Id \Longrightarrow SQ$ be the unit of the adjunction. Then, for every endofunctor T, the category CoLog^Q_T has a terminal object*

$$(QTS, T, QT\epsilon\colon QTSQ \Longrightarrow QT).$$

The theorem basically says that in an adjoint situation there is always the most general "language" for the category $\mathsf{Coalg}(T)$; we call this object the *full one-step semantics* for T.

3.4.3 Remark. We may be interested in one-step semantics of equationally presentable functors only or, in other words, we may want to restrict CoLog^Q to triples (T, L, δ) where L is equationally presentable. Equationally presentable one-step semantics form a coreflexive subcategory of CoLog^Q, $\otimes$-composition of two equationally presentable is again equationally presentable and, similarly to previous, for every functor T, it has the most general full one-step semantics that is still in the category $\mathsf{Coalg}(T)$.

On the other hand, one can be interested in the subcategory of one-step semantics satisfying Hennessy-Milner property. The $\otimes$-operation is again closed on this category.

One of the limitation of Alexander Kurz's framework is that it can only talk about properties of computations that happen in one step. The higher perspective given by the categories CoLog seems to offer a way around of this issue. Achim Jung and Liang-Ting Chen believe that the monoidal objects of $(\mathsf{CoLog}^Q, \otimes, \mathbb{I})$ should provide a suitable *"multi-step"* semantics for transition systems. For details see [Che13] and [CJ14b].

3.5 Example – Jónsson-Tarski duality

In this section we present a fragment of Jónsson-Tarski duality that fits into Alexander Kurz's framework. Full details and proofs can be found in [CJ14a].

3.5.1 Definition. A topological space is a *Stone space* if it is compact 0-dimensional and Hausdorff. We denote the category of Stone spaces and continuous maps by **Stone**.

The original Stone duality is a duality between the category of Stone spaces and the category of Boolean algebras, and is witnessed by the dual adjunction of functors Clp and Ult. The functor Clp assigns to a Stone space the Boolean algebra of its closed-open subsets and Ult assigns to a Boolean algebra its Stone space of ultrafilters.

3.5.2 For a Stone space $X = (X, \tau)$, the *Vietoris space* $\mathbb{V}X$ is the space $(\mathscr{K}X, \mathbb{V}\tau)$ where $\mathscr{K}X$ is the set of all compact/closed subsets of X and $\mathbb{V}\tau$ is the topology generated by the elements of the form $\boxtimes U$ and $\Diamond U$. Where, for all $U \in \tau$,

$$K \in \boxtimes U \quad \stackrel{\text{def}}{\equiv} \quad K \subseteq U \quad \text{and} \quad K \in \Diamond U \quad \stackrel{\text{def}}{\equiv} \quad K \cap U \neq \emptyset.$$

And, for a continuous map $f : X \to Y$ between two Vietoris spaces, define $\mathbb{V}(f) \colon \mathbb{V}X \to \mathbb{V}Y$ as

$$\mathbb{V}(f) \colon K \mapsto f[K].$$

Fact. $\mathbb{V}$ *is a well-defined endofunctor on Stone spaces.*

3.5.3 On the algebraic side, define an endofunctor $\mathbb{M}\colon \mathbf{Bool} \to \mathbf{Bool}$ as

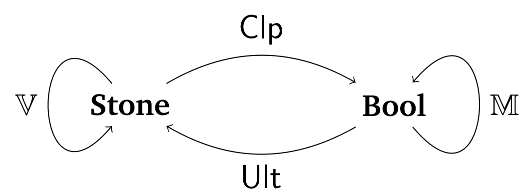
$$\begin{aligned} \mathbb{M}\colon A &\mapsto \mathbb{BA}\langle\, \Box a : a \in A \mid \Box 1 = 1,\ \Box(a \wedge b) = \Box a \wedge \Box b\ \rangle \text{ and} \\ \mathbb{M}(f)\colon \Box a &\mapsto \Box f(a), \end{aligned}$$

where the resulting Boolean algebra on the right-hand side is just the term algebra generated by the $\Box a$'s and factorised by the smallest congruence containing all instances of the equations on the right-hand side of the presentation. Note that the equational presentation of $\mathbb{M}$ is finitary.

Next, we can define an interpretation of $\mathbb{M}$ in $\mathbb{V}$ as the natural *isomorphism* $\delta\colon \mathbb{M}\circ\mathsf{Clp} \Longrightarrow \mathsf{Clp}\circ\mathbb{V}$ given by:

$$\delta_X\colon \Box U \longmapsto \{K \in \mathscr{K}X \mid K \subseteq U\}.$$

The whole picture looks as follows



Now, we have everything we need to use the tools described earlier in the text. Together with the fact that Clp and Ult are dual to each other, we have that the logic of $\mathsf{Alg}(\mathbb{M})$ is sound and complete and has the Hennessy-Milner property with respect to $\mathsf{Coalg}(\mathbb{V})$.

How does the equational logic of $\mathsf{Alg}(\mathbb{M})$ looks like? Since **Bool** is a variety, it has to have a presentation $\mathbf{Bool} \cong \mathtt{Alg}\,\langle\Sigma,\ E\rangle$ where Σ is a signature and E a set of equations. The logic inherited from the presentation consists of (intuitively):

1. the (logical) operations $\wedge, \vee, \mathsf{t}, \mathsf{f}$ and $\neg$ from the signature Σ,

2. the standard propositional tautologies, such as $\neg\neg a = a$, $\neg(a \vee b) = \neg a \wedge \neg b$, etc.

And the presentation of the functor $\mathbb{M}$ enriches the logic by

3. the unary operation $\Box$, and

4. the axiom schemas:

$$\Box 1 = 1, \quad \text{and} \quad \Box(a \wedge b) = \Box a \wedge \Box b.$$

This should look quite familiar; it is similar to the axioms of Hennesy-Milner logic.

3.5.4 Definition. A *modal algebra* (A, t) is a Boolean algebra A and an unary operation t on A, called *modal operation*, such that t preserves binary meets and the top. A Boolean algebra homomorphism $f : (A, t) \to (A', t')$ is a *modal algebra homomorphisms* if it commutes with the modal operations:

$$f(t(x)) = t'(f(x)) \quad \text{for all } x \in A.$$

Observe that the category of $\mathbb{M}$-algebras is equivalent to the category of modal algebras. Indeed, for a modal algebra (A, t) define a $\mathbb{M}$-algebra on generators as

$$\alpha_t : \Box x \mapsto t(x),$$

and, for converse, for a $\mathbb{M}$-algebra $(A,\ \alpha : \mathbb{M}A \to A)$ define a modal algebra (A, t_α) as

$$t_\alpha : x \mapsto \alpha(\Box x).$$

We can immediately see that $\alpha_{t_\alpha} = \alpha$ and $t_{\alpha_t} = t$. The transformations are well defined because both modal operation and $\Box$ distribute over meets and preserve top. Moreover, a Boolean algebra homomorphism $f : A \to B$ is a modal algebra homomorphism between (A, t) and (B, r) if and only if it is a $\mathbb{M}$-algebra homomorphism between (A, α_t) and (B, α_r). Also, the commutativity of the square

$$\begin{array}{ccc} \mathbb{M}A & \xrightarrow{\mathbb{M}f} & \mathbb{M}B \\ \downarrow{\scriptstyle \alpha_r} & & \downarrow{\scriptstyle \alpha_r} \\ A & \xrightarrow{f} & B \end{array}$$

is precisely the condition for commutativity of f and modal operations.

The previous discussion allows us to recover the famous Jónsson-Tarski duality between the categories of *descriptive general frames* and modal algebras: The category of $\mathbb{V}$-coalgebra is equivalent to the category of descriptive general frames (as proved in [KKV04]) and the category of $\mathbb{M}$-algebras is equivalent to the category of *modal algebras* as shown above. Moreover, the category of modal algebras gives an algebraic semantics to Hennessy-Milner logic. Therefore, the duality of modal algebras and $\mathbb{M}$-algebras allows us to think of the equational logic of $\mathbb{M}$-algebras as if it was an algebraization of Hennessy-Milner logic.

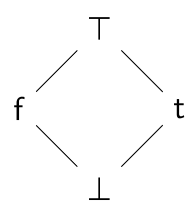
Chapter 4

Four valued view of the world

If we take a look at the truth relation of the classical modal logic more carefully, we can see that it can be sometimes too strict. A proposition is true only if it holds in all states of a Kripke frame; otherwise it is false. Sometimes, we would like to get more precise information. For example, we would like to differentiate between the situations when the proposition holds in all states, is false in all states, when it is sometimes true and sometimes false or we do not have enough information to decide it.

The other reason for not using classical two-valued logic is that we often do not have such a strong decision procedure. Sometimes we can not decide for some states whether the proposition is true or false. Also, we can not have a recursive procedure deciding if a program stops on a given input or does not stop.

The above inspired Belnap [Bel77] to create an informational-logical structure $\mathscr{FOUR}$ that looks like this:



As usual, we have the classical logical values t and f representing true and false but, on top of that, we also have the information values $\top$ and $\bot$ that represent the situation when we have a contradicting information and no information respectively.

This lattice has two orders. First, the logical order $\mathsf{f} < \{\bot, \top\} < \mathsf{t}$ and, second, the information order $\bot \sqsubset \{\mathsf{f}, \mathsf{t}\} \sqsubset \top$. Both orders carry a lattice structure. For example, if a proposition is both true and false, we have a contradiction (written as $\mathsf{f} \sqcup \mathsf{t} = \top$) and, on the other hand, the true and false values have nothing in common ($\mathsf{f} \sqcap \mathsf{t} = \bot$). Similarly, it is illogical to have a contradiction and no information ($\top \wedge \bot = \mathsf{f}$ and from symmetry also $\top \vee \bot = \mathsf{t}$).

4.1 Algebraic approach

Properties of $\mathscr{FOUR}$ inspired Arieli and Avron [AA96] to introduce the following definitions. A *bilattice* has the following structure

$$(A; \wedge, \vee, \otimes, \oplus, \neg, \mathrm{t}, \mathrm{f}, \top, \bot)$$

where both $(A; \wedge, \vee, \mathrm{t}, \mathrm{f})$ and $(A; \otimes, \oplus, \top, \bot)$ are bounded distributive lattices. The two distributive lattice structures give two partial orders, the *knowledge* and the *logical* (or *truth*) order:

$$a \otimes b = a \iff a \leq_k b, \qquad a \wedge b = a \iff a \leq_t b.$$

Negation obeys the following rules:

1. $a \leq_k b \implies \neg a \leq_k \neg b$,
2. $a \leq_t b \implies \neg b \leq_t \neg a$, and
3. $\neg\neg a = a$.

Then, a matrix $\langle A, F\rangle$ is an *implicative bilattice* if A is a bilattice, $F \subseteq A$ is a *bifilter* (that is a filter with respect to both orders) and we have a *weak implication* $\supset$ defined by

$$x \supset y \overset{\text{def}}{\equiv} \begin{cases} y & \text{if } x \in F, \\ \mathrm{t} & \text{if } x \notin F. \end{cases}$$

Alternatively, one can define implicative bilattices to be the bilattices with an implication satisfying a list of axioms. An important example of an implicative bilattice is the matrix $\langle \mathscr{FOUR}, \{\mathrm{t}, \top\}\rangle$.

Let **Fm** be the set all well formed terms in the language $\mathscr{L} = \{\wedge, \vee, \otimes, \oplus, \neg, \supset, \mathrm{t}, \mathrm{f}, \top, \bot\}$ and generated by a fixed set of variables. A *valuation* $v\colon \mathbf{Fm} \to A$ in an implicative bilattice $\langle A, F\rangle$ is an $\mathscr{L}$-homomorphism determined inductively by the values of the variables. Now, we can define a truth relation. For a $\Gamma \subseteq \mathbf{Fm}$ and a $\varphi \in \mathbf{Fm}$, we say that

$$\Gamma \models \varphi \text{ holds in } \langle A, F\rangle$$

if, for every valuation v in $\langle A, F\rangle$ such that $v[\Gamma] \subseteq F$, also $v(\varphi) \in F$.

The above defined truth relation is finitely axiomatizable. The resulting entailment relation $\vdash$ is sound and complete with respect to the class of all implicative bilattices. Moreover, we have

4.1.1 Theorem ([JR13]). *The following statements are equivalent*

1. $\Gamma \vdash \varphi$,
2. $\Gamma \models \varphi$ *holds in* $\langle \mathscr{FOUR}, \{\mathrm{t}, \top\}\rangle$, *and*
3. $\Gamma \models \varphi$ *holds in all implicative bilattices.*

4.1.1 Implications

As the name "weak implication" suggest that one can also define *strong implication*

$$x \to y \stackrel{\text{def}}{\equiv} (x \supset y) \wedge (\neg y \supset \neg x).$$

Neither weak nor strong implications behave exactly as we are used to from propositional logic. The first mentioned enjoys the classical deductive properties. We have the deduction theorem and modus ponens for weak implication (the second as a part of the axiomatization of $\vdash$):

$$\frac{\varphi \vdash \psi}{\vdash \varphi \supset \psi} \quad \text{and} \quad \varphi, (\varphi \supset \psi) \vdash \psi$$

On the other hand, strong implication also satisfies several useful properties: First, similarly to the classical propositional implication

$$\Gamma \models \varphi \to \psi \iff \Gamma \models \neg\psi \to \neg\varphi.$$

Second, for $y \in \{\mathsf{t}, \top\}$, it does not have to necessarily hold that $x \to y \notin \{\mathsf{t}, \top\}$ because $\mathsf{t} \to \top = \mathsf{f}$ in $\mathcal{FOUR}$. The reason is that it can happen that even if ψ is valid, $\varphi \to \psi$ does not have to be [JR13].

4.1.2 Twist-structures

Every Boolean algebra $A = (A; \sqcap, \sqcup, \sim, 1, 0)$ gives us a *twist-structure*

$$A^{\bowtie} = (A \times A; \wedge, \vee, \otimes, \oplus, \neg, \supset, \mathsf{t}, \mathsf{f}, \top, \bot)$$

where

1. $(a, b) \wedge (a', b') = (a \sqcap a', b \sqcup b'), \quad (a, b) \vee (a', b') = (a \sqcup a', b \sqcap b'),$
   $\mathsf{t} = (1, 0)$ and $\mathsf{f} = (0, 1)$;
2. $(a, b) \otimes (a', b') = (a \sqcap a', b \sqcap b'), \quad (a, b) \oplus (a', b') = (a \sqcup a', b \sqcup b'),$
   $\top = (1, 1)$ and $\bot = (0, 0)$;
3. $\neg(a, b) = (b, a)$; and
4. $(a, b) \supset (a', b') = (\sim a \sqcup a', a \sqcap b')$.

A twist-structure over a Boolean algebra is always an implicative bilattice. Conversely, every implicative bilattice can be obtained as a twist-structure of a Boolean algebra [BJR11; JR13]. Therefore, the four-valued axiomatization of the logic presented above is also sound and complete with respect to twist-structures.

4.2 Topological approach

The computation motivation to four-valued logic and the slogan *"where there is a computation there is also a topology,"* suggest a completely different approach to four-valued logic. As Jung and Moshier showed in [JM06], bitopological spaces can provide a natural semantics for four-valued logics. They also developed a theory of d-frames – an equational axiomatization of bitopological spaces. D-frames then play the same role to bitopological spaces as frames[1] play to topological spaces.

4.2.1 Bitopological spaces

We can read the original Stone duality for Boolean algebras logically. The elements of a Boolean algebra correspond to propositional predicates, the ultrafilters are models and the Stone topology is generated by the sets $\Phi_+(a) = \{F \mid a \in F\}$ containing all the models/states where a holds.

In the case of a four-valued logic, we do not have to have a full information to decide where a formula holds and where it fails. Instead, we can usually only make observations and, as the computation continues and we learn more and more, we may decide the logical value of a proposition later. This naturally introduces two topologies: The positive (upper) topology represents positive observations and the negative (lower) topology represents the observably false propositions.

Then, for an observation a, if the open set $\Phi_+(a)$ represents the states where the proposition a holds and the open set $\Phi_-(a)$ is where the proposition fails, then $\Phi_+(a) \cap \Phi_-(a)$ represents where a is contradictory and the complement of $\Phi_+(a) \cup \Phi_-(a)$ represents where we do not have enough information to decide.

4.2.2 D-frames

A *d-lattice* is a quadruple $(L_+, L_-; \mathsf{con}, \mathsf{tot})$ such that $(L_+; \sqcap, \sqcup, 1, 0)$ and $(L_-; \sqcap, \sqcup, 1, 0)$ are bounded distributive lattices and the relations $\mathsf{con} \subseteq L_+ \times L_-$ and $\mathsf{tot} \subseteq L_+ \times L_-$ satisfy the following rules:

$$
\begin{array}{lrl}
(\mathsf{con}\text{–}\downarrow) & \alpha \in \mathsf{con} \text{ and } \beta \sqsubseteq \alpha \implies & \beta \in \mathsf{con}, \\
(\mathsf{tot}\text{–}\uparrow) & \alpha \in \mathsf{tot} \text{ and } \beta \sqsupseteq \alpha \implies & \beta \in \mathsf{tot}, \\
(\mathsf{con}\text{–}\wedge,\vee) & \alpha, \beta \in \mathsf{con} \implies & \alpha \vee \beta \in \mathsf{con} \text{ and } \alpha \wedge \beta \in \mathsf{con}, \\
(\mathsf{tot}\text{–}\wedge,\vee) & \alpha, \beta \in \mathsf{tot} \implies & \alpha \vee \beta \in \mathsf{tot} \text{ and } \alpha \wedge \beta \in \mathsf{tot}, \\
(\mathsf{con},\mathsf{tot}\text{–}tt,f\!f) & tt \in \mathsf{con} \text{ and } tt \in \mathsf{tot}, & f\!f \in \mathsf{con} \text{ and } f\!f \in \mathsf{tot}, \\
(\mathsf{con}\text{–}\mathsf{tot}) & \alpha \in \mathsf{con}, \beta \in \mathsf{tot} \text{ and} & \\
 & (\alpha \sqcup \beta = \alpha \wedge \beta \text{ or } \alpha \sqcup \beta = \alpha \vee \beta) \implies & \alpha \sqsubseteq \beta.
\end{array}
$$

[1] *Frames* are complete lattices satisfying the equation: $b \wedge (\bigvee_i a_i) = \bigvee_i (b \wedge a_i)$. For an more information see [PP11].

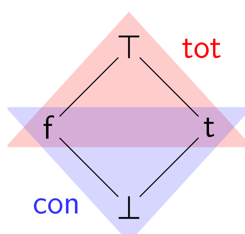
Where $(x_+, x_-) \vee (y_+, y_-) = (x_+ \sqcup y_+, x_- \sqcap y_-)$, $(x_+, x_-) \wedge (y_+, y_-) = (x_+ \sqcap y_+, x_- \sqcup y_-)$ and $\mathit{ff} = (1,0), \mathit{tt} = (0,1) \in L_+ \times L_-$.

The idea is that the L_+ elements corresponds to positive observations and the elements of L_- corresponds to negative observations. Then, a pair of elements $(a, b) \in L_+ \times L_-$ is in con if the information given by (a, b) is *consistent*, i.e. no model can satisfy both a and b, and is in tot if it is *total*, i.e. all models satisfy either a or b (or both). This intuition motivates the axioms for con and tot listed above.

For example, (con–↓) just means that, if we have a consistent information, taking a smaller part of it must still be consistent. Or, in the language of computation: a systems as it computes gradually explore more and more information and, if at some point all the explored data is consistent, then it is clear that until that point in all the previous stages of computation the explored data must have been also consistent.

4.2.1 Example.

1. The only nontrivial four element d-lattice is the Belnap's 4-valued lattice enriched by d-lattice structure: $\mathcal{FOUR} = ((0 < 1), (0 < 1);\ \mathsf{con} = \{\mathsf{t}, \mathsf{f}, \bot = (0,0)\},\ \mathsf{tot} = \{\mathsf{t}, \mathsf{f}, \top = (1,1)\})$ where $\mathsf{t} = (0,1)$, $\mathsf{f} = (1,0)$. $\mathcal{FOUR}$ is depicted in the following picture.



2. Any bitopological space (X, τ_+, τ_-) carries a d-lattice structure:

$$\Omega X = (\tau_+, \tau_-;\ \mathsf{tot}_X,\ \mathsf{con}_X),$$

where

$$\mathit{tt}_X = (\emptyset, X), \quad \mathit{ff}_X = (X, \emptyset),$$

$$U\, \mathsf{con}_X V \ \stackrel{\text{def}}{\equiv}\ U \cap V = \emptyset \quad \text{and} \quad U\, \mathsf{tot}_X V \ \stackrel{\text{def}}{\equiv}\ U \cup V = X,$$

and, on top of that, it always satisfies the following axiom

(con–$\bigsqcup^{\uparrow}$) $\qquad A \subseteq \mathsf{con}$ and A is $\sqsubseteq$-directed $\implies \bigsqcup^{\uparrow} A \in \mathsf{con}$.

The second example motivates the following definition:

4.2.2 Definition. D-lattices where both L_+ and L_- are frames and satisfy (con–$\bigsqcup^{\uparrow}$) are called *d-frames*.

D-frames play the same role to bitopological spaces as frames play to topological spaces. The same as frames are models of geometric intuitionistic logic, d-frames can be considered to be the models of positive geometric four-valued logic. Or, we can think of elements of d-frames as predicates being approximated. For more details see [JM06; Kli12].

4.2.3 Stone type duality

4.2.3 Definition. For a d-frame $\mathscr{L} = (L_+, L_-; \mathsf{con}, \mathsf{tot})$, define $\Sigma(\mathscr{L})$ to be the bitopological space with points all the pairs of completely prime filters (F_+, F_-), $F_+ \subset L_+$ and $F_- \subset L_-$, satisfying

(dp$_{\mathsf{con}}$) $\quad (a,b) \in \mathsf{con} \implies a \notin F_+$ or $b \notin F_-$, and

(dp$_{\mathsf{tot}}$) $\quad (a,b) \in \mathsf{tot} \implies a \in F_+$ or $b \in F_-$.

$\Sigma(\mathscr{L})$ is equipped with two topologies – the plus and the minus topology. They are generated by the sets of the form

$$\Phi_+(a) = \{ (F_+, F_-) \mid a \in F_+ \} \quad \text{and} \quad \Phi_-(b) = \{ (F_+, F_-) \mid b \in F_- \},$$

for $a \in L_+$ and $b \in L_-$.

Also, Σ acts on d-frame homomorphisms (that is the homomorphisms that preserve the d-frame structure). For a d-frame homomorphisms $f : \mathscr{L} \to \mathscr{M}$, set:

$$\Sigma(f) \colon \Sigma(\mathscr{M}) \to \Sigma(\mathscr{L}) \quad \text{to} \quad (F_+, F_-) \longmapsto (f^{-1}[F_+], f^{-1}[F_-]).$$

The mappings $\Sigma \colon$ **d-Frm** $\to$ **biTop** defined this way form a functor. Similarly, we can make $\Omega \colon$ **biTop** $\to$ **d-Frm** from the previous example to be also a functor by specifying how it acts on morphisms. For a bicontinuous map $f : (X; \tau_+^X, \tau_-^X) \to (Y; \tau_+^Y, \tau_-^Y)$ set

$$\Omega(f) \colon \Omega(Y) \to \Omega(X) \quad \text{to} \quad (U_+, U_-) \in \tau_+^Y \times \tau_-^Y \longmapsto (f^{-1}[U_+], f^{-1}[U_-]).$$

These functors witness a duality similarly to the duality between topological spaces and frames:

4.2.4 Theorem ([JM06]). *The functors Ω and Σ form a dual adjunction between the category of bitopological spaces and the category of d-frames.*

4.2.4 Stone bispaces and d-frames

The same as Stone spaces are dual to Boolean algebras, bitopological Stone spaces are dual to distributive lattices. Let us first start with the definitions.

4.2.5 Definition. Let $\mathscr{L} = (L_+, L_-; \mathsf{con}, \mathsf{tot})$ be a d-frame. We say that a is *well-below b (in L_+)*, and write $a \lhd_+ b$, for $a, b \in L_+$, if there exists a $c \in L_-$ such that $(a,c) \in \mathsf{con}$ and $(b,c) \in \mathsf{tot}$. Similarly, $a \lhd_- b$ if there exists a $c \in L_+$ such that $(c,a) \in \mathsf{con}$ and $(c,b) \in \mathsf{tot}$. Next, the elements of pairs in $\mathsf{con} \cap \mathsf{tot}$ are called *complemented*.

The intuition behind the previous definitions comes from bitopological spaces. If $\mathscr{L}$ is spacial, say $\mathscr{L} = \Omega(X; \tau_+, \tau_-)$, and $U, V \in \tau_+$, then U is well-below V if and only if

$$\overline{U}^{\tau_-} \subseteq V,$$

Also, U and a $W \in \tau_-$ are complemented if and only if

$$U \cup W = X \quad \text{and} \quad U \cap W = \emptyset.$$

4.2.6 Definition. Let $\mathscr{L} = (L_+, L_-; \mathrm{con}, \mathrm{tot})$ be a d-frame. We say that $\mathscr{L}$ is *compact* if, for any directed $\{\alpha_i \mid i \in I\} \subseteq L_+ \times L_-$ such that $\bigsqcup^{\uparrow}_{i\in I} \alpha_i \in \mathrm{tot}$, already $\alpha_i \in \mathrm{tot}$ for some $i \in I$.

We say that $\mathscr{L}$ is *d-regular* if

$$a = \bigsqcup\{c \in L_+ \mid c \lhd_+ a\}, \quad b = \bigsqcup\{c \in L_- \mid c \lhd_- b\}$$

for all $a \in L_+$ and $b \in L_-$. Similarly, we say that $\mathscr{L}$ is *zero-dimensional* if

$$a = \bigsqcup\{\text{ complemented } c \in L_+ \mid c \sqsubseteq a\}, \quad b = \bigsqcup\{\text{ complemented } c \in L_- \mid c \sqsubseteq b\}$$

for all $a \in L_+$ and $b \in L_-$.

Finally, we say that a d-frame is *Stone* if it is zero-dimensional and compact. Similarly, a bitopological space X is *compact*, *d-regular*, *zero-dimensional* or *Stone* if $\Omega(X)$ is. We denote the category of Stone d-frames and bitopological Stone spaces by **d-Stone** and **biStone** respectively.

Again, similarly to the monotopological case:

4.2.7 Theorem ([JM06]). *The functors Ω and Σ restricted to the subcategories of compact (T_0) d-regular bitopological spaces and compact d-regular d-frames form a dual equivalence of the categories.*

Moreover, the same is true when restricted to the subcategories **biStone** *and* **d-Stone***.*

4.2.8 Let $\mathrm{con} \cap \mathrm{tot} \colon$ **biStone** $\to$ **DLat** be the functor sending a bispace to the distributive lattice of its complemented elements:

$$\mathrm{con} \cap \mathrm{tot}\colon \quad (X; \tau_+, \tau_-) \quad \longmapsto \quad (\{U \in \tau_+ \mid X \setminus U \in \tau_-\}; \cap, \cup, X, \emptyset),$$

and let $\mathrm{spec}\colon$ **DLat** $\to$ **biStone** be the functor mapping distributive lattices to their spectra:

$$\mathrm{spec}\colon \quad (D; \sqcap, \sqcup, 1, 0) \quad \longmapsto \quad (\mathtt{PFilt}(D); \tau_+^D, \tau_-^D)$$

where $\mathtt{PFilt}(D)$ is the set of all prime filters of D, and τ_+^D and τ_-^D are topologies generated by the elements of the form

$$\Phi_+^D(a) = \{F \mid a \in F\} \quad \text{and} \quad \Phi_-^D(a) = \{F \mid a \notin F\},$$

for $a \in D$, respectively. Both functors are defined on morphisms simply as preimages. This gives a bitopological analogue to Priestly duality:

Theorem ([JM06]). *The functors* $\mathrm{con} \cap \mathrm{tot}$ *and* spec *form a dual equivalence of categories.*

Chapter 5

Research projects

5.1 4-valued coalgebraic modal logic

To get a logic well-suited for reasoning about behaviours of general state based systems, it seems reasonable to try to combine all the ideas mentioned in the previous chapters. Since there are more ways to introduce four-valued logics (we described two of them), there are also more possible ways to extend them to obtain four-valued modal logics.

5.1.1 Continuing with the algebraic approach

A possible approach is to continue where Arieli and Avron started. Although, Achim Jung and Umberto Rivieccio were not the first to try to extend Arieli-Avron logic with modal operation, their four-valued modal logic is so far the most elegant (see [JR13]).

Unfortunately, the logic of Jung and Rivieccio is not presented via an algebra-coalgebra duality as the logics fitting into Alexander Kurz's framework (from Section 3.3) and, therefore, it is not clear how to extend this logic to all coalgebras. However, it seems that something can be done. A starting point could be the following version of Jónsson-Tarski duality (taken from Jung and Rivieccio [JR13]):

5.1.1 Theorem. *The category of bimodal spaces and bimodal functions is dually equivalent to the category of bimodal algebras and algebra homomorphisms.*

(Where bimodal algebras are models of Jung-Rivieccio's four-valued modal logic) And this can be further translated to

5.1.2 Theorem. *The category of Stone* $(\mathbb{V} \times \mathbb{V})$*-coalgebras is dually equivalent to the category of Boolean* $\mathbb{M}_{\Box_+,\Box_-}$*-algebras. Where* $\mathbb{M}_{\Box_+,\Box_-}$ *is an endofunctor over Boolean algebras defined as*

$$\mathbb{M}_{\Box_+,\Box_-} : B \longmapsto \mathbb{BA}\Big\langle\, \Box_+ a,\ \Box_- a : a \in B \mid \Box_+ a \sqcap \Box_+ b = \Box_+(a \sqcap b),\ \Box_+ 1 = 1, \\ \Box_- a \sqcap \Box_- b = \Box_-(a \sqcap b),\ \Box_- 1 = 1 \Big\rangle.$$

One can also define an interpretation $\delta_{\Box_+,\Box_-} : M_{\Box_+,\Box_-} \circ \texttt{Clp} \Longrightarrow \texttt{Clp} \circ (\mathbb{V} \times \mathbb{V})$ of $M_{\Box_+,\Box_-}$ in $(\mathbb{V} \times \mathbb{V})$ similarly to how we did it before for Hennessy-Milner logic. But now, the inherited truth relation from $\delta_{\Box_+,\Box_-}$ (as we defined it in 3.3.2) is just two-valued:

$$(X,\xi), x \models \varphi \ \stackrel{\text{def}}{\equiv}\ x \in [\![\varphi]\!]_\xi,$$

where (X,ξ) is a coalgebra and $[\![\varphi]\!]_\xi$ is a subset of X.

Question: *Can we generalise Kurz's framework such that it admits a four-valued truth relation and the truth relation of the four-valued modal logic of Jung and Rivieccio becomes just a special case of it?*

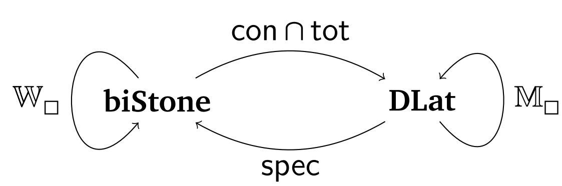
We expect to be able to do that after we inspect carefully how the truth relation of the Jung and Rivieccio's logic and the truth relation arising from $\delta_{\Box_+,\Box_-}$ differ.

5.1.2 Continuing with the topological approach

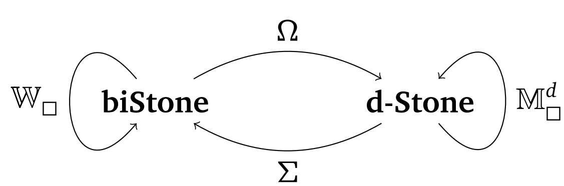
An another option is to continue topologically. Taking an inspiration from the Jónsson-Tarski algebra-coalgebra duality from 3.5, we can take the category of bitopological Stone spaces **biStone** and dually equivalent to it the category of bounded distributive lattices **DLat** (for details see 4.2.6 and 4.2.8), define a functor $\mathbb{W}_\Box$: **biStone** → **biStone** and a functor $\mathbb{M}_\Box$: **DLat** → **DLat** by

$$\mathbb{M}_\Box : B \longmapsto \mathbb{DL}\Big\langle\ \Box a : a \in B \mid \Box a \sqcap \Box b = \Box(a \sqcap b),\ \Box 1 = 1 \Big\rangle$$

and obtain the following schema



Where, in the picture, con ∩ tot and spec are the functors witnessing the dual equivalence of **biStone** and **DLat** as in 4.2.8. Also, there is a natural isomorphism $\delta_\Box : \mathbb{M}_\Box \circ \mathsf{con} \cap \mathsf{tot} \Longrightarrow \mathsf{con} \cap \mathsf{tot} \circ \mathbb{W}_\Box$. Similarly, one can define a functor $\mathbb{M}^d_\Box$: **d-Stone** → **d-Stone** and get



Where Ω and Σ are the functors providing the duality of **biStone** and **d-Stone** as in 4.2.7. And again, there is a natural isomorphism $\delta^d_\Box \colon \mathbb{M}^d_\Box \circ \Omega \Longrightarrow \Omega \circ \mathbb{W}_\Box$. (Details of $\mathbb{W}_\Box$ and $\mathbb{M}^d_\Box$ are given in the Appendix.)

In the first case, the logic $\mathbb{M}_\Box$-algebras is positive logic enriched with box modality. Unfortunately, since positive logic does not have negation, this logic is very simple. When factorised by logical equivalence, it consist of the following formulas only (ordered by their logical strength):

$$\mathsf{f} \;<\; \Box\mathsf{f} < \;\Box\Box\mathsf{f}\; < \;\Box\Box\Box\mathsf{f}\; < \;\Box\Box\Box\Box\mathsf{f}\; < \quad \dots \quad < \;\mathsf{t}.$$

Because of that, the four-valued positive geometric logic of **d-Stone** enriched with box modality is also quite weak (it can be interpreted in $\mathbb{W}_\Box$). What we would really like to have is for the functor $\mathbb{M}_{\Box,\Diamond} \colon \mathbf{DLat} \to \mathbf{DLat}$ defined as

$$\begin{aligned}\mathbb{M}_{\Box,\Diamond} \colon B \longmapsto \mathbb{DL}\Big\langle\; \Box a,\ \Diamond a : a \in B \mid \;& \Box a \sqcap \Box b = \Box(a \sqcap b),\ \Box 1 = 1,\\ & \Diamond a \sqcup \Diamond b = \Diamond(a \sqcup b),\ \Diamond 0 = 0,\\ & \Box a \sqcap \Diamond b \sqsubseteq \Diamond(a \sqcap b),\ \Box(a \sqcup b) \sqsubseteq \Box a \sqcup \Diamond b \Big\rangle\end{aligned}$$

to have a functor $\mathbb{W}_{\Box,\Diamond} \colon \mathbf{biStone} \to \mathbf{biStone}$ such that $\mathbb{M}_{\Box,\Diamond}$ can be interpreted in $\mathbb{W}_{\Box,\Diamond}$ by a natural isomorphism $\delta_{\Box,\Diamond} \colon \mathbb{M}_{\Box,\Diamond} \circ \mathsf{con} \cap \mathsf{tot} \Longrightarrow \mathsf{con} \cap \mathsf{tot} \circ \mathbb{W}_{\Box,\Diamond}$. To get a reasonably powerful 4-valued positive modal logic, we would also like to have a functor $\mathbb{M}^d_{\Box,\Diamond} \colon \mathbf{d\text{-}Stone} \to \mathbf{d\text{-}Stone}$ which can be interpreted in $\mathbb{W}_{\Box,\Diamond}$ by a natural isomorphism $\delta^d_{\Box,\Diamond} \colon \mathbb{M}^d_{\Box,\Diamond} \circ \Omega \Longrightarrow \Omega \circ \mathbb{W}_{\Box,\Diamond}$.

Question: *Is it possible to define a functor $\mathbb{W}_{\Box,\Diamond}$ such that the functor $\mathbb{M}_{\Box,\Diamond}$ can be interpreted in it? Also, is it possible to define a functor $\mathbb{M}^d_{\Box,\Diamond}$ that would be then interpreted in $\mathbb{W}_{\Box,\Diamond}$?*

5.1.3 Algebraic versus topological approach

We have seen two approaches to four-valued logics. Although they both are well founded, it is not immediately clear how are those concepts related. Just by comparing the structures we see that the core logic of both implicative bilattices and d-frames logics is the same. It contains the constants $\mathsf{t}, \mathsf{f}, \top, \bot$, the logical operations $\wedge, \vee$ and the knowledge operations $\sqcup$ and $\sqcap$ resp. $\oplus$ and $\otimes$. Also, all the operations distribute over each other and one can define the logical operations from the knowledge operations and vice versa.

At first sight, it seems that this is where the similarities end. Implicative bilattices add two more logical operations: negation and implication, whereas d-frames add consistency and totality relations and directed information joins. However, there is also a good reason to believe that they have more in common. Any d-frame $\mathscr{L} = (L_+, L_-; \mathsf{con}, \mathsf{tot})$ admits a partial two-valued implication $\prec$. Take, $\alpha, \beta \in \mathsf{con}$, then

$$\alpha \prec \beta \quad \stackrel{\text{def}}{\equiv} \quad (\beta_+, \alpha_-) \in \mathsf{tot}.$$

Also, whenever there is an isomorphisms $i \colon L_- \to L_+$ such that

$$(x, y) \in \mathsf{con} \iff x \sqcap i(y) = 0$$
$$(x, y) \in \mathsf{tot} \iff x \sqcup i(y) = 1,$$

then $\mathscr{L}$ also admits a reasonably behaving negation [JM06]:

$$\neg(a_+, a_-) = (i(a_-), i^{-1}(a_+)).$$

On the other hand, one can always define con and tot relations for implicative bilattices. Remember, every implicative bilattice can be viewed as a twist-structures $B^{\bowtie}$ for some Boolean algebra B. This representation allows us to define $\mathsf{con} \subseteq B \times B$ and $\mathsf{tot} \subseteq B \times B$ satisfying all the d-lattices conditions for consistency and totality relations:

$$(a, b) \in \mathsf{con} \ \stackrel{\text{def}}{\equiv} \ a \sqcap b = 0_B \qquad \text{and} \qquad (a, b) \in \mathsf{tot} \ \stackrel{\text{def}}{\equiv} \ a \sqcup b = 1_B.$$

In summary, it seems that the greatest difference is that implicative bilattices have (four-valued) implication that is defined on the whole structure and d-frames have directed joins.

Question: *Is there an equivalence between a nontrivial subcategory of implicative bilattices and a nontrivial subcategory of d-frames?*

To solve this, we can try to extend the two (two-valued) implication $\prec$ to the whole d-frame. Where, by extending we mean that we redefine $\prec$ such that it gives t on con whenever the original $\prec$ gave t. If a such extending of $\prec$ could be found, we can then compare its properties with the properties of the implications $\supset$ and $\to$ in implicative bilattices.

5.2 Implication and Esakia duality

One of the reasons why the relation between implicative bilattices and d-frames is not so clear is because the role of implication is not so well understood yet. It is not even much clearer how the Heyting algebras implication fits into the Stone duality for distributive lattices (as described in 4.2.8). Concretely, we know that

1. the category of distributive lattices is dually equivalent to the category of bitopological Stone spaces;

2. the category of Boolean algebras is dually equivalent to the category of Stone spaces, witch is really just the category of bitopological Stone spaces such that their plus and minus topologies are equal; and that

3. the category of Heyting algebras is dually equivalent to the category of *Esakia spaces*, that is the bitopological Stone spaces such that a closure in the plus topology of a pairwise clopen subset is still pairwise clopen, where a subset is pairwise clopen if it is compact (with respect to both topologies) and its complement is also compact [Bez+10].

In 4.2.8 we defined a functor spec: **DLat** → **biStone** mapping a distributive lattice D to a bitopological space with plus and minus topologies generated by

$$\Phi_+(a) = \{ F \in \mathtt{PFilt}(D) \mid a \in F \} \quad \text{and} \quad \Phi_-(a) = \{ F \in \mathtt{PFilt}(D) \mid a \notin F \}.$$

It is clear that $\wedge$ in the distributive lattice is interpreted as $\cap$ in the corresponding space: $\Phi_+(a \wedge b) = \Phi_+(a) \cap \Phi_+(b)$. Moreover, if the distributive lattice D is a Boolean algebra, then negation is interpreted as a complement: $\Phi_-(\neg a) = (\mathtt{PFilt}(D) \setminus \Phi_+(a)) \in \tau_-$.

Question: *If D is a Heyting algebra, is it possible to find a similar description for implication in Esakia spaces?*

5.3 Modal μ-calculus

Ordinary modal logic allows us to express only properties about a finite number of steps of a computation. For example, we can not write a formula that says that "the computation never stops". To be able to do that one has to add fixedpoint operators to modal logics. Take Hennessy-Milner logic with the least fixpoint μ operator. Then, the formula $\mu x.\ \Diamond \mathrm{t} \wedge \Box x$ expresses exactly what we wanted; see how it unfolds:

$$\begin{aligned}
&\Diamond \mathrm{t} \wedge \Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x),\\
&\Diamond \mathrm{t} \wedge \Box(\Diamond \mathrm{t} \wedge \Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x)),\\
&\Diamond \mathrm{t} \wedge \Box(\Diamond \mathrm{t} \wedge \Box(\Diamond \mathrm{t} \wedge \Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x))),\\
&\Diamond \mathrm{t} \wedge \Box(\Diamond \mathrm{t} \wedge \Box(\Diamond \mathrm{t} \wedge \Box(\Diamond \mathrm{t} \wedge \Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x)))),\\
&\quad \vdots
\end{aligned}$$

which is the same as

$$\begin{aligned}
&\Diamond \mathrm{t} \wedge \Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x),\\
&\Diamond \mathrm{t} \wedge \Box\Diamond \mathrm{t} \wedge \Box\Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x),\\
&\Diamond \mathrm{t} \wedge \Box\Diamond \mathrm{t} \wedge \Box\Box\Diamond \mathrm{t} \wedge \Box\Box\Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x),\\
&\Diamond \mathrm{t} \wedge \Box\Diamond \mathrm{t} \wedge \Box\Box\Diamond \mathrm{t} \wedge \Box\Box\Box\Diamond \mathrm{t} \wedge \Box\Box\Box\Box(\mu x.\ \Diamond \mathrm{t} \wedge \Box x),\\
&\quad \vdots
\end{aligned}$$

Unfortunately, semantics of modal μ-calculi is usually only given by fixpoint operators over certain posets or by game semantics (see [Ven06b] and [KP11]). Also, we only have similarly unsatisfactory results for logics of general coalgebras. In fact, an extensive study of fixpoint operators has been provided only for Moss' modal logics and not for modal logics arising from algebra-coalgebra dualities or from predicate liftings.

Since the carriers of our coalgebras can also be Stone spaces (like in Section 3.5 or in [KKV04]), one can try to take advantage of that. The idea is that an evaluation of a fixpoint formula could correspond to a sequence of points in a space and such a sequence has to have a limit point because Stone spaces are compact. Similar ideas have been used many times since Scott's discovery of domains [Sco70].

Question: *Is it possible to add the least fixpoint operator to Hennessy-Milner logic such that an evaluation of a fixpoint formula would correspond to a sequence of points in a Stone space?*

A first step to answer this question could be to start with a simpler logic. We can try to add the least fixpoint operator to positive logic with just box modality. As we discussed in 5.1.2, the topological counterparts to $\mathbb{M}_{\Box}$-algebras are $\mathbb{V}_{\Box}$-coalgebras over bitopological Stone spaces. The question is then reformulated to: *Does an evaluation of a formula of a positive logic with box modality and the least fixpoint correspond to a sequence of points in a bitopological Stone space?*

An overview

State based system are one of the main subjects of studies of computer science. However complicated their hardware representation can get in the future, their observational properties will always stay the same – they will be able to read from input, write on output, change their inner state and so on. In order to be able to understand their behaviour better, we need more expressive languages than we currently have. We hope to by able to do that by extending coalgebraic logics to be four-valued and/or by extending them with fixpoint operations. In either way, answering any of the above mentioned questions gets us closer to to our goal. Also, anytime we get more powerful logic, a proper mathematical analysis of its properties is needed. For example, one has to examine which new properties of systems can we capture that we could not capture before.

Appendix A

Appendix – Upper Vietoris construction

A.1 Bitopological upper Vietoris functor

A.1.1 Definition. Let $(X;\tau_+,\tau_-)$ be a bitopological space. The relation $\leq_+ \subseteq X \times X$ is defined as follows, for all $x, y \in X$,

$$x \leq_+ y \quad \stackrel{\text{def}}{\equiv} \quad \forall U_+ \in \tau_+.\ x \in U_+ \implies y \in U_+.$$

Similarly we can define the relation $\leq_-$. We say that X is *weakly balanced* if $\leq_+ \subseteq \geq_-$ holds, that is

$$(\forall x, y \in X) \quad x \leq_+ y \implies x \geq_- y. \tag{wb}$$

We can define a bitopological version of upper Vietoris functor:

A.1.2 Theorem. *Let $(X;\tau_+,\tau_-)$ be a bitopological space. Let $\mathbb{W}_\Box X = (\mathscr{K}X; \boxtimes\tau_+, \diamondsuit\tau_-)$ be the Vietoris powerbispace of X, where the set of points $\mathscr{K}X$ consists of τ_+-compact saturated subsets of X, the topology $\boxtimes\tau_+$ is generated by the sets of the form $\boxtimes U_+$ for all $U_+ \in \tau_+$ and the topology $\diamondsuit\tau_-$ is generated by the sets of the form $\diamondsuit V_-$ for all $V_- \in \tau_-$. Where,*

$$K \in \boxtimes U_+ \ \stackrel{\text{def}}{\equiv}\ K \subseteq U_+, \quad \text{and} \quad K \in \diamondsuit V_- \ \stackrel{\text{def}}{\equiv}\ K \cap V_- \neq \emptyset.$$

Then the following holds:

1. *If X is d-regular, $\mathbb{W}_\Box X$ is also d-regular.*
2. *If X is compact and weakly balanced then $\mathbb{W}_\Box X$ is also compact.*
3. *If X is zero-dimensional, $\mathbb{W}_\Box X$ is also zero-dimensional.*
4. *$\mathbb{W}_\Box X$ is always weakly balanced.*
5. *$\boxtimes\tau_+$ is always T_0.*

We can see that our construction is functorial and has the standard embedding property:

A.1.3 Proposition. *Let* $\mathbf{biTop}_{\leq}$ *be the category of all weakly balanced bitopological spaces and bicontinuous maps. Then the mapping* $\mathbb{W}_{\Box}\colon \mathbf{biTop}_{\leq} \to \mathbf{biTop}_{\leq}$ *sending each bispace to its powerbispace (as described above) and each bicontinous map* $f : X \to Y$ *to the map* $\mathbb{W}_{\Box}(f)\colon K \mapsto {\uparrow_+} f[K] = \{y \in Y \mid \exists x \in f[K] \text{ s.t. } x \leq_+ y\}$ *forms a functor.*

A.1.4 Proposition. *Let* $(X; \tau_+, \tau_-)$ *be a weakly balanced bitopological space such that* τ_+ *is* T_0*. Then the mapping* $e\colon x \mapsto {\uparrow_+}\{x\}$ *is an embedding of* X *into* $\mathbb{W}_{\Box} X$*.*

A.2 D-frame upper Vietoris functor

Direct translation of the Vietoris functor for frames separation suggests the following definition. For a d-frame $L = (L_+, L_-; \mathsf{con}, \mathsf{tot})$ define $\mathbb{M}^d_{\Box} L = (\mathbb{V}_+ L_+, \mathbb{V}_- L_-; \mathsf{con}_*, \mathsf{tot}_*)$. Where

$$\mathbb{V}_+ L_+ = \mathbb{F}\mathrm{r}\left\langle \Box a : a \in L_+ \mid \Box(a \sqcap b) = \Box a \sqcap \Box b,\ \bigsqcup\nolimits^{\uparrow} \Box a_i = \Box(\bigsqcup\nolimits^{\uparrow} a_i),\ \Box 1 = 1 \right\rangle$$
$$\mathbb{V}_- L_- = \mathbb{F}\mathrm{r}\left\langle \Diamond a : a \in L_- \mid \Diamond(a \sqcup b) = \Diamond a \sqcup \Diamond b,\ \bigsqcup\nolimits^{\uparrow} \Diamond a_i = \Diamond(\bigsqcup\nolimits^{\uparrow} a_i),\ \Diamond 0 = 0 \right\rangle$$

and

$$\mathsf{tot}_* = \mathtt{TOT}\left\langle \mathsf{tot}_1 \right\rangle \stackrel{\text{def}}{\equiv} {\uparrow}\mathbb{DL}_{\vee,\wedge}\left\langle \mathsf{tot}_1 \right\rangle \qquad \text{for } \mathsf{tot}_1 = \{tt, ff\} \cup \{(\Box a, \Diamond b) : (a, b) \in \mathsf{tot}\},$$
$$\mathsf{con}_* = \mathtt{CON}\left\langle \mathsf{con}_1 \right\rangle \stackrel{\text{def}}{\equiv} {\downarrow}\mathbb{DCPO}_{\bigsqcup^{\uparrow}}\left\langle \mathbb{DL}_{\vee,\wedge}\left\langle \mathsf{con}_1 \right\rangle \right\rangle \quad \text{for } \mathsf{con}_1 = \{tt, ff\} \cup \{(\Box a, \Diamond b) : (a, b) \in \mathsf{con}\}.$$

(Note that $\mathbb{DL}_{\vee,\wedge}$ resp. $\mathbb{DCPO}_{\bigsqcup^{\uparrow}}$ just means that we close the relation under $\vee$ and $\wedge$ resp. directed suprema.)

We follow Johnstone's steps, by showing that the construction preserves d-regularity, compactness and zero-dimensionality and then we can show that it is interpreted in the bitopological construction.

A.2.1 Theorem. *Let* L *be a d-frame. Then we have that,*

1. *if* L *is d-regular then so* $\mathbb{M}^d_{\Box} L$ *is;*
2. *if* L *is zero-dimensional then so* $\mathbb{M}^d_{\Box} L$ *is;*
3. *if* L *is compact and d-regular then* $\mathbb{M}^d_{\Box} L$ *is also compact.*

Note. *The theorem is imbalanced. We assume that* L *satisfies (*con–tot*) but we do not prove that the same holds for* $\mathbb{M}^d_{\Box} L$*.*

A.2.2 Proposition. L *can be embedded in* $\mathbb{M}^d_{\Box} L$*.*

A.2.3 Lemma (about points). *Let $L = (L_+, L_-; \mathsf{con}, \mathsf{tot})$ be a d-compact d-regular d-frame. Then, points of $\mathbb{M}^d_\Box L$ correspond precisely to elements of L_-.*

A.2.4 Theorem. $\mathbb{M}^d_\Box \circ \Omega \cong \Omega \circ \mathbb{W}_\Box$: **biKReg** $\to$ **dKReg**. *Where* **biKReg** *and* **dKReg** *are the categories of compact d-regular bispaces and d-frames respectively.*

A.2.5 Corollary. *For any d-compact and d-regular d-frame L, $\mathbb{M}^d_\Box L$ satisfies (*con–tot*).*

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