Tightness relative to some (co)reflections in topology

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Abstract.

We address what might be termed the reverse reflection problem: given a monoreflection from a category $\mathbf{A}$ onto a subcategory $\mathbf{B}$, when is a given object $B \in \mathbf{B}$ the reflection of a proper subobject? We start with a well known specific instance of this problem, namely the fact that a compact metric space is never the Čech-Stone compactification of a proper subspace. We show that this holds also in the pointfree setting, i.e., that a compact metrizable locale is never the Čech-Stone compactification of a proper sublocale. This is a stronger result than the classical one, but not because of an increase in scope; after all, assuming weak choice principles, every compact regular locale is the topology of a compact Hausdorff space. The increased strength derives from the conclusion, for in general a space has many more sublocales than subspaces. We then extend the analysis from metric locales to the broader class of perfectly normal locales, i.e., those whose frame of open sets consists entirely of cozero elements. We include a second proof of these results which is purely algebraic in character.

At the opposite extreme, we show that an extremally disconnected locale is a compactification of each of its dense sublocales. Finally, we analyze the same phenomena, also in the pointfree setting, for the 0-dimensional compact reflection and for the Lindelöf reflection.

Introduction

It is well known that the Čech-Stone compactification of a non-compact metrizable space is never metrizable. Since a subspace of a metrizable space is metrizable, this can be interpreted as saying that a metrizable space cannot be the Čech-Stone compactification of a proper subspace. This holds also in the pointfree context, i.e., a metrizable frame cannot be the Čech-Stone compactification of a proper sublocale. In this paper we discuss this and related phenomena concerning the relation between objects that are (isomorphic to) compact and similar reflections and their subobjects.

In the context of a given reflection $\rho_X : X \to R(X)$ of a category $\mathbf{A}$ onto a subcategory $\mathbf{B}$, we call a subobject X of $Y \in \mathbf{B}$ *tight* if each morphism $X \to B \in \mathbf{B}$ can be uniquely extended to a morphism $Y \to B$, which is to say that Y is isomorphic to $R(X)$. (For a more precise formulation see Section 2.) The concrete cases we are interested in are

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- the Čech-Stone compactification in completely regular spaces and frames,
- the compact 0-dimensional reflections in 0-dimensional spaces and frames,
- the Lindelöf (co)reflection in both cases.

After introductory Sections 1 and 2, we discuss in Section 3 the Čech-Stone compactification and its pointfree variant. First we prove that a compact perfectly normal frame has no non-trivial tight sublocale. This is a substantial generalization of the fact mentioned above, both in scope (the class of perfectly normal spaces, also known as σ-regular spaces, is larger than that of metrizable spaces) and in strength (there are more sublocales than subspaces). At the other extreme, every dense sublocale is tightly embedded in an extremally disconnected frame, and this feature characterizes such frames.

The facts concerning metrizable frames can be readily reformulated as the statement that any frame L with countably based frame $\mathfrak{J}_{\mathrm{cr}}L$ of completely regular ideals is compact. Since the proof in Section 3 is topological in nature, one may naturally ask how to prove this algebraic fact by purely algebraic means. We offer such a proof in Section 4.

In Section 5 we focus our attention on the 0-dimensional case. It turns out that, once again in analogy with perfectly normal, there are no non-trivial tight sublocales. This is particularly noteworthy in light of the fact that extremal disconnectivity is a strong form of 0-dimensionality, and yet the former have all their dense sublocales tight and the latter have none of them tight.

Finally, Section 6 is devoted to the Lindelöfication. We learn that for any Lindelöf cozero frame a tight sublocale is trivial; here the generalization to perfectly normal frames is particularly meaningful, since in the metrizable case there is nothing to be proved. The counterpart of the extremally disconnected case is now slightly changed: it turns out that all dense sublocales are tight precisely in the extremally disconnected P-frames, a class of interest in its own right.

1 Preliminaries

In a poset we will use the standard notation of $\vee$ and $\bigvee$ for suprema (joins) and $\wedge$ and $\bigwedge$ for infima (meets). The smallest (largest) element will be denoted by 0 or $\bot$ (1 or $\top$). We write ${\downarrow}A$ (resp. ${\uparrow}A$) for $\{x : \exists\, a \in A\ (x \le a)\}$ ($\{x : \exists\, a \in A\ (x \ge a)\}$), and speak of a *down-set (*up-set) if $A = {\downarrow}A$ ($A = {\uparrow}A$). We abbreviate ${\downarrow}\{a\}$ to ${\downarrow}a$ and ${\uparrow}\{a\}$ to ${\uparrow}a$. The pseudocomplement of a, the largest element b such that $a \wedge b = 0$, will be denoted by a^*, provided, of course, that it exists.

1.1. Frames A *frame* L is a complete lattice satisfying the law

$$a \wedge \bigvee B = \bigvee_{b\in B} (a \wedge b) \qquad \text{(frm)}$$

for all $a \in L$ and $B \subseteq L$. A complete lattice satisfying the dual distributive law, with the suprema and infima interchanged, is referred to as a *coframe*.

By (frm), the mapping $(x \mapsto x \wedge b) : L \to L$ preserves suprema, and hence has a right Galois adjoint $b \to -$. This introduces the Heyting algebra structure, characterized by the equivalence

$$a \wedge b \le c \quad \text{iff} \quad a \le b \to c,$$

to be used without further comment.

In a σ-*frame* one assumes only countable joins and the law (frm) is postulated for countable B.

A *frame homomorphism* (resp. σ-*frame homomorphism*) $h : L \to M$ preserves all (resp. countable) joins, including the empty join $\bigvee \emptyset = \bot$, and all finite meets, including the empty meet $\bigwedge \emptyset = \top$. The category of frames and frame homomorphisms (resp. σ-frames and σ-frame homomorphisms) will be denoted by

$$\mathbf{Frm} \quad \text{resp.} \quad \sigma\mathbf{Frm}.$$

A typical frame is the lattice $\Omega(X)$ of open sets of a topological space X. If $f : X \to Y$ is a continuous map then we have a frame homomorphism $\Omega(f) = (U \mapsto f^{-1}[U]) : \Omega(Y) \to \Omega(X)$. This results in a contravariant functor

$$\Omega : \mathbf{Top} \to \mathbf{Frm}.$$

Not every frame (locale) is isomorphic to $\Omega(X)$ for some space X. Those that are are termed *spatial*. A frame L is spatial iff for all $a \in L$,

$$a = \bigwedge_{a \leq p \text{ prime}} p,$$

where an element p is *prime* if $p < 1$ and $x \wedge y = p$ implies $x = p$ or $y = p$.

1.2. Regularity and complete regularity. We say that a is *rather below* b, and write

$$a \prec b$$

if $a^* \vee b = 1$. We write

$$a \prec\!\!\prec b$$

for a *completely below* b, which means that there is a family $\{a_r\}$, indexed by the diadic rational numbers r, $0 \leq r \leq 1$, such that $a = a_0$, $b = a_1$ and $a_r \prec a_s$ for $r < s$. Thus $\prec\!\!\prec$ is the largest interpolative subrelation of $\prec$.

A frame L is *regular* (*completely regular*) if

$$\forall a \in L, \quad a = \bigvee_{x \prec a} x \quad (\text{resp.} \quad a = \bigvee_{x \prec\!\!\prec a} x).$$

A frame is *normal* if $a \vee b = 1$ implies the existence of u, v such that $u \vee b = a \vee v = 1$ and $u \wedge v = 0$. The $\prec$ and $\prec\!\!\prec$ relations coincide in a normal frame, so that a regular normal frame is completely regular.

Note that a space X is regular, completely regular, or normal in the standard sense iff $\Omega(X)$ is a regular, completely regular, or normal frame, respectively.

In the sequel, all spaces and frames will be assumed to be completely regular unless explicitly stipulated otherwise. We do not assume spatiality.

1.2.1. The σ-case. A σ-frame is regular if each $a \in L$ is a countable join $a = \bigvee_{n=1}^{\infty} a_n$ with $a_n \prec a$. Let us refer to this property, whether in spaces or frames, as σ-*regularity*. It is a standard fact (see, e.g., [6]) that each σ-regular frame is normal, and hence also completely regular.

1.3. Compactness and compactification. A subset $A \subseteq L$ is a *cover* if $\bigvee A = \top$, and L is *compact* if each cover of L contains a finite subcover. This definition obviously coincides with the standard definition of compactness in spaces.

For us, the following fact is important.

Each compact regular frame is spatial and completely regular.

The standard Čech-Stone compactification of a topological space X will be denoted by βX; the reader may think of any construction he likes. It has been extended in [7] to frames L as follows. A *completely regular ideal*, or *cr-ideal*, in L is a non-void down-set J such that $a \vee b \in J$ whenever $a, b \in J$, and such that for each $a \in J$ there is some $b \in J$ with $a \prec\!\!\prec b$. We have the cr-ideals

$$\sigma(a) = \{x : x \prec\!\!\prec a\}, \quad a \in L.$$

Set

$$\mathfrak{J}_{\mathrm{cr}} L = \{J : J \text{ a cr-ideal in } L\}.$$

Then $\mathfrak{J}_{\mathrm{cr}} L$ is a compact (completely) regular frame, we have a frame homomorphism $v_L : \mathfrak{J}_{\mathrm{cr}} L \to L$ given by the formula $v_L(J) = \bigvee J$, with right adjoint $\sigma_L = \sigma : L \to \mathfrak{J}_{\mathrm{cr}} L$ defined as above, and v_L is an isomorphism iff L is compact. Furthermore, $\mathfrak{J}_{\mathrm{cr}}$ can be made a functor from the category of completely regular frames into that of compact regular frames by setting $\mathfrak{J}_{\mathrm{cr}} h(J) = {\downarrow} h[J]$. Then $v = (v_L : \mathfrak{J}_{\mathrm{cr}} L \to L)_L$ is a natural transformation constituting a coreflection.

1.3.1. Other constructions. Besides the standard compactification we will consider further analogous constructions: zero dimensional compactification, Lindelöfication, etc. They will be introduced as required.

1.4. Sublocales. There are several equivalent representations of subobjects of frames. (See, e.g., [14, 15]). Perhaps the simplest is to define a *sublocale homomorphism*, briefly a *sublocale*, of L to be an onto frame homomorphism $L \to M$. Thus for instance we have a subspace Y of a space of a space X represented as the homomorphism $h_Y = (U \mapsto U \cap Y) : \Omega(X) \to \Omega(Y))$. It should be noted, however, that a space typically has many sublocales that are not induced by subspaces.

The sublocales $\widehat{a} = (x \mapsto a \wedge x) : L \to {\downarrow} a$, $a \in L$, are referred to as *open sublocales*, and those of the form $\tilde{a} = (x \mapsto a \vee x) : L \to {\uparrow} a$, $a \in L$, are called *closed sublocales.* In the spatial case they correspond to open and closed subspaces, respectively. For any sublocale $h : L \to M$ there is a smallest closed sublocale $\tilde{a}$ through which it factors. That means that h factors through any closed sublocale $\tilde{b}$ iff $\tilde{a}$ factors through $\tilde{b}$ iff $b \leq a$. $\tilde{a}$ is called the *closure of* h.

A sublocale $h : L \to M$ is *dense* if $h(a) = \bot$ implies $a = \bot$. Note that a subspace is dense in the standard sense iff the above mentioned $h_Y = (U \mapsto U \cap Y) : \Omega(X) \to \Omega(Y)$ is dense.

1.5. Metrizable frames. Metrizability for frames was defined first by Isbell in [12] as the existence of a countably generated uniformity; consequently, it precisely extends the classical notion of metrizability in spaces. It has since been characterized in more "metric" terms in [16, 17, 8]. As for the uniformity, we do not need it here. For our purposes it suffices to know (see, e.g., [15, 8]) that

1.5.1. *a compact regular frame is metrizable iff it has a countable join basis.*

1.6. Perfect normality and Vedenissoff's Theorem. Recall that in a space X a cozero set is a preimage $f^{-1}[\mathbb{R} \smallsetminus \{0\}]$, often denoted by $\mathsf{coz}(f)$, with f a continuous real function on X. This can be naturally extended to frames using the *pointfree reals* ([5, 13], see also [14]). However, to avoid introducing too many new concepts we will use a simple equivalent characterization instead. An element $a \in L$ is a *cozero* if it is a countable join $a = \bigvee_{n=1}^{\infty} a_n$, with $a_n \prec\!\!\prec a$ for all n. An equivalent formulation is $a = \bigvee_{n=1}^{\infty} a_n$ with $a_1 \prec\!\!\prec a_2 \prec\!\!\prec \cdots \prec\!\!\prec a_n \prec\!\!\prec \cdots$. The set of all cozero elements of L is obviously a σ-subframe of L, and will be denoted by

$$\mathsf{Coz}\, L.$$

A space is said to be *perfectly normal* if it is T_0, normal, and every open set is an F_σ. Such spaces are characterized in Vedenisoff's Theorem ([9, 1.5.19]).

1.6.1 Theorem. *The following are equivalent for a T_1-space X*

(1) *X is perfectly normal.*

(2) *Each open set of X is a cozero.*

(3) *Disjoint open sets A and be can be precisely separated by a continuous function $f : X \to [0,1]$, in the sense that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.*

As is the tradition in pointfree topology, the topological term is used for the corresponding frame attribute, i.e., a frame is said to be perfectly normal if every element is a cozero (see [11]). By 1.2.1 one easily sees that this is the same as being σ-regular. Since we will generalize a certain fact from metric spaces to perfectly normal locales, it is appropriate to recall that there exist non-metric perfectly normal spaces, even compact ones. One such is the *Alexandroff double arrow space*, also known as the *two-arrow space*, or *split interval*, presented by P. S. Alexandroff in the twenties. See, e.g., [9, 3.10C].

1.7. The Boolean part of a frame, 0-dimensionality. Set

$$B(L) = \{a \in L : a \text{ is complemented}\},$$

the sublattice of all complemented elements of L. We easily see that $B(L)$ is a Boolean algebra. A frame L is *0-dimensional* (or *totally disconnected*) if it is join-generated by $B(L)$.

2 Tight embeddings of subspaces and spaces

2.1. We will be concerned only with tightness in the setting of spaces and frames, but it will perhaps do no harm to define the concept somewhat more generally.

Let $\mathbf{B}$ be a monoreflective subcategory of $\mathbf{A}$, with reflector $\rho = (\rho_A : A \to F(A))_{A\in\mathrm{obj}\mathbf{A}}$. A morphism $f : A \to B$ in $\mathbf{A}$ is said to be *tight with respect to the reflection* ρ if there is an isomorphism $g : F(A) \to B$ such that $f = g \circ \rho_A$. In all the special cases we will discuss the reflection morphisms are dense embeddings. Thus tightness may be understood as a special type of density.

We will, however, mostly deal with special frames representing and generalizing in a contravariant way those particular spaces from which the original motivation came. Hence we will have an *epicoreflection* ρ and a *tight surjection* $f : B \to A$ of B onto A with an isomorphism $g : B \to F(A)$ such that $f = \rho_A \circ g$.

2.2. Tightness in the Čech-Stone and $\mathfrak{J}_{\mathbf{cr}}$-compactifications. In particular we will speak of a tight embedding of a subspace Y into a compact regular space X, or briefly, of a *tight subspace* $Y \subseteq X$, if each continuous map $f : Y \to Z$ with Z compact can be uniquely extended to a continuous map $g : X \to Z$. Likewise we shall speak of a *tight sublocale* $h : L \to S$ of a compact locale L if each frame homomorphism $M \to S$ with M compact factors through h.

2.3. Tightness versus C^*-embeddings and quotients. Actually, in order to test the tightness of the frame surjection $h : L \to S$, L compact, it is sufficient to show that each frame homomorphism $\Omega[0,1] \to S$ factors through h. When h has this property it is said to be a *C^*-quotient map*; the corresponding pointed notion, that of a *C^*-embedded subspace* $Y \subseteq X$, plays a major role in general topology. (See, for example, [10] and [1]). However, in contrast to tightness, the locale X into which Y is C^*-embedded need not be compact. We reiterate for emphasis.

In the context of the compact regular coreflection, a tight sublocale is a dense C^*-quotient out of a compact frame.

2.4. Observations. Compositions of onto homomorphisms are onto; we have the following straightforward facts.

(1) If $g : L \to T$ and $h : T \to S$ are dense then $h \circ g : L \to S$ is dense.

(2) If $g : L \to T$ and $h : T \to S$ are onto and if $h \circ g : L \to S$ is tight then $g : L \to T$ is tight.

(3) If $L \to T$ is a C^*-quotient and L is compact then, when the map is factored as $L \to \operatorname{cl} S \to S$, the map $\operatorname{cl} S \to S$ is tight.

(4) $g : L \to S$ is tight iff it the composition $g \circ v_L : \mathfrak{J}_{\mathrm{cr}} L \to S$ is tight in $\mathfrak{J}_{\mathrm{cr}} L$.

If we can do so without risk of confusion, we shall abbreviate the last statement by saying that *S is tight in $\mathfrak{J}_{cr}L$.*

3 Čech-Stone and $\mathfrak{J}_{\mathbf{cr}}$ compactifications

3.1. The present investigation was motivated by the well-known fact that the compactification βX of a metric noncompact space X is not metrizable. In other words,

a compact metric space contains no proper tight subspace.

We will present a substantial extension of this fact, namely that this (and more) holds for any compact perfectly normal frame

3.2. Theorem. *For every compact perfectly normal frame L, any tight $h : L \to M$ is an isomorphism.*

Proof. Suppose a tight $h : L \to M$ is not an isomorphism. Hence, it is not one-one and by the regularity of the frames involved, $h(a) = \top$ for some $a < \top$. Actually, we may assume that a is maximal since every element of a compact regular frame lies beneath a maximal element. Now h factors through $\widehat{a}$ by [1, 2.1.1], and we observed in 2.4(2) that $\widehat{a}$ is tight. Furthermore, a compact regular frame is spatial, that is, $L \cong \Omega(X)$ for some compact perfectly normal space X, and up to isomorphism, $\widehat{a} = (U \mapsto U \cap V)$, where $V \equiv \{x_0\}$ for some $x_0 \in X$. In other words, $\widehat{a}$ is the frame map of the Čech-Stone compactification of V.

Since X is perfectly normal there exists a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = \{x_0\}$. Since x_0 cannot be isolated, f is not bounded away from 0 and hence there exists a strictly increasing sequence $\{k_n\}$ of natural numbers for which all the subsets $f^{-1}(1/k_{n+1}, 1/k_n)$ are non-empty, and we can take $x_n \in V$ such that $1/k_{n+1} < f(x_n) < 1/k_n$.

Further note that the disjoint sets

$$A = \{x_1, x_3, x_5, \ldots\} \quad \text{and} \quad B = \{x_2, x_4, x_6, \ldots\}$$

are closed in V. First, since $y \in V$ implies $f(y) > 0$, $y \notin f^{-1}(0, 1/k_n)$ for some n, so that $y \in f^{-1}(1/k_n, \infty)$. On the other hand, if also $y \notin A$ then $y \in X \smallsetminus \{x_1, x_3, \ldots, x_{2m+1}\}$ for any m, and for large enough m the intersection of these two neighbourhoods of y is disjoint from A. Of course, the same type of argument applies to B. On the other hand, neither A nor B can be closed in X because, by compactness, that would make their f-images closed in $[0,1]$, which is not the case. Therefore $\overline{A} \cap \overline{B} = \{x_0\}$.

Now take the continuous function $w : (0,\infty) \to [0,1]$ defined by

$$w(t) = \begin{cases} 1 \ \text{ for } \ f(x_1) \le t, \\ \dfrac{t - f(x_{2n})}{f(x_{2n+1}) - f(x_{2n})} \ \text{ for } f(x_{2n+1}) \le t \le f(x_{2n}) \text{ and} \\ \dfrac{t - f(x_{2n+2})}{f(x_{2n+1}) - f(x_{2n+2})} \ \text{ for } f(x_{2n+2}) \le t \le f(x_{2n+1}) \end{cases},$$

and let $g : X \to \mathbb{R}$ be the continuous extension of $w \circ f$ provided by the fact that $X = \beta V$. Then g has value 1 at each point of A and 0 at each point of B, so that g can take no value at x_0 which would allow it to be continuous. □

3.2.1. Corollary. *For every compact metric frame L, that is, every countably based compact L, any tight $h : L \to M$ is an isomorphism.*

3.2.2. Remarks. 1. Note that Theorem 3.2, even specialized as in Corollary 3.2.1, is a strict generalization of the classical result mentioned in 3.1. That is because it applies to all sublocales, not just those induced by subspaces.

2. It should be noted that 3.2 can be obtained as a consequence of a classical result of Čech, namely that for any Tychonoff space T, if βT contains a closed G_δ subset disjoint from T then it contains a copy of $\beta\mathbb{N}$ (see [9], 3.6.G(a)). In the setting of the proof above, $T = V$ and the zero set $Z(f) = \{x_0\}$ is clearly disjoint from $V = f^{-1}[\mathbb{R} \smallsetminus \{0\}]$ so that $\{x_0\}$ contains a copy of $\beta\mathbb{N}$, a contradiction. Of course the present proof, which seems to be new, is considerably simpler as it requires no knowledge of $\beta\mathbb{N}$. For an alternative proof see [10], which also uses $\beta\mathbb{N}$.

3.3. Now we will present an important class of frames in which, by contrast with cozero spaces, every dense sublocale is tight.

3.3.1. Recall the *Booleanization* of a frame L:

$$\mathsf{b}_L = (a \mapsto a^{**}) : L \to \mathfrak{B}L = \{a : a = a^{**}\}.$$

It is a dense sublocale, and by Isbell's basic result, it is the smallest dense sublocale, which is to say that for every dense sublocale $h : L \to M$ there is a $g : M \to \mathfrak{B}L$ such that $\mathsf{b}_L = g \circ h$. Consequently by 2.4,

$$\textit{if } \mathsf{b}_L \textit{ is tight then each dense sublocale of } L \textit{ is tight.} \qquad (*)$$

Further recall that a frame L is *extremally disconnected* if for each $a \in L$, $a^* \vee a^{**} = \top$. This terminology is in agreement with the definition in spaces, that is, with the requirement that for each open U the closure $\overline{U}$ is open. Thus (recall 1.7)

$$L \textit{ is extremally disconnected iff } \mathfrak{B}L = B(L).$$

Theorem 3.3.2 is essentially Theorem 8.4.1 of [1], rephrased in terms of tightness. We offer a proof for the sake of a self-contained treatment.

3.3.2. Theorem. *The following are equivalent for a compact completely regular frame L.*

(1) *L is extremally disconnected.*

(2) *Every dense sublocale $h : L \to M$ is tight.*

(3) *Every dense open sublocale $\widehat{a} = (x \mapsto x \wedge a) : L \to {\downarrow}a$ is tight.*

(4) *Every open sublocale is tight in its closure.*

Proof. (1) $\Longrightarrow$ (2): By $(*)$ above it suffices to show that $\mathsf{b}_L : L \to \mathfrak{B}L$ is tight. Since L is extremally disconnected we have $B(L) = \mathfrak{B}L$, hence also $\mathfrak{J}_{\mathrm{cr}}B(L) = \mathfrak{J}_{\mathrm{cr}}\mathfrak{B}L$, and so we have this commutative diagram.

$$\begin{array}{ccc} \mathfrak{J}_{\mathrm{cr}}B(L) & = & \mathfrak{J}_{\mathrm{cr}}\mathfrak{B}L \\ \downarrow f & & \downarrow g \\ L & \xrightarrow{\mathsf{b}_L} & \mathfrak{B}L \end{array}$$

where f and g are given by the join in L and $\mathfrak{B}L$, respectively.

Define a mapping $\phi : L \to \mathfrak{J}_{\mathrm{cr}}B(L)$ by setting $\phi(a) = \{x : x \in B(L), x \le a\}$. We claim that $\phi \circ f$ is the identity map on $\mathfrak{J}_{\mathrm{cr}}B(L)$. To verify this claim consider $J \in \mathfrak{J}_{\mathrm{cr}}B(L)$; obviously $J \subseteq \phi(\bigvee J)$. On the other hand, if $B(L) \ni x \le \bigvee J$ then $x^* \vee \bigvee J = \top$, so that by compactness $x^* \vee y_1 \vee \cdots \vee y_n = \top$ for some $y_k \in J$. Since J is an ideal, we have $y = y_1 \vee \cdots \vee y_n \in J$, and $x \le y \in J$. This proves the claim, and the claim shows that f is an isomorphism. Thus, $\mathsf{b}_L = g \circ f^{-1}$ and if K is compact and $h : K \to \mathfrak{B}L$ is a homomorphism then we have, first, an $\overline{h} : K \to \mathfrak{J}_{\mathrm{cr}}\mathfrak{B}L$ such that $g \circ \overline{h} = h$, and then $\mathsf{b}_L f \circ \overline{h} = g \circ f^{-1} \circ f \circ \overline{h} = g \circ \overline{h} = h$.

(2) $\Longrightarrow$ (3) trivially.

(3) $\Longrightarrow$ (1): For an $a \in L$ consider the open sublocale

$$g = (x \mapsto x \wedge (a \vee a^*)) : L \to {\downarrow}(a \vee a^*),$$

and the frame map $h : \mathbf{2} \times \mathbf{2} \to {\downarrow}(a \vee a^*)$, where $\mathbf{2}$ is the two-element Boolean algebra $\{0 < 1\}$, are given by the rule $h(1,0) = a$ and $h(0,1) = a^*$. Since $\mathbf{2} \times \mathbf{2}$ is compact regular we have by tightness a frame map $k : \mathbf{2} \times \mathbf{2} \to L$ such that $g \circ k = h$. Now for $b = k(1,0)$ and $c = k(0,1)$ one has

$$(a \vee a^*) \wedge b = g(b) = gk(1,0) = h(1,0) = a$$

so that $a \leq b$. Similarly, $a^* \leq c$. Since $(1,0)$ and $(0,1)$ are complements of one another, we see that $c = b^*$ and hence $c = b^* \leq a^*$ and $c^* \leq a^{**}$, hence $c = a^*$ and we obtain $\top = c \vee c^* = a^* \vee a^{**}$.

Finally, (4) $\Longrightarrow$ (3) trivially, while in light 2.4(3), (1) $\Longrightarrow$ (4) is a consequence of the equivalence of part (6) with the rest of the conditions in Theorem 8.4.1 of [1]. □

4 An algebraic proof

4.1. Consider the following reformulations of Corollary 3.2.1.

4.1.1. *Any frame L with countably based $\mathfrak{J}_{cr}L$ is compact,*

4.1.2. *If the frame $\mathfrak{J}_{cr}L$ of completely regular ideals of L is countably based then each $J \in \mathfrak{J}_{cr}L$ is principal, i.e., of the form $\sigma(a)$, $a \in L$.*

Since both these formulations constitute purely algebraic statements, it is at least curious, perhaps even perplexing, that the proof we presented in Section 3 depends so completely on topological reasoning. To shed some light on this matter we give here an algebraic proof not based on the spatiality of the frames in question. In the following sections we take up parallel issues and analyze them algebraically, without the topological detour, but under somewhat different hypotheses.

A standard classical result holds that a subspace $Y \subseteq X$ is C^*-embedded iff for every $U, V \in \mathsf{Coz}\, Y$ such that $U \cup V = Y$ there exists a pair $U', V' \in \mathsf{Coz}\, X$ such that $U' \cap Y \subseteq U$, $V' \cap Y \subseteq V$, and $U' \cup V' = X$ ([10]). The pointfree version of this result appears as part of Theorem 7.1.1 in [1]; translated into the terms of this article, we have this.

4.2. Theorem. *A dense sublocale $m : L \to M$ is tight iff for every pair $a, b \in \mathsf{Coz}\, M$ such that $a \vee b = \top$ there exists a pair $c, d \in \mathsf{Coz}\, L$ such that $c \vee d = \top$ and $m(c) \leq a$ and $m(d) \leq b$.*

The proof of Theorem 4.4 below requires a simple lemma.

4.3. Lemma. *Let $\bot = c_0 \prec c_1 \prec c_2 \prec \ldots$ be a sequence in L with $c \equiv \bigvee c_n$. Let*

$$d_n \equiv c_{2n}^* \wedge c_{2n+3}, \quad a \equiv \bigvee_{n \geq 0} d_{2n}, \quad \text{and} \quad b \equiv \bigvee_{n \geq 0} d_{2n+1}.$$

Then the following hold.

(1) $d_n \wedge d_m = \bot$ *for all m and n such that $|m - n| > 1$.*

(2) $\bigvee_{0 \leq n \leq k} d_n = c_{2k+3}$.

(3) $a \vee b = c$

Proof: (1) If $|m - n| > 1$, say $m \geq n + 2$, then

$$d_n \wedge d_m = c_{2n}^* \wedge c_{2n+3} \wedge c_{2m}^* \wedge c_{2m+3} \leq c_{2n+3} \wedge c_{2m}^* \leq c_{2n+3} \wedge c_{2n+4}^* = \bot.$$

(2) yields to a straightforward induction on k. (3) We have

$$a \vee b = \bigvee d_{2n} \vee \bigvee d_{2n+1} = \bigvee d_n = \bigvee c_n = c. \quad \square$$

4.4. Theorem: *A compact perfectly normal frame has no proper tight dense sublocales.*

Proof: Consider a dense sublocale $m : L \to M$ of the compact perfectly normal frame L. Since a frame homomorphism takes cozero elements to cozero elements, M must also be a perfectly normal frame. Therefore Theorem 4.2 tells us that $m_*(x) \vee m_*(y) = \top$ for all $x, y \in M$ such that $x \vee y = \top$, where $m_* : M \to L = (y \mapsto \bigvee_{m(a) \leq y} a)$ is the right adjoint map of m.

Suppose for the sake of argument that m is not one-one. Then m is not codense, i.e., there exists some $\top > e \in L$ such that $m(e) = \top$. Because L is compact and regular, there is a maximal element c such that $e \leq c < \top$. Now m factors through the open quotient $L \to \downarrow c = (d \mapsto d \wedge c)$, hence q is also a dense tight sublocale by 2.4(3), and without loss of generality we may take m to be the open quotient map of c.

Express c in the form $c = \bigvee c_n$ for a sequence $\bot = c_0 \prec c_1 \prec c_2 \prec \ldots$. Note that there must be infinitely many indices n for which $c_n < c_{n+1}$, for otherwise c would be complemented, and no proper dense element of a frame is complemented. So assume that $c_n < c_{n+1}$ for all n, and let a, b, and d_n have the meaning given to them in Lemma 4.3.

Because $a \vee b = c$ it follows that $(c \to a) \vee (c \to b) = m_*(a) \vee m_*(b) = \top$. Therefore either $c \to a \nleq c$ or $c \to b \nleq c$, say $c \to b \nleq c$. Since

$$(c \to b) \vee a \geq b \vee a = c \text{ and } (c \to b) \vee a \nleq c,$$

it follows from the maximality of c that $\top = (c \to b) \vee a = (c \to b) \vee \bigvee d_{2n}$. Consequently, the compactness of L implies the existence of an index m for which $(c \to b) \vee \bigvee_{0 \leq n \leq m} d_{2n} = \top$. But if we meet both sides of this equation with c we get $b \vee \bigvee_{0 \leq n \leq m} d_{2n} = c$, and this cannot be correct. For if we put

$$x \equiv \bigvee_{0 \leq n \leq m} d_{2n} \leq c_{4m+3}, \quad y \equiv \bigvee_{0 \leq n \leq m} d_{2n+1} \leq c_{4m+5}, \quad z \equiv \bigvee_{m+1 \leq n} d_{2n+1} \leq c^*_{4m+6},$$

then since $b \vee \bigvee_{0 \leq n \leq m} d_{2n} = x \vee y \vee z$ we have the contradiction

$$\begin{aligned} c_{4m+6} &= c_{4m+6} \wedge c = c_{4m+6} \wedge (b \vee \bigvee_{0 \leq n \leq m} d_{2n}) \\ &= (c_{4m+6} \wedge x) \vee (c_{4m+6} \wedge y) \vee (c_{4m+6} \wedge z) \leq c_{4m+5}. \qquad \square \end{aligned}$$

5 The 0-dimensional variant

5.1. Now we will be concerned with tightness in the context of the 0-dimensional compact coreflection in the category

0dFrm

of 0-dimensional frames, that is, frames generated by their complemented elements. Since $x \prec\!\!\prec x$ for complemented elements x, each L in **0dFrm** is completely regular. Both the compact reflection in 0-dimensional spaces ([2], [3]) and the compact coreflection in **0dFrm** ([4]) are the work of the second author.

For our purposes it will be sufficient to outline a simple construction of the coreflection along the lines of the construction of the compact coreflection in frames.

Recall the Boolean part $B(L)$ of the frame L from 1.7. In Section 3 we discussed the extremally disconnected frames L in which $\mathfrak{B}L = B(L)$. Of course, these Boolean algebras do not coincide in general 0-dimensional frames. The compact zero-dimensional coreflection of **0dFrm** onto the subcategory of compact 0-dimensional frames is given by the coreflection maps

$$k_L : \mathfrak{J}B(L) \to L, \quad k_L(J) = \bigvee J$$

where $\mathfrak{J}M$ is the frame of all ideals in M (see, e.g., [4]) with the associated locale map sending a to $\alpha(a) = \{b \in B(L) : b \leq a\}$. (Of course, $\mathfrak{J}B(L)$ coincides with $\mathfrak{J}_{\rm cr}B(L)$ because $B(L)$ is Boolean.)

5.2. Set

$$SL = \left\{\bigvee M : M \subseteq B(L),\ M \text{ countable}\right\}.$$

Hence, SL is the sub-σ-frame of L generated by $B(L)$. Note that it is, in some sense, the 0-dimensional counterpart of $\mathsf{Coz}\, L$.

5.2.1. Proposition. *For a compact 0-dimensional frame L with $SL = L$, a tight $L \to M$ is an isomorphism.*

Proof. Since the coreflection involved is given by $\mathfrak{J}B(L)$, it suffices to show that any 0-dimensional frame for which every ideal in $B(L)$ is countably generated is compact.

Consider such an L. Suppose that there exists a proper ideal $J \subseteq B(L)$ such that $\bigvee J = \top$, and let a_n, $n = 0, 1, 2, \ldots$, be the generators provided by the hypothesis. Then $a_0 \vee \cdots \vee a_n < \top$ for all n, so that by the standard adjustment of the generators we may assume that $a_n > 0$ for all n, and that $a_k \wedge a_l = 0$ whenever $k \neq l$. Now this obviously determines an embedding $h : \mathfrak{P}\mathbb{N} \to L$ ($\mathfrak{P}$ designates the set of all subsets), $h(M) = \bigvee\{a_n : n \in M\}$, and we let P represent the image of h. Next, for any ideal I in P, the ideal $\widetilde{I} = \{b \in B(L) : b \leq a \text{ for some } a \in P\}$ on $B(L)$ is countably generated by hypothesis and this implies I is countably generated as well - a contradiction by the familiar fact that $\mathfrak{P}\mathbb{N}$ does not have the property in question. □

5.2.2. Corollary. *Any 0-dimensional L with countable $B(L)$ is compact.*

5.2.3. Note. Corollary 5.2.2 can be obtained as a consequence of Čech's original result applied to $\beta\mathbb{N}$, the maximal ideal space of $\mathfrak{P}\mathbb{N}$, but it really is a purely algebraic fact. Suppose a maximal ideal $\mathfrak{M}$ of $\mathfrak{P}\mathbb{N}$ is countably generated but not principal, so that we have $S_n \in \mathfrak{M}$ with $S_n \subseteq S_{n+1}$ generating $\mathfrak{M}$. Then $A = \bigcup\{S_{2n+1} \smallsetminus S_{2n} : n = 0, 1, \ldots\}$ and $B = \bigcup\{S_{2n+2} \smallsetminus S_{2n+1} : n = 0, 1, \ldots\}$ are disjoint, so that $A \in \mathfrak{M}$ or $B \in \mathfrak{M}$ ($\mathfrak{M}$ is prime), meaning that there exists k such that $A \subseteq S_k$ or $B \subseteq S_k$, clearly a contradiction.

5.3. Quite analogously as in 3.3.2 we obtain

Theorem. *The following are equivalent for a compact 0-dimensional frame L.*

(1) *L is extremally disconnected,*

(2) *every dense sublocale $h : L \to M$ is tight,*

(3) *every open dense sublocale $\widehat{a} = (x \mapsto x \wedge a) : L \to {\downarrow} a$ is tight.*

6 The Lindelöf case

6.1. A frame L is said to be *Lindelöf* if every subset $A \subseteq L$ such that $\bigvee A = \top$ has a countable subset $A_0 \subseteq A$ such that $\bigvee A_0 = \top$. The full subcategory of Lindelöf objects is coreflective in the category of completely regular frames, and the *Lindelöf coreflection* or *Lindelöfication* is given by

$$\mathfrak{H}(\mathsf{Coz}\,L) \to L, \quad J \mapsto \bigvee J \text{ in } L,$$

where $\mathfrak{H}$ is the functor assigning to each σ-frame its frame of σ-ideals; this is the free frame over $\mathsf{Coz}\,L$ in the sense that any σ-frame morphism $\mathsf{Coz}\,L \to M$, M a frame, lifts to a unique frame morphism $\mathfrak{H}(\mathsf{Coz}\,L) \to M$. Further, any frame surjection $h : L \to M$ with Lindelöf L is called a *Lindelöfication of* M; as a sublocale, we say that $L \to M$ is tight if the map is isomorphic to $\mathfrak{H}(\mathsf{Coz}\,M) \to M$.

Theorem 6.2 records the relevant information from the literature, extracted from Theorem 8.2.12 and Corollary 8.2.13 of [1].

6.2. Theorem. *The following are equivalent for a dense surjection $m : L \to M$.*

(1) *Every frame map $K \to M$, K Lindelöf, factors through m.*

(2) *Every frame map $\Omega\mathbb{R} \to M$ factors through m, i.e., m is a C-quotient map.*

(3) *If $\bigvee_{\mathbb{N}} c_n = \top$ in $\mathsf{Coz}\,M$ then there exist $\{d_n\} \subseteq \mathsf{Coz}\,L$ with $m(d_n) = c_n$ for all n and $\bigvee_{\mathbb{N}} d_n = \top$. In particular, $\bigvee_{\mathbb{N}} m_*(c_n) = \top$.*

(4) *For $\{c_n : n \in \mathbb{N}\} \subseteq \mathsf{Coz}\,M$, $m_*\left(\bigvee_{\mathbb{N}} c_n\right) = \bigvee_{\mathbb{N}} m_*(c_n)$.*

(5) *m is coz-iso, i.e., $\mathsf{Coz}\,L \approx \mathsf{Coz}\,M$.*

Furthermore, the regular Lindelöf coreflection $\mathfrak{H}(\mathsf{Coz}\,L) \to L$ is the unique Lindelöfication of M which satisfies the conditions above.

6.3. Corollary. *In the context of the Lindelof coreflection, a tight sublocale is a dense C-quotient out of a Lindelof frame.*

We have the following counterpart of 3.2 and 5.2.1.

6.4. Corollary. *A tight sublocale of a Lindelöf perfectly normal frame is trivial.*

6.4.1. Note. Of course, as in 3.2, this implies that any countably based Lindelöf frame L has this property, but that is totally trivial. For if L is countably based then so is M, which makes it Lindelöf and makes h an isomorphism.

6.5. The parallel Lindelöf version of 3.3.2 is less directly analogous and is, therefore, perhaps more interesting. Recall that a completely regular frame L is a *P-frame* if every cozero element is complemented, that is, if $\mathsf{Coz}\,L = B(L)$.

6.5.1. Lemma. *If the Booleanization is a C-quotient of a regular Lindelöf frame L then L is a P-frame.*

Proof. In this case $\mathsf{b}_L : L \to \mathfrak{B}L$ is the regular Lindelöf coreflection. Since $\mathsf{Coz}\,L \approx \mathsf{Coz}\,\mathfrak{B}L = \mathfrak{B}L$ by 6.2(5), L is a P-frame. □

6.5.2. Theorem. *The following are equivalent for a regular Lindelöf frame L.*

(1) *L is an extremally disconnected P-frame.*

(2) *Every dense sublocale of L is tight.*

(3) *Every dense open sublocale of L is tight, and L contains no proper dense cozero element.*

(4) *Every dense open sublocale of L is tight, and every cozero quotient of L is a C-quotient.*

When these conditions obtain, every cozero sublocale of L is tight in its closure.

Proof. (1) $\Longrightarrow$ (2): As before, it suffices to show that $\mathsf{b}_L : L \to \mathfrak{B}L$ is tight. Consider then the following commuting square given by the Lindelöf coreflection functor.

$$\begin{array}{ccc} \mathfrak{H}(\mathsf{Coz}\, L) & \xrightarrow{f} & \mathfrak{H}(\mathsf{Coz}\, \mathfrak{B}L) = \mathfrak{H}(\mathfrak{B}L) \\ \downarrow h & & \downarrow k \\ L & \xrightarrow{\mathsf{b}_L} & \mathfrak{B}L \end{array}$$

Here f is the map resulting from b_L, with h and k the coreflection maps. Now h is an isomorphism because L is Lindelöf, and hence $\mathsf{b}_L = k \circ f \circ h^{-1}$. On the other hand, $\mathsf{Coz}\, L = B(L)$ because L is a P-frame and $B(L) = \mathfrak{B}L$ by extremal disconnectedness so that f is an isomorphism (in fact, it is the identity map), showing b_L is as claimed. (2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) If every dense open quotient is tight then every such quotient is a C-quotient and hence a C^*-quotient, so that by 3.2.2 L is extremally disconnected and the Booleanization $\mathsf{b}_L : L \to \mathfrak{B}L$ is a C^*-quotient. Furthermore, b_L is coz-codense because L contains no proper dense cozero element, with the result that b_L is a C-quotient map by [1, 7.2.3], hence L is a P-frame by 6.5.1.

Finally, L is a P-frame iff every cozero quotient is a C-quotient by [1, 8.4.7], and this fact clearly shows (4) to be equivalent to the other numbered conditions. And when a cozero quotient $L \to {\downarrow} a$ is factored as $L \to \mathrm{cl}\, {\downarrow} a \to {\downarrow} a$, the map $\mathrm{cl}\, {\downarrow} a \to {\downarrow} a$ becomes a C-quotient. This is tight because a closed quotient of a Lindelöf frame is Lindelöf. $\square$

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