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Faculty of Information Technology

HABILITATION THESIS

Categorical Finite Model Theory

Game Comonads and Arboreal Categories

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Categorical Finite Model Theory: Game Comonads and Arboreal Categories
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Colophon
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Abstract

This habilitation thesis surveys almost a decade of my research in the area of game comonads, which is a new research area in the intersection of categorical logic and finite model theory. The unifying idea running through this approach is that by formalising basic concepts of finite model theory in the language of comonads and elementary path categories (e.g. arboreal categories), we obtain an elegant abstract theory which (1) specialises to old and new theorems and examples and (2) reveals connections with other branches of category theory. We discuss applications of this approach to the study of homomorphism counting theorems, graph parameters, composition methods (e.g. the theorems of Mostowski and Feferman–Vaught) and model-theoretic types.

The thesis also contains an extensive introduction to the theory of game comonads, built step-by-step from relevant concepts of finite model theory and comonad theory. This introduction should be accessible to anyone with minimal knowledge of category theory and logic.

Keywords

category theory, finite model theory, comonads, model comparison games, bisimulation, unravelling/unfolding, composition methods, homomorphism counting, types, graph parameters

Declaration

This habilitation thesis is part of the Habilitation Proceedings at Czech Technical University in Prague, conducted in accordance with the Rules of Habilitation Proceedings and Proceedings to Appoint Professors of Czech Technical University in Prague, effective as of April 22, 2022. In accordance with Section 72, Subsections 3(a) and 3(b) of Act No.111/1998Coll, it is presented as both a written work presenting new scientific findings (Chapters 3, 4, and 9) and a collection of published scientific papers (Chapters 5, 6, 7, and 8) accompanied by a commentary (Chapter 2).

On the use of AI

Large Language Models (LLMs), such as Claude Opus and Gemini Pro, were used to improve the quality of the text. However, they were asked to only provide grammar corrections and were specifically instructed not to alter the author's style. At no point of the process were the LLMs asked to generate new text or content.

Dedicated to my PhD fathers
Achim Jung and Aleš Pultr.

Preface

Game comonads and arboreal categories are a novel branch of categorical logic, establishing a connection between category theory and finite model theory, where the latter is a branch of mathematical logic focusing on the interaction of logic and algorithms.

In general, categorical logic is concerned with the structural analysis of important notions of logic, such as the semantics of first-order theories (hyperdoctrines), model theory (locally presentable and accessible categories), foundations (topos theory), and algebraic theories (Lawvere theories and monads). In the same spirit, game comonads provide a structural analysis of some aspects of finite model theory. Notably, logics important in finite model theory lack many desirable properties. They typically lack compactness and, furthermore, the question of completeness is also uninteresting, due to the Trakhtenbrot's theorem which says that the set of first-order sentences valid in finite structures is not recursively enumerable. Therefore, in some sense, game comonads provide a *structural approach to logics without compactness and completeness*.

The word *'game'* in 'game comonads' refers to model comparison games, such as the Ehrenfeucht–Fraïssé game or the bisimulation games, studied in model theory, modal logic and theory of concurrency. These games are an essential tool in these respective fields, which allows one to capture (and hence also study) logical equivalence with respect to some logic fragment. In some sense these model comparison games enable one to convert a syntactic notion (logical equivalence) into a combinatorial notion (a winning condition in some game).

Game comonads originate in the observation of Samson Abramsky and Anuj Dawar that some of the most common games in finite model theory can be described using simple constructions on (first-order) relational structures. Moreover, they satisfy familiar axioms of *comonads* from category theory, justifying the rest of the name 'game comonads'. The formalism of arboreal categories was initiated by Abramsky and Reggio in order to capture common synthetic properties of the categories of coalgebras of these comonads.

Therefore, game comonads and arboreal categories help one to abstract away from intricate combinatorial considerations when studying logical equivalences and, instead, concentrate on common features the different equivalences enjoy. Furthermore, by 'categorifying' a statement from finite model theory one obtains an abstract statement about arbitrary comonads. The original statement and more are recovered by instantiating it with different comonads. Moreover, the proof of this abstract statement requires the use of different methods, which often reveals connections with other areas of category theory.

The study of game comonads is still in its inception and many insights and connections are yet to be discovered. The purpose of this text is to give an overview of the basics of the theory of game comonads, with additional emphasis on the author's own research. Requiring only basic familiarity with logic and category theory, the text can serve as an introduction to the area.

Acknowledgement

I would like to thank my closest coworkers from whom I've learned so much over the years. In particular, I would like to thank Luca Reggio, Dan Marsden, Nihil Shah, Anuj Dawar and Samson Abramsky from the game comonads programme and to Jakub Opršal, Max Hadek, Henning Urbat and Anna Laura Suarez from my other recent projects.

Further, I owe a great deal to Dušan Knop (the head of my department) for fighting many battles at the university level on my behalf. I am also grateful to Wieslaw Kubis for letting me work on this project during my time in his group and to Mai Gehrke for (among others) supporting me (and pushing me) in the journey from pointfree topology to finite model theory.

Finally, I would like to remember Věra Trnková who (together with Aleš Pultr) taught me introductory and advanced courses of category theory at the university and gave me the chance to attend the famous Prague Category Theory seminar, as a young master student.

One would say that, by obtaining a PhD, the person who graduates is now considered a "proper researcher". However, the situation is not so clear at that point, one still has to grapple with what comes next. This is usually a series of postdocs interleaved with attempts to find a permanent position, establishing one's own research direction, obtaining grants, and so on.

Now that I teach core courses at my university, have a permanent position, and have secured some research grants, I feel (finally!) that I have truly graduated. This is why I dedicate this manuscript to my former PhD advisors, Achim and Aleš, to whom I owe so much.

Finally, I would like to thank my family for their patience and for providing enough distractions to remind me that there is also a world outside of academia.

Tomáš Jakl
in Prague
23rd September 2026

Contents

Contents **ix**

1 Introduction **1**
1.1 On the content of this manuscript . . . 5

2 Preliminaries **9**
2.1 Model comparison games . . . 9
The back-and-forth Ehrenfeucht–Fraïssé game . . . 9
Basic variants of the back-and-forth game . . . 10
Forth-only and counting variants . . . 13
Pebble games and restricted variable fragments . . . 15
Bisimulation game and modal logic . . . 16
2.2 Comonad theory overview . . . 18
Comonad and adjunctions . . . 20
2.3 Game comonads . . . 23
Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$ . . . 24
Pebbling comonad $\mathbb{P}_k$ . . . 25
Modal comonad $\mathbb{M}_k$ . . . 25
2.4 Comonads and logic, positive existential and counting fragments . . . 26
2.5 Coalgebras and combinatorial properties . . . 28
2.6 Comonads and standard logic fragments . . . 31
2.7 Comonads for other logic fragments . . . 32
2.8 Logic extensions using translations . . . 33
2.9 Transfer theorems for the EM category . . . 35
Factorisation systems . . . 35
Locally finitely presentable categories . . . 40
2.10 Back-and-forth games abstractly . . . 41

Synthetic model comparison games **45**

3 Elementary path categories **47**
3.1 Elementary path categories and logic . . . 47
Logical equivalences . . . 48
One-way logical refinement . . . 48
3.2 Additional assumptions . . . 49
Basic axioms . . . 49
Axioms about quotients . . . 50
Axioms about colimits of path embeddings . . . 50
3.3 Back-and-forth systems . . . 52
Initial position . . . 53
Operations on b&f system . . . 54
3.4 Bisimilarity and behavioural equivalence . . . 55
Open pathwise embeddings . . . 55
Composing with b&f systems . . . 56
Bisimilarity and behavioural equivalence . . . 57
Open morphisms à la Joyal–Nielsen–Winskel . . . 58
Pullbacks of open morphisms . . . 60

4 Arboreal categories **63**
4.1 Implicitly given paths . . . 63
4.2 Path functor . . . 64
4.3 Arboreal categories . . . 67
4.4 Concrete arboreal categories . . . 69
4.5 The representation theorem . . . 70
Preliminaries . . . 71
The motivation . . . 72
The functor of paths . . . 72
From functors to categories . . . 74
Factorisation system of $\mathsf{Ar}(\Phi)$ . . . 75
Subobjects and paths in $\mathsf{Ar}(\Phi)$ . . . 77
Tree colimits in $\mathsf{Ar}(\Phi)$ . . . 78
The correspondence . . . 80
4.6 Examples . . . 85
Game Comonads . . . 85
Models of concurrency . . . 85
Arboreal categories of types . . . 86
4.7 Consequences of the representation theorem . . . 86
4.8 Four ways of expressing bisimulations . . . 92
Simplifications . . . 93
Implications for logic . . . 96

Comonadic meta-theorems **99**

5 Homomorphism counting theorems **101**
5.1 Motivation . . . 101
5.2 A Categorical Lovász Theorem . . . 102
5.3 A Categorical Dvořák & Grohe theorem . . . 106
The case of finite rank comonads . . . 108
Related work . . . 109
5.4 Equality elimination . . . 109

6 Density comonads **113**
6.1 The density construction . . . 113
The comonad structure . . . 114
6.2 Discrete density comonads . . . 115
6.3 The abstract classification theorem . . . 115
6.4 Concrete examples . . . 118
Examples . . . 120
6.5 Comonad morphisms, a brief overview . . . 121
Comonad morphisms from composites of functors . . . 122
6.6 Graph Parameters . . . 122
Grading graph parameters . . . 123
Comparison with game comonads . . . 124
Nowhere dense comonads . . . 126
6.7 Homomorphism-Counting Theorems . . . 126
6.8 Game comonads from proof systems . . . 129
6.9 Dense graph parameters and more . . . 133

Composition methods and types 135

7 Lifting theorems 137
7.1 Eilenberg–Moore laws . . . 137
7.2 Kleisli laws . . . 139
7.3 Extended Kleisli liftings . . . 143
Historical remarks . . . 145
7.4 Beyond unary operations . . . 146
7.5 Heteromorphisms . . . 147
7.6 Lifting adjunctions . . . 152
On compositions of conjugations . . . 154
Right adjoints versus extended Kleisli liftings . . . 156
Comonad morphisms . . . 156

8 Preservation theorems and logic 159
8.1 Preservation of equivalence with respect to counting and positive existential fragments . . . 160
8.2 Preservations of the other fragments . . . 161
8.3 Preservations of open morphisms . . . 162
8.4 Preservations from minimal decompositions . . . 163
Preservation of the back-and-forth relation . . . 164
Preservation of (open) pathwise-embeddings . . . 166
Simplifying the axioms . . . 166
8.5 Strong path functors . . . 167
Path embeddings and conjugations . . . 167
All assumptions together . . . 169
Preservation of paths . . . 170
8.6 Logic extensions and Kleisli liftings . . . 171
8.7 Beyond unary operations . . . 174
8.8 Product preservation theorems for free . . . 175

9 Model-Theoretic Types 177
9.1 Motivation . . . 177
9.2 Assignments . . . 178
The coreflection . . . 179
Elementary path category structure of $\mathfrak{X}_P$ . . . 181
Related work . . . 182
9.3 Structured and unstructured assignments . . . 182
Equivalence of types . . . 183
Adjunctions and comonads for structures with constants . . . 183
9.4 Formulas and quantification . . . 185
9.5 Preservation for assignments . . . 186
Comparing standard logic fragments and their formulas . . . 191

Bibliography 193

Glossary of Symbols 201

Index 205

1 Introduction

Finite model theory is a well-established branch of theoretical computer science, originating in the 1980s. Its main focus is on the study of the semantic restriction of various logic fragments to the class of finite structures. This is motivated by questions in formal language theory, automata theory, database theory, descriptive complexity, etc.

Due to this semantic restriction to the finite structures, these logics lose many desirable properties. Namely, as was observed very early on, classical tools from model theory such as completeness, compactness, the Löwenheim-Skolem theorem, Beth definability, Craig interpolation, and Łoś-Tarski preservation fail in the finite. Some tools, on the other hand, still apply, such as model comparison games (e.g. the Ehrenfeucht–Fraïssé game), the van Benthem theorem (see Rosen [109]), and the Homomorphism Preservation Theorem (see Rossman [110]).

However, finite model theory is not primarily about classifying which theorems of classical model theory hold in the finite realm and which fail. As emphasised by Kolaitis [78, Section 3], equally important are also the questions of:

- connections between logic and computational complexity (field of study called *descriptive complexity theory*),
- the expressive power of logics on finite structures, and
- the asymptotic properties of logical formulas (0–1 laws, structural limits).

Note that the first concerns algorithmic meta-theorems, such as the famous Courcelle's theorem [39], and characterising complexity classes using logic fragments. The second, on the other hand, develops tools suitable for separating logics. Therefore, the first and second together aim to tackle the famous P vs NP problem.

Over the years, finite model theory has developed its own set of techniques, methods, and key results, which are designed to tackle the aforementioned research questions. The tools and notions that play the largest role in achieving these goals are probably the following:

(i) model comparison games
(ii) composition methods (Feferman–Vaught–Mostowski-type theorems)
(iii) types
(iv) interpretations and transductions
(v) locality theorems
(vi) homomorphism counting theorems
(vii) 0–1 laws
(viii) arithmetic circuits
(ix) automata theory

This points to the stark contrast with classical model theory, where these tools either do not play such a large role (the first half of the list above, except for ((iii))) or are not usable in the infinite (the second half of the list).

Origins of game comonads. Especially important for us are model comparison games such as the Ehrenfeucht–Fraïssé game, bisimulation game, or the pebble game. In particular, the latter establishes equivalence in the fragment of first-order logic of formulas of bounded variable count. Dawar et al. [43] observed that the existence of a winning strategy in the forth-only k-pebble game from a structure A to B corresponds to the existence of a homomorphism from $\mathbb{P}_k A$, the unravelling of A according to the rules of the pebble game, to B. Schematically:

$$\text{forth-only k-pebble game from } A \text{ to } B$$

$$\approx$$

$$\mathbb{P}_k A \to B$$

Later, Abramsky, Dawar, and Wang [3] showed that $\mathbb{P}_k A$ is a *comonad* on the category of relational structures of the same signature, establishing a link between finite model theory and category theory. Moreover, $\mathbb{P}_k$ by its coalgebras characterises tree-width, an important structural graph parameter [98].

Soon after, many more new *game comonads* were discovered; for example, the Ehrenfeucht–Fraïssé comonad, the modal comonad [14], the guarded comonad [6], the Hella comonad [38], and many others. They all follow the same pattern. On the one hand, they are built by a semantic encoding of the rules of a model comparison game, hence establishing a link with the logic fragment that corresponds to the game. On the other hand, their coalgebras have a *tree-like structure* by which they characterise some combinatorial parameter.

Categorical theorems. Orthogonal to identifying new game comonads came the efforts of understanding traditional finite model theory (FMT) results through the lens of category theory, piggybacking on the link with logic that game comonads provide. The usual structure of these results is as follows:

$$\text{Theorem in FMT} \;=\; \text{Categorical skeleton} \;+\; \text{Combinatorial core}$$

The task is to identify the correct level of abstraction for the categorical FMT theorem, i.e. the 'categorical skeleton' in the equation above, so that its proof follows by standard categorical techniques and the theorem itself specialises to new examples in finite model theory (see below).

Theorem in FMT	**Categorical techniques**
homomorphism counting theorem	comonadicity + finite rank
Composition method	(op)monoidal (co)monads + relative adjunctions
equi-rank Homomorphism Preservation Theorems	model saturation (small object argument)
Hodges' word-construction	finitary functors

Once the categorical FMT theorem is established, the remaining work lies in checking which comonads satisfy its assumptions, thereby obtaining new (concrete) FMT results. For example, in Chapter 5 we categorify the homomorphism counting results of Dvořák and Grohe as a single theorem about arbitrary comonads. The original theorems are recovered by instantiating the categorical theorem with the Ehrenfeucht–Fraïssé and pebble comonads. Moreover, we also obtain new theorems, for example, by instantiating the categorical homomorphism counting theorem with the modal and pebble-relation comonads, which yield homomorphism counting results for modal logic and restricted conjunction fragments, respectively. Usually, this part of digesting concrete results by instantiating the categorical theorems requires verification of some structural properties of our comonads, which is the 'combinatorial core' of the argument in the equation above.

Arboreal categories. The fact that coalgebras of game comonads typically have a tree-like structure led to the discovery of arboreal categories by Abramsky and Reggio [10, 11]. The setting of arboreal categories captures the essential features of the different categories of coalgebras of game comonads, thereby providing a common formalism for defining forth-only, bijective, back-and-forth,[1] existential, positive, etc. games. In fact, in any arboreal category, the notion of back-and-forth equivalence coincides with bisimilarity as described using spans of open morphisms, hence providing a link between finite model theory and the semantics of concurrency.

1: See also Otto [99] for a unifying approach to back-and-forth model comparison games, using a clever encoding into the standard bisimulation game for Kripke structures.

Importantly, this common language of arboreal categories abstracts away the intricate details of the different model comparison games. Each type of game (e.g. the back-and-forth Ehrenfeucht–Fraïssé, pebble, bisimulation, etc. game) is expressed as a single (simple) categorical notion for arboreal categories which, when instantiated with different examples of arboreal categories (e.g. arising from different game comonads), recovers the different examples of the same type of game from the literature.

Already the fact that many types of games can be captured using a simple common categorical formalism is an interesting feat on its own. Moreover, the fact that arboreal categories provide a common language for the different model comparison games arising in finite model theory can be used to prove logic meta-theorems, see e.g. [13, 66, 105]. Finally, arboreal categories lead to a rich theory in their own right; see, e.g., Chapter 4 for a representation theorem or [68] for the connections with the theory of fibrations.

In summary, *arboreal categories provide a synthetic approach to model comparison games in finite model theory* by defining all games of the same type at once, in terms of their common features. This is in contrast with the more usual *analytic* approach to games where each game is defined separately in concrete terms, and then its common properties are understood only intuitively or appear as theorems about these games.[2]

2: The distinction between 'synthetic' and 'analytic' comes from geometry. Euclid's approach to geometry is the principal example of the 'synthetic' approach, where he takes the notions of line, circle, and plane as basic whereas points are the derived notion. The 'analytic' approach, on the other hand, assumes that Euclid's basic geometric notions are definable from the concrete description of the space using its points and their coordinates.

Categorical finite model theory, an outlook. Being able to talk about model comparison games abstractly, in terms of comonads and arboreal categories, gives one the ability to talk about logical equivalences and, consequently, formulas, types, composition methods, and so on. In this

way, we are able to express many of the essential notions of finite model theory (listed as items ((i))–((ix)) above[3]) and, moreover, do so completely syntax-free whilst bringing a new set of tools to the area—the tools developed in category theory over many decades.

3: Note that 'locality' is also within reach, now that the theory of types has been developed. This is the topic of the author's forthcoming work.

In his 2007 survey, Kolaitis [78] calls for an abstract finite model theory:

> *"Lindström showed that first-order logic is a maximal logic possessing both the Compactness Theorem and the Löwenheim-Skolem Theorem. In other words, there is no logic that is strictly more expressive than first-order logic and possesses both the Compactness Theorem and the Löwenheim-Skolem Theorem. This result, obtained in 1969, became the catalyst for the development of abstract model theory.*
>
> *Is there an abstract finite model theory? . . . Is there a Lindström-type characterization of first-order logic on finite structures? Similarly, is there a Lindström-type characterization of least fixed-point logic on finite structures?"*

On the one hand, if we interpret Kolaitis's call to the letter, the current research directions of game comonads and arboreal categories do not seem to tackle this problem. On the other hand, game comonads provide tools to compare logic fragments important in finite model theory, e.g. using comonad morphisms (Chapter 8). One can argue that this approach is still within the intentions Kolaitis had in mind since, by doing so, we study the category of logics and their morphisms. This way, the relationships of different logic fragments are understood by how they interact with one another.

The emerging field of *categorical finite model theory*, encapsulating game comonads, arboreal categories, and the categorical [57] and cohomological [1, 37] approaches to CSP, etc., brings categorical tools to the study of finite model theory. See the recent survey by Dawar [41], which outlines how game comonads fit into greater finite model theory research and what problems are natural to tackle using this formalism. In particular, the following questions and tasks come naturally from this endeavour:

- *Establish categorical algorithmic meta-theorems.* In his invited CSL 2007 talk, Dawar [40] describes that decidability meta-theorems (e.g. Courcelle's theorem) rely on an iterated use of the composition method, locality, and automata theory. Our categorical methods account for the first two, with the last in the list and their intricate combination left as future work. See also the author's preliminary work on Courcelle's algorithm [64].
- *Combine the comonadic and cohomological approaches.* The Invertible Map Game, an important notion in finite model theory characterising linear-algebraic logic, cannot be captured by comonads [81]. This failure led Abramsky and Conghaile to discover the cohomological approach to forth-only games [1, 37]. However, this approach now sits independently of game comonads, and it is not clear how to model the interaction of the two.
- *Capture fixed-point logics and other logics in the framework.* There are still some logics important in finite model theory which do not have their comonadic counterpart, e.g. the fixed-point logics. Also, fragments such as the guarded and loosely-guarded fragments have

rvation by Elias
cation).

comonads whose coalgebras do not form an arboreal category.[4] Does that mean that there are better comonads for these fragments, or does the notion of arboreal categories need a generalisation to account for these fragments?

- *What is the connection with concurrency theory?* Back-and-forth equivalence in arboreal categories exactly corresponds to the Joyal–Nielsen–Winskel equivalence, designed for establishing the bisimilarity of two concurrent systems [10]. It is likely that more useful connections will show up once the two theories are put side by side.

Furthermore, categorical finite model theory develops its own tools which, in turn, motivate research questions specific to this field of research:

- *What is the correspondence with other fields of categorical model theory?* Arboreal categories take a lot of inspiration from the theory of presentable and accessible categories [20]. The recent representation theorem (Chapter 4) and the upper bound on the expressibility of an arboreal-finitely presentable adjunction [105] bring these two theories even closer. Still, these are only preliminary results and some more formal understanding is still missing.
- *Develop the theory of presentations of comonads.* Similarly to how monads are presented using algebraic theories, comonads too would benefit from some common formalism for presenting them. This would allow one to specify new game comonads by tweaking presentations of other game comonads and, thereby, obtain comonads with intended logical properties.
- *The relationship of coinduction and model-theoretic types.* In Subsection"Simplifications" on page 93, we observe that the space of model-theoretic types arises as the quotient of the cofree coalgebra by its largest bisimilarity relation. This points to the coinductive nature of types, which is partially exploited in Chapter 9. However, we only scratch the surface of the interaction of these two notions in this manuscript.

1.1 On the content of this manuscript

This manuscript is an overview of the author's own work related to game comonads. In particular, the following papers (ordered chronologically) serve as inspiration.

[42] Anuj Dawar, Tomáš Jakl, and Luca Reggio. 'Lovász-Type Theorems and Game Comonads'. In: *Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. New York, NY, USA: Association for Computing Machinery, 2021

[4] Samson Abramsky, Tomáš Jakl, and Thomas Paine. 'Discrete Density Comonads and Graph Parameters'. In: *Coalgebraic Methods in Computer Science. CMCS 2022*. Ed. by Helle Hvid Hansen and Fabio Zanasi. Vol. 13225. Lecture Notes in Computer Science. Springer International Publishing, 2022

[64] Tomáš Jakl, Dan Marsden, and Nihil Shah. 'A game comonadic account of Courcelle and Feferman-Vaught-Mostowski theorems'. In preparation, arXiv preprint. 2022. arXiv: 2205.05387

[65] Tomáš Jakl, Dan Marsden, and Nihil Shah. *Generalizations of Bilinear Maps – Technical Report*. 2022. arXiv: 2205.05382

[66] Tomáš Jakl, Dan Marsden, and Nihil Shah. 'A categorical account of composition methods in logic'. In: *38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2023

[67] Tomáš Jakl, Dan Marsden, and Nihil Shah. 'A categorical account of composition methods in logic (extended version)'. In: *Logical Methods in Computer Science* Volume 21, Issue 4, 10 (Oct. 2025)

[68] Tomáš Jakl and Luca Reggio. *On the Axioms of Arboreal Categories*. To appear in the proceedings of CMCS 2026. arXiv: 2603.21841

Below is an overview of the different chapters of this manuscript, with an attribution of the origins of their content.

- **Chapter 2** gives a gentle introduction to the theory of game comonads, which complements the other existing expositions [2, 12] by also reviewing the principal notions of finite model theory and category theory.

  This chapter is original, except for Lemmas 2.9.6 and 2.9.8 which are borrowed from the joint work with Reggio [68].

- **Chapter 3** introduces elementary path categories as a minimal synthetic language for talking about model comparison games. The aim of elementary path categories is to describe the synthetic language directly, without worrying too much about the inherent structure the arboreal categories have. We use this language to compare different notions of bisimilarity, namely the back-and-forth equivalence, bisimilarity, and behavioural equivalence.

  This chapter is original. The notion of elementary path categories is introduced here for the first time.

- **Chapter 4** is where we introduce arboreal categories and prove their representation theorem. We also show that the four different notions of bisimilarity from Chapter 3 are equivalent in arboreal categories.

  This chapter is based on unpublished ongoing work with Luca Reggio.

- **Chapter 5** is about Lovász-type homomorphism counting theorems which, undoubtedly, are the most influential application of game comonads.

  This chapter is based on [42], however, only the content of Section 5.2 and the proof of Proposition 5.4.3 are taken from therein.

- **Chapter 6** introduces density constructions of comonads as a means of generating new comonads from a functor. As an application, we devise a simple construction (of *discrete density comonads*) which allows us to represent the most important classes of interest in graph theory by means of coalgebras of a comonad. Later, we also use density constructions to recover the Ehrenfeucht–Fraïssé comonad from a simple proof system.

  All except for the last two sections originate in the joint work with Abramsky and Paine [4].

- **Chapter 7** focuses on the question of when a functor between base categories, e.g. the functor of taking the disjoint union of two structures, lifts to the categories of coalgebras of some comonads. This chapter lays the groundwork for the follow-up chapters.

  This chapter is based on joint work with Marsden and Shah [65–67]. The text is new except for Lemma 7.2.2 and proofs in Sections 7.3, and 7.5.

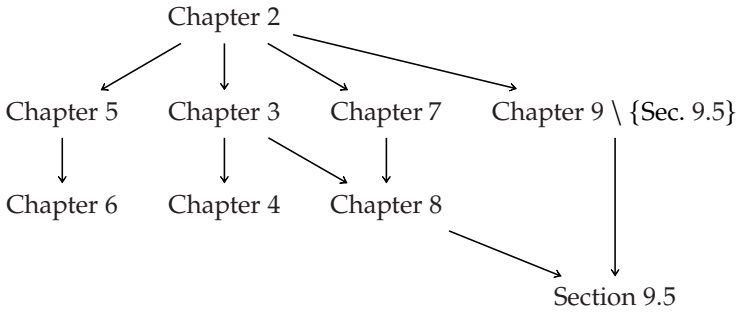
- **Chapter 8** uses results from Chapter 7 and the language of elementary path categories to re-prove the famous Feferman–Vaught–Mostowski theorems, i.e. we port the composition method to game comonads. Furthermore, comonad morphisms are studied as a means to compare different logics characterised by comonads. Lastly, strong functors are introduced, which give even stronger preservation properties than comonad morphisms.

  This chapter is based on joint work with Marsden and Shah [66]; however, its text is new except for some parts of Sections 8.4 and 8.8.

- **Chapter 9** finishes this manuscript with a discussion on how to talk about model comparison types in the setting of elementary path and arboreal categories. Preservation theorems (Section 9.5) are proved using strong functors from Chapter 8.

  This is new unpublished work of the author.

Below is the graph of dependencies between the chapters. Note that occasional references going across chapters might occur, but still, as the graph indicates, many of the topics can be read independently of the others. For example, one does not need to know anything about homomorphism counting theorems (Chapter 5) when reading about preservation theorems (Chapter 8).



Omitted material. Recall from above that the research in game comonads and arboreal categories can be split into the following main areas:

1. *'Comonad Engineering'*, i.e. devising new comonads for various logic fragments.
2. *Categorising results in finite model theory*, e.g. homomorphism counting theorems, van Benthem–Rosen theorems, etc.
3. *Studying synthetic model comparison games*, e.g. developing the theory of arboreal and elementary path categories.

Since the author of this text focuses mostly on the latter two aspects in his research, this manuscript completely omits the important direction of 'comonad engineering', which makes all of the follow-up work grounded.

Also, since this manuscript serves as a habilitation thesis, whose purpose is to showcase the author's own work, many important results from the latter two categories listed above had to be omitted too. In particular, we make no exposition of Homomorphism Preservation Theorems [13], linear arboreal categories [9], the van Benthem–Rosen theorem [7], Hodges' word-constructions and upper bounds on the expressibility of arboreal adjunctions [105], or no-go theorems [72].

Finally, the author's comonadic locality theorems (unpublished) and the author's work on Courcelle's theorem [64] are also omitted, since they have not matured for a textbook exposition yet.

Prerequisites. Although most of the material is written in categorical language, the required category-theoretic background is quite minimal. Apart from a few exceptions, we assume only a basic familiarity with functors, natural transformations, adjunctions, limits, colimits, monomorphisms, and epimorphisms. Everything else is introduced as needed. Similarly, no prior knowledge of finite model theory is assumed.

Preliminaries 2

The purpose of this chapter is to give a gentle high-level introduction to the theory of game comonads. To understand the original motivations we also review the relevant parts of finite model theory. In particular we focus on the correspondence of model comparison games, logics and comonads. We also overview basic category-theoretic facts concerning comonads. Although the categorical toolkit that we rely on is quite minimal, it would be besides the point to review all of it. In particular, we still assume familiarity with basic notions from category theory, such as functors, natural transformations, adjunctions, limits, colimits, etc. as covered e.g. in [15, 25, 108].

The principal references for finite model theory are Libkin [80] and Ebbinghaus–Flum [47]; and for game comonads, we refer the reader to the original papers, that is, Abramsky–Shah [14] and Abramsky–Dawar–Wang [3], as well as the recent survey Abramsky–Reggio [12].

2.1 Model comparison games 9
The back-and-forth Ehrenfeucht–Fraïssé game 9
Basic variants of the back-and-forth game 10
Forth-only and counting variants 13
Pebble games and restricted variable fragments 15
Bisimulation game and modal logic 16
2.2 Comonad theory overview 18
Comonad and adjunctions 20
2.3 Game comonads 23
Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$ 24
Pebbling comonad $\mathbb{P}_k$ . . . 25
Modal comonad $\mathbb{M}_k$ 25
2.4 Comonads and logic, positive existential and counting fragments 26
2.5 Coalgebras and combinatorial properties 28
2.6 Comonads and standard logic fragments 31
2.7 Comonads for other logic fragments 32
2.8 Logic extensions using translations 33
2.9 Transfer theorems for the EM category 35
Factorisation systems . . . 35
Locally finitely presentable categories 40
2.10 Back-and-forth games abstractly 41

2.1 Model comparison games

Model comparison games are an important tool in model theory. They are especially useful in finite model theory, where they are often used to prove inexpressibility results and to separate logic fragments. These games are played on structures of the same type, i.e. in the same signature. In our case, we only consider (finite and finitary) relational signatures. A σ**-structure** in signature $\sigma = \{R_1, \ldots, R_n\}$ is a tuple $(A, R_1^A, \ldots, R_n^A)$ where A is a set and for each $R \in \sigma$ we have a relation

$$R^A \subseteq A^n$$

where n is the arity of R. For example, for $\sigma = \{E(\cdot,\cdot), P(\cdot)\}$, a σ-structure (A, R^A, P^A) is a directed graph (where edges are given by the binary relation E) with a selected subset (given by the unary predicate P).

In this text, whenever we say 'structures', we implicitly mean σ-structures in the same relational signature σ.

The back-and-forth Ehrenfeucht–Fraïssé game

Given two structures A, B, the **(back-and-forth) Ehrenfeucht–Fraïssé game** is a game between two players, **Spoiler** and **Duplicator**. The task of Spoiler is to provide evidence that A and B are different, while Duplicator tries to prevent that.

In each round of the game, given some position of the game, Spoiler makes a move and Duplicator responds with their own move, which yields a new position for the next round. The **positions** are given as pairs $(\overline{a}, \overline{b})$ of sequences of the same length consisting of elements from

the structures A and B, that is, pairs $(\overline{a}, \overline{b})$ where $\overline{a} = [a_1, \ldots, a_n]$ and $\overline{b} = [b_1, \ldots, b_n]$ and, for every i, $a_i \in A$ and $b_i \in B$.

One round of the game is played as follows. Starting from position $([a_1, \ldots, a_{n-1}], [b_1, \ldots, b_{n-1}])$, either

- Spoiler chooses an element $a_n \in A$ and then Duplicator chooses $b_n \in B$; or
- Spoiler chooses an element $b_n \in B$ and then Duplicator chooses $a_n \in A$.

The position for the next round is $(\overline{a} = [a_1, \ldots, a_n], \overline{b} = [b_1, \ldots, b_n])$.

We say that **Duplicator wins the round** if the arising mapping $a_i \mapsto b_i$ (for $i = 1, \ldots, n$) obtained from the resulting position $(\overline{a}, \overline{b})$ induces a **partial isomorphism**. Formally, we require that the mapping $a_i \mapsto b_i$ is an isomorphism

$$A \restriction \overline{a} \quad \cong \quad B \restriction \overline{b}$$

where $A \restriction \overline{a}$ is the induced substructure of A on vertices $\{a_1, \ldots, a_n\}$.

We say that **Duplicator wins** or that **Duplicator has a winning strategy** in the k**-round** Ehrenfeucht–Fraïssé game if, starting from the position of empty sequences $([], [])$, it is possible for Duplicator to win the next k rounds of the game, no matter how Spoiler plays.[1] We say that **Spoiler has a winning strategy** if Duplicator does not have a winning strategy.

1: Note that, for Duplicator, winning rounds $1, \ldots, k$ is the same as winning only round k. However, the requirement to win every round allows us to generalise the game to unbounded number of turns, which we need later on.

The following folklore theorem expresses that the Ehrenfeucht–Fraïssé game between σ-structures captures logical equivalence $\equiv_{\mathtt{FO}^=}$, with respect to the first-order logic $\mathtt{FO}^=$, consisting of sentences in the language of σ.

Theorem 2.1.1 (Ehrenfeucht–Fraïssé [48, 51]) *Given structures A, B,*

$A \equiv_{\mathtt{FO}^=} B$	*if and only if,*	*for every k, Duplicator has a winning strategy in the k-round Ehrenfeucht–Fraïssé game between A and B.*

Remark 2.1.2 In the setting of game comonads it is more natural to talk about logics without equality. The logics with equality can be easily recovered, for example, by extending the signature, see Section 2.8 below. This is the reason why we explicitly decorate $\mathtt{FO}^=$ with the equality sign and we reserve the denotation $\mathtt{FO}$ for the fragment of $\mathtt{FO}^=$ without equality. Similarly, later we introduce the infinitary versions of $\mathtt{FO}$ with and without equality, which we denote by $\mathscr{L}^=_{\infty\omega}$ and $\mathscr{L}_{\infty\omega}$, respectively.

Basic variants of the back-and-forth game

In the setting of finite model theory and game comonads, there are a number of important variants of the Ehrenfeucht–Fraïssé game, capturing different fragments of first-order logic.

Bounded quantifier rank. First, we consider the variant where the number of rounds k is fixed beforehand. This corresponds to the restriction $\mathtt{FO}^{=}_{k}$ of $\mathtt{FO}^{=}$, consisting only of formulas of quantifier rank $\leq k$. The **quantifier rank** $\mathtt{qrank}(\varphi)$ of a formula φ is defined inductively as follows.

$$\begin{aligned}
\mathtt{qrank}(x = y) &= 0 \\
\mathtt{qrank}(R(x_1, \ldots, x_u)) &= 0 \\
\mathtt{qrank}(\phi \wedge \psi) &= \max\{\mathtt{qrank}(\phi), \mathtt{qrank}(\psi)\} \\
\mathtt{qrank}(\phi \vee \psi) &= \max\{\mathtt{qrank}(\phi), \mathtt{qrank}(\psi)\} \\
\mathtt{qrank}(\exists x.\phi) &= \mathtt{qrank}(\phi) + 1 \\
\mathtt{qrank}(\forall x.\phi) &= \mathtt{qrank}(\phi) + 1
\end{aligned}$$

Proposition 2.1.3 *Given structures A, B, $A \equiv_{\mathtt{FO}^{=}_{k}} B$ if and only if Duplicator has a winning strategy in the k-round Ehrenfeucht–Fraïssé game.*

Proof (inspired by [80, Theorem 3.9] or [47, Theorem 2.2.8]). We inductively prove a more general fact, about logics with constants. In particular, we prove that for every pair of n tuples $\overline{a} = [a_1, \ldots, a_n]$ and $\overline{b} = [b_1, \ldots, b_n]$ in A and B, respectively,

$$A, \overline{a} \equiv_{\mathtt{FO}^{=}_{k}} B, \overline{b} \iff A, \overline{a} \simeq_k B, \overline{b} \tag{2.1}$$

where $A, \overline{a} \equiv_{\mathtt{FO}^{=}_{k}} B, \overline{b}$ expresses that, for every formula $\varphi(\overline{x})$ of quantifier rank k with n free variables, $A \models \varphi(\overline{a})$ iff $B \models \varphi(\overline{b})$, and $A, \overline{a} \simeq_k B, \overline{b}$ means that Duplicator has a winning strategy in the k-round Ehrenfeucht–Fraïssé game between A and B starting from $\overline{a}$ and $\overline{b}$.

The key insight is that, one can encode all information about the position $\overline{a}$ and what can be reached from this position in round k of the game, with a formula $\varphi^k_{\overline{a}}(\overline{x})$ which, importantly, has quantifier rank k. The formulas are defined inductively, by

$$\varphi^0_{\overline{a}}(\overline{x}) = \bigwedge_{R^A(a_{i_1}, \ldots, a_{i_n})} R(x_{i_1}, \ldots, x_{i_n}) \wedge \bigwedge_{\neg R^A(a_{i_1}, \ldots, a_{i_n})} \neg R(x_{i_1}, \ldots, x_{i_n})$$

$$\varphi^{k+1}_{\overline{a}}(\overline{x}) = \bigwedge_{a \in A} \exists y\, \varphi^k_{\overline{a},a}(\overline{x}, y) \wedge \forall v \bigvee_{a \in A} \varphi^k_{\overline{a},a}(\overline{x}, y)$$

where $R^A(a_{i_1}, \ldots, a_{i_n})$ in the conjunction defining $\varphi^0_{\overline{a}}(\overline{x})$ ranges over all atomic formulas, including equality formulas.

Observe that, by definition, $A, \overline{a} \simeq_k B, \overline{b}$ iff $B \models \varphi^k_{\overline{a}}(\overline{b})$. Furthermore, since $\varphi^k_{\overline{a}}(\overline{x})$ has quantifier rank k, we obtain the left-to-right implication of (2.1).

Conversely, for the right-to-left implication of (2.1), let $\varphi(\overline{x})$ be a formula such that $A \models \varphi(\overline{a})$. The only interesting case is when $\varphi(\overline{x}) = \exists y\, \psi(\overline{x}, y)$ as the case when φ is a negation or conjunction of formulas is handled by induction on the complexity of φ. From $A \models \varphi(\overline{a})$ we obtain that there is some $a \in A$ such that $A \models \psi(\overline{a}, a)$. Also, from $A, \overline{a} \simeq_k B, \overline{b}$ there is some $b \in B$ such that $A, \overline{a}a \simeq_{k-1} B, \overline{b}b$. By induction $A, \overline{a}a \equiv_{\mathtt{FO}^{=}_{k-1}} B, \overline{b}b$ and so $B \models \psi(\overline{b}, b)$ and, therefore, also $B \models \exists y\, \psi(\overline{b}, y)$ i.e. $B \models \varphi(\overline{b})$. □

Observe that, by applying Proposition 2.1.3 to all k, we recover Theorem 2.1.1.

No equality. As hinted at in Remark 2.1.2, we also consider the fragment $\mathtt{FO}$, consisting of first-order sentences (in a given fixed language σ) without equality. This corresponds to the **relaxed Ehrenfeucht–Fraïssé game** where we weaken the winning condition for Duplicator. Instead of requiring that the position $(\overline{a}, \overline{b})$ induces a partial isomorphism $a_i \mapsto b_i$ we require that it induces a **partial correspondence**, that is, for any m-ary $R \in \sigma$ and $i_1, \dots, i_m \in \{1, \dots, n\}$,

$$(a_{i_1}, \dots, a_{i_m}) \in R^A \iff (b_{i_1}, \dots, b_{i_m}) \in R^B. \tag{2.2}$$

The proof of Proposition 2.1.3 can be easily adapted to yield the following.

Proposition 2.1.4 *Given structures A, B,*

- *$A \equiv_{\mathtt{FO}_k} B$ if and only if Duplicator has a winning strategy in the k-round Ehrenfeucht–Fraïssé game; and*
- *$A \equiv_{\mathtt{FO}} B$ if and only if, for every k, Duplicator has a winning strategy in the k-round Ehrenfeucht–Fraïssé game.*

To see that Proposition 2.1.4 strictly subsumes Theorem 2.1.1 and Proposition 2.1.3, define A^I as an extension of A in the expanded signature

$$\sigma^I = \sigma \cup \{I\}$$

where I is a new binary relational symbol. We interpret I in A^I as equality, i.e. $I^{A^I}(x, y)$ iff $x = y$. Then, observe that $A^I \equiv_{\mathtt{FO}} B^I$ (in signature σ^I) iff $A \equiv_{\mathtt{FO}^=} B$ (in signature σ) and similarly for $\mathtt{FO}_k$ and $\mathtt{FO}_k^=$. Indeed, notice that (2.2) applied to the new binary relation I says that $a_i = a_j$ iff $b_i = b_j$, which simply says that $\{(a_i, b_i) \mid i \in \{1, \dots, n\}\}$ is a graph of a bijective function and, therefore, the partial correspondence actually is a partial isomorphism.

Infinitary conjunctions and disjunctions. Finally, we might consider a variant of the Ehrenfeucht–Fraïssé game where there is no k after which the game stops, that is, the game is **unbounded** and Duplicator has to win every round forever. On the logic side, this corresponds to allowing arbitrary conjunctions and disjunctions.[2] This logic is denoted by

$$\mathscr{L}_{\infty\omega}$$

since in the model theory literature, the logic $\mathscr{L}_{\kappa\lambda}$ usually denotes first-order logic with conjunctions and disjunctions of size $<\kappa$ and quantification blocks which can bound $<\lambda$ variables at once.

2: This is necessary to be able to represent the model-theoretic type of a given position in a game, cf. the proof of Theorem 2.1.1.

Assumption 2.1.5 In this text, whenever we speak about first-order logic and write $\mathscr{L}_{\infty\omega}$, we mean *infinitary first-order logic without equality*. The variant of $\mathscr{L}_{\infty\omega}$ with equality is denoted by $\mathscr{L}^=_{\infty\omega}$.
Also, when we speak about a game variant, unless said otherwise, we automatically mean the relaxed variant.

As before, we obtain the following, by adapting the proof of Theorem 2.1.1, see e.g. [47, Theorem 3.2.7].

Proposition 2.1.6 *Given structures A, B,*

- *$A \equiv_{\mathscr{L}_{\infty\omega}} B$ if and only if Duplicator has a winning strategy in the (relaxed) unbounded Ehrenfeucht–Fraïssé game; and*
- *$A \equiv_{\mathscr{L}^{=}_{\infty\omega}} B$ if and only if, for every k, Duplicator has a winning strategy in the non-relaxed unbounded Ehrenfeucht–Fraïssé game.*

Note that the restrictions of $\mathscr{L}_{\infty\omega}$ and $\mathscr{L}^{=}_{\infty\omega}$ to formulas of bounded quantifier rank yield the logics $\mathtt{FO}_k$ and $\mathtt{FO}^{=}_k$. This is because, for a fixed k, every formula in $\mathscr{L}_{\infty\omega}$ and $\mathscr{L}^{=}_{\infty\omega}$ of quantifier rank $\leq k$ is equivalent to a formula in $\mathtt{FO}_k$ and $\mathtt{FO}^{=}_k$, respectively. Therefore, adding infinitary conjunctions and disjunctions does not increase the expressive power in this case.

Forth-only and counting variants

The variations of the Ehrenfeucht–Fraïssé game in the previous subsection concerned fairly minor modifications, involving only changes to the number of rounds or the winning condition. Next, we look at two variations that alter the Ehrenfeucht–Fraïssé game more fundamentally.

The **forth-only Ehrenfeucht–Fraïssé game** from A to B is a non-symmetric version of the back-and-forth game, in which Spoiler is required to always play on A and, therefore, Duplicator always responds in B. Furthermore, the winning condition for Duplicator does not require that the mapping $a_i \mapsto b_i$ induced by $(\overline{a}, \overline{b})$ is a partial isomorphism, instead, it requires that it is a **partial homomorphism**. In the relaxed variant, we require that $(\overline{a}, \overline{b})$ induces a **forth-only partial correspondence**, i.e. the left-to-right implication of (2.2) only.

Having a winning strategy for Duplicator in such a game corresponds to the preservation of **positive existential** formulas, that is, formulas that use neither universal quantification nor negation. Given a logic $\mathscr{L}$, we define

$$\exists^{+}\mathscr{L}$$

to be the fragment of $\mathscr{L}$ which only consists of positive existential formulas.

The fact that the forth-only game is non-symmetric means that we must also replace logical equivalence $\equiv_{\mathscr{L}}$ with its non-symmetric variant. To this end, for structures A, B and logic $\mathscr{L}$, we define the **logical refinement** $\Rrightarrow_{\mathscr{L}}$ relation by:

$$A \Rrightarrow_{\mathscr{L}} B \iff \forall \phi \in \mathscr{L} \quad A \models \phi \text{ implies } B \models \phi$$

Observe that $A \equiv_{\mathscr{L}} B$ iff $A \Rrightarrow_{\mathscr{L}} B$ and $B \Rrightarrow_{\mathscr{L}} A$. Furthermore, if $\mathscr{L}$ is closed under negation (i.e. $\phi \in \mathscr{L}$ implies $\neg\phi \in \mathscr{L}$), then $A \Rrightarrow_{\mathscr{L}} B$ is equivalent to $A \equiv_{\mathscr{L}} B$. Therefore, the relation $\Rrightarrow_{\mathscr{L}}$ is only interesting for logics that are not closed under negation, such as the positive existential fragments.

We can now state the forth-only variant of the Ehrenfeucht–Fraïssé theorem.

Proposition 2.1.7 *Given structures A, B, then*

- $A \Rrightarrow_{\exists^+\mathtt{FO}} B$ *iff, for every k, Duplicator wins the k-round (relaxed) forth-only Ehrenfeucht–Fraïssé game from A to B;*
- $A \Rrightarrow_{\exists^+\mathtt{FO}_k} B$ *iff Duplicator wins the k-round (relaxed) forth-only Ehrenfeucht–Fraïssé game from A to B; and*
- $A \Rrightarrow_{\exists^+\mathscr{L}_{\infty\omega}} B$ *iff Duplicator wins the unbounded (relaxed) forth-only Ehrenfeucht–Fraïssé game from A to B.*

It is not difficult to check that the positive existential fragments of $\mathtt{FO}_k$ and $\mathtt{FO}_k^=$ are equally expressive. Therefore, also both types of the relaxed and non-relaxed variants of the forth-only Ehrenfeucht–Fraïssé game are equally difficult for Duplicator.

The last variant of the Ehrenfeucht–Fraïssé game that we consider is the **bijective Ehrenfeucht–Fraïssé game** between structures A and B, which is defined as follows. In round n, starting from position $(\overline{a}, \overline{b})$,

- first, Duplicator picks a bijection $i\colon A \to B$, and
- then Spoiler picks an element $a_{n+1} \in A$.

The position in the next round will be $([a_1, \ldots, a_{n+1}], [b_1, \ldots, b_{n+1}])$ where $b_{n+1} = i(a_{n+1})$. The winning condition for Duplicator stays the same as in the back-and-forth game.

This game corresponds to the **counting extension** of $\mathscr{L}_{\infty\omega}$, denoted simply by

$$\mathtt{C}$$

which is an infinitary first-order logic where instead of classical quantifiers we include quantifiers $\exists^{\geq n}$, with the semantics

$$A \models \exists^{\geq n} x.\phi(x) \quad \text{iff} \quad \begin{array}{l}\text{there exist at least } n \text{ different } a \in A \\ \text{such that } A \models \phi(a).\end{array}$$

Observe that the classical existential quantifier is modelled by $\exists^{\geq 1}$ and universal quantification $\forall x.\phi$ is obtained by the usual negation translation $\neg(\exists^{\geq 1} x.\neg\phi)$. Therefore, $\mathtt{C}$ is an extension of $\mathscr{L}_{\infty\omega}$.

On the other hand, every counting quantification $\exists^{\geq n} x.\ \phi$ can be replaced by $\exists x_1, \ldots, x_n.\ \bigwedge_{i\neq j}(x_i \neq x_j) \wedge \bigwedge_i \phi(x_i)$. Hence, the expressive power of $\mathtt{C}$ and $\mathscr{L}_{\infty\omega}$ is the same. The syntactic distinction becomes important once we consider fragments of these two logics. Denote by $\mathtt{C}_k$ the restriction of $\mathtt{C}$ to only include formulas of quantifier rank $\leq k$. This fragment is now strictly more expressive than $\mathtt{FO}_k$.

Proposition 2.1.8 (Hella [59]) *Given* finite *structures A, B,*

$A \equiv_{\mathtt{C}_k} B$ *if and only if* *Duplicator has a winning strategy in the counting k-round Ehrenfeucht–Fraïssé game between A and B.*

Note that whether $\mathtt{C}_k$ contains equality or not does not change the power of the logic, see [42] or Section 5.4 for details. Consequently, the relaxed

and non-relaxed variants of the counting games have the same expressive power too. However, this is a non-trivial fact and, therefore, we still sometimes make the distinction between the two syntactic variants $\mathsf{C}_k^=$ and C_k and the non-relaxed and relaxed variants of the counting games, respectively.

Pebble games and restricted variable fragments

Pebble games are variants of the Ehrenfeucht–Fraïssé games with a key role in finite model theory. In the k**-pebble** variant, each structure is assigned a collection of k pebbles. Then, in each turn, Duplicator and Spoiler 'move the pebbles' onto some elements of the structure.

Formally, a position at every round is a pair of partial functions

$$(\alpha \colon [k] \rightharpoonup A, \beta \colon [k] \rightharpoonup B),$$

where $[k] = \{1, \dots, n\}$. The initial position of the game is the pair $(\emptyset, \emptyset)$ consisting of partial functions with empty domains.

The rounds have the following structure.

- First, Spoiler chooses a pebble index $p \in [k]$. Then, either
- Spoiler chooses an element $a \in A$ and then
  Duplicator chooses $b \in B$; or
- Spoiler chooses an element $b \in B$ and then
  Duplicator chooses $a \in A$.

The position in the next round will be (α', β') where $\alpha' = \alpha[p \mapsto a]$ i.e. $\alpha'(p) = a$ and $\alpha'(x) = \alpha(x)$ for $x \neq p$, and $\beta' = \beta[p \mapsto b]$.

In the case of the **bijective** k**-pebble** game, the round structure changes as follows.

- First, Spoiler chooses a pebble index $p \in [k]$,
- then Duplicator chooses a bijection $i \colon A \to B$ and
- then Spoiler picks an element $a \in A$.

The position in the next round will be $(\alpha[p \mapsto a], \beta[p \mapsto i(a)])$. Note that Duplicator loses on their turn if the cardinalities of A and B differ.

The winning condition for Duplicator after every round is the same as the winning condition for Duplicator in the Ehrenfeucht–Fraïssé game on the position $(\overline{a}, \overline{b})$ where $a_i = \alpha(\pi(i))$ and $b_i = \beta(\pi(i))$ for some enumeration $\pi \colon [n] \to \mathrm{dom}(\alpha)$ of the domain of α. For example, if $\alpha(1) = a$, $\alpha(3) = a'$ and it is undefined elsewhere, then $\overline{a} = [a, a']$.

Pebble games correspond to the restrictions of our logics to fragments that use a limited number of variables. In particular, given a logic $\mathscr{L}$, define the k**-variable fragment**

$$\mathscr{L}^k$$

to be the fragment of $\mathscr{L}$ where each formula does not use more than k variables. In particular, $\mathscr{L}_{\infty\omega}^k$ and C^k are the k-variable fragments of $\mathscr{L}_{\infty\omega}$ and C, respectively.

Example 2.1.9 The logic $\mathcal{L}^3_{\infty\omega}$ can express that a graph is connected, for example, with the sentence

$$\forall x, y. \bigvee_{n \in \mathbb{N}} \mathtt{path}_n(x, y)$$

where $\mathtt{path}_n(x, y)$, for $n > 0$, is the formula of quantifier rank $n - 1$

$$\exists z.E(x, z) \wedge (\exists x.E(z, x) \wedge (\exists z.E(x, z) \wedge (\ldots E(x, y))))$$

expressing that there is a path of length n.

On the other hand, connectivity is not expressible in $\mathtt{FO}^=$ with any number of variables. Using combinatorial or locality-type arguments it can be shown that Duplicator has a winning strategy in the k-round game between structures A and B where A is a long cycle (with length exponential in k) and $B = A + A$ [80]. Since A is connected and B is not, Proposition 2.1.3 gives that connectivity is not expressible in $\mathtt{FO}^=$.

The following fact establishes the correspondence of the k-variable fragments with k-pebble games.

Proposition 2.1.10 *Given structures A, B, we have*

- $A \Rrightarrow_{\exists^+\mathcal{L}^k_{\infty\omega}} B$ *iff Duplicator has a winning strategy in the forth-only k-pebble game from A to B; and*
- $A \equiv_{\mathcal{L}^k_{\infty\omega}} B$ *iff Duplicator has a winning strategy in the back-and-forth k-pebble game between A and B.*

Furthermore, if A, B are finite then

- $A \equiv_{\mathsf{C}^k} B$ *iff Duplicator has a winning strategy in the bijective k-pebble game between A and B.*

As before, these facts extend to non-relaxed versions of these games with respect to the variants of these logics with equality, which we exploit in Section 2.8 below.

Remark 2.1.11 Note that $\equiv_{\mathsf{C}^k}$-equivalence has many equivalent characterisations in terms of combinatorics, logic, algebra, and linear optimisation. For example, $\equiv_{\mathsf{C}^{k+1}}$-equivalence is the same as equivalence with respect to the k-dimensional Weisfeiler–Leman (k-WL) equivalence. The k-WL equivalence relations are a widely studied family of approximations of the graph isomorphism relation [33] (see [74] for a recent exposition).

Bisimulation game and modal logic

Finally, we review the correspondence between games and modal logic. Note that this correspondence works in the setting of **pointed Kripke frames**, which are σ-structures in a **modal signature** σ, that is, a signature where each relation is of arity 1 or 2. To this end, we fix $\sigma = \langle P_1, \ldots, P_n, R_1, \ldots, R_m \rangle$ where each P_i is unary and each R_i is binary. Furthermore, the requirement that the Kripke frames are **pointed structures** means that, as part of the structure, there is a **designated point** $a_0 \in A$.

Bisimulation game. We start by describing the back-and-forth version of the **bisimulation game** between pointed Kripke structures (A, a_0) and (B, b_0). As in the back-and-forth Ehrenfeucht–Fraïssé game, the positions are pairs of sequences $(\overline{a}, \overline{b})$ reached during the game according to the following rules.

Positions in the games are pairs (a, b) of elements of A and B, respectively. The initial position is (a_0, b_0) and if, in round n, the position is (a_n, b_n) then

- first, Spoiler chooses a binary $R \in \sigma$ and then either
- Spoiler chooses an element $a_{n+1} \in A$ such that $R^A(a_n, a_{n+1})$ and then Duplicator chooses $b_{n+1} \in B$ such that $R^B(b_n, b_{b+1})$; or
- Spoiler chooses an element $b_{n+1} \in B$ such that $R^B(b_n, b_{b+1})$ and then Duplicator chooses $a_{n+1} \in A$ such that $R^A(a_n, a_{n+1})$.

Either way, the new position is (a_{n+1}, b_{n+1}).

The winning condition for Duplicator at position (a_i, b_i) (including the starting position) is that, for every unary $P \in \sigma$,

$$P^A(a_i) \iff P^B(b_i) \tag{2.3}$$

The forth-only variant of the bisimulation game, called **simulation game**, modifies the winning condition so that the equivalence in (2.3) changes to just left-to-right implication. Also, as with the other forth-only games we described so far, Spoiler always plays on A and, consequently, Duplicator always responds on B.

Finally, the **bijective bisimulation game** is defined similarly to the bijective pebble game, that is, in round n at position (a_n, b_n):

- First, Spoiler chooses a binary relation $R \in \sigma$,
- then Duplicator chooses a bijection i between the sets of R-successors of the current positions, that is, a bijection between the sets $\{a' \mid R^A(a_n, a')\}$ and $\{b' \mid R^B(b_n, b')\}$.
- then, Spoiler picks an R-successor a_{n+1} of a_n.

The position in the next round will be $(a_{n+1}, i(a_{n+1}))$. Similarly to the bijective pebble game, Duplicator loses on their turn if the cardinalities of the R-successors of a_n and b_n differ.

Modal logic, denoted `ML`, is an extension of propositional logic with modalities. The syntax of modal logic is given by the following recursive description:

$$\varphi ::= P \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi' \mid \Box_R\varphi \mid \Diamond_R\varphi$$

where P ranges over propositional variables, i.e. unary predicates in σ, and R ranges over binary predicates in σ.

For a pointed Kripke structure (A, a_0), the semantics of modal formulas

is given inductively as follows.

$$\begin{array}{lcl}
(A, a_0) \models P & \iff & P^A(a_0) \\
(A, a_0) \models \neg\varphi & \iff & \text{not } (A, a_0) \models \varphi \\
(A, a_0) \models \varphi \wedge \varphi' & \iff & (A, a_0) \models \varphi \quad \text{and} \quad (A, a_0) \models \varphi' \\
(A, a_0) \models \varphi \vee \varphi' & \iff & (A, a_0) \models \varphi \quad \text{or} \quad (A, a_0) \models \varphi' \\
(A, a_0) \models \Box_R \varphi & \iff & \forall a_1 \text{ if } R^A(a_0, a_1) \text{ then } (A, a_1) \models \varphi \\
(A, a_0) \models \Diamond \varphi & \iff & \exists a_1 \text{ s.t. } R^A(a_0, a_1) \text{ and } (A, a_1) \models \varphi
\end{array}$$

Instead of the quantifier rank restriction, we consider the **modal depth** restriction $\mathtt{ML}_k$ of $\mathtt{ML}$, where modal depth is defined analogously to $\mathtt{qrank}$, except that we count the nesting of modalities $\Box_R$ and $\Diamond_R$ instead of quantifiers. Since the box modality $\Box_R$ plays the role of universal quantifiers, the existential positive fragment $\exists^+\mathtt{ML}$ of $\mathtt{ML}$ consists of formulas which do not use $\Box_R$ or negation.

Finally, **counting modal logic** $\#\mathtt{ML}$ is the infinitary variant of $\mathtt{ML}$ which replaces modalities $\Box_R$ and $\Diamond_R$ with **graded modalities** $\Diamond_R^n$, for every $n \in \mathbb{N}$, whose semantics is given as follows.

$$\begin{array}{lll}
(A, a_0) \models \Diamond^n \varphi & \text{iff} & \text{there exist at least } n \text{ different } a_1 \in A \\
& & \text{such that } R^A(a_0, a_1) \text{ and } (A, a_1) \models \varphi
\end{array}$$

The logic fragments defined above correspond to the bisimulation and simulation games as follows.

Proposition 2.1.12 *Given pointed Kripke structures (A, a_0) and (B, b_0), we have:*

- $(A, a_0) \rightrightarrows_{\exists^+\mathtt{ML}_k} (B, b_0)$ *iff Duplicator has a winning strategy in the k-round simulation game from (A, a_0) to (B, b_0); and*
- $(A, a_0) \equiv_{\mathtt{ML}_k} (B, b_0)$ *iff Duplicator has a winning strategy in the k-round bisimulation game between (A, a_0) and (B, b_0).*

Furthermore, if A, B are finite then

- $(A, a_0) \equiv_{\#\mathtt{ML}_k} (B, b_0)$ *iff Duplicator has a winning strategy in the bijective k-round bisimulation game between (A, a_0) and (B, b_0).*

The third is proved (without restricting to modal depth k) in [16], the second is a standard fact, known as the Hennessy–Milner theorem, see e.g. [91, Proposition 10.6] or [54, Lemma 31]. Note that if we do not restrict the number of rounds in the bisimulation game then the corresponding logic is the *infinitary* modal logic, i.e. a version of modal logic with infinitary conjunctions and disjunctions [54, Theorem 41].

2.2 Comonad theory overview

The theory of game comonads brings a new structural approach to the study of logics in finite model theory. It stems from the observation that each of the main game types (Ehrenfeucht–Fraïssé, pebble, bisimulation) introduced in Section 2.1 induces a semantic construction satisfying the

axioms of comonads from category theory. Then, due to the correspondence of logics and games, the theory of (co)monads can be used as a new abstract tool for a systematic and syntax-study of logic fragments.

We introduce the different comonads, dubbed *game comonads*, corresponding to each type of the game in Section 2.3 below. In this section, we review the basics of the (abstract) theory of comonads which are relevant for our study of game comonads.

A **comonad in the coextension form**[3] on a category $\mathscr{C}$ is a triple $(\mathbb{C}, \varepsilon, (-)^*)$ where

3: Often referred to as the 'Kleisli form'; however, it first appeared in this formulation as an exercise in Manes [89, page 32].

- $\mathbb{C}$ is a mapping on objects $\mathrm{obj}(\mathscr{C}) \to \mathrm{obj}(\mathscr{C})$,
- the **counit** ε is an indexed family of morphisms, i.e. for each $A \in \mathrm{obj}(\mathscr{C})$, there is a fixed

$$\varepsilon_A \colon \mathbb{C}(A) \to A,$$

- the **coextension** operation $(-)^*$ maps a morphism $f : \mathbb{C}(A) \to B$ to a morphism $f^* : \mathbb{C}(A) \to \mathbb{C}(B)$.

Furthermore, counits and coextensions are required to satisfy the following equations, for any $f \colon \mathbb{C}(A) \to B$ and $g \colon \mathbb{C}(B) \to C$:

$$(\varepsilon_A)^* = \mathrm{id}_{\mathbb{C}(A)}, \quad \varepsilon_B \cdot f^* = f, \quad (g \cdot f^*)^* = g^* \cdot f^*. \tag{2.4}$$

Or, equivalently, we express (2.4) as commutativity of the following diagrams.

$$\mathbb{C}(A) \underset{\varepsilon_A^*}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{C}(A) \qquad \begin{array}{ccc} \mathbb{C}(A) & \xrightarrow{f^*} & \mathbb{C}(B) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varepsilon_B} \\ & & B \end{array} \qquad \begin{array}{ccc} \mathbb{C}(A) & \xrightarrow{f^*} & \mathbb{C}(B) \\ & {\scriptstyle (g\cdot f^*)^*}\searrow & \downarrow{\scriptstyle g^*} \\ & & \mathbb{C}(C) \end{array}$$

Comonads can be equivalently presented in their **comonoid form**,[4] that is, as triples $(\mathbb{C}, \varepsilon, \delta)$ where

4: The word 'comonoid' refers to the fact that the equations (2.5) specify that $(\mathbb{C}, \varepsilon, \delta)$ forms a comonoid in the category of endofunctors $\mathscr{C} \to \mathscr{C}$.

- $\mathbb{C}$ is an endofunctor on $\mathscr{C}$,
- a **counit** ε is a natural transformation $\mathbb{C} \Rightarrow \mathrm{Id}$ (where $\mathrm{Id} \colon \mathscr{C} \to \mathscr{C}$ is the identity functor), and
- a **comultiplication** δ is a natural transformation $\mathbb{C} \Rightarrow \mathbb{C}^2$

such that the following diagrams commute.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\delta} & \mathbb{C}^2 \\ {\scriptstyle\delta}\Downarrow & & \Downarrow{\scriptstyle \delta\mathbb{C}} \\ \mathbb{C}^2 & \xrightarrow[\mathbb{C}\delta]{} & \mathbb{C}^3 \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\delta} & \mathbb{C}^2 \\ & {\scriptstyle \mathrm{id}}\searrow & \Downarrow{\scriptstyle \varepsilon\mathbb{C}} \\ & & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\delta} & \mathbb{C}^2 \\ & {\scriptstyle \mathrm{id}}\searrow & \Downarrow{\scriptstyle \mathbb{C}\varepsilon} \\ & & \mathbb{C} \end{array} \tag{2.5}$$

It is then standard that these two presentations are equivalent [89]. In fact, given a comonad in the coextension form $(\mathbb{C}, \varepsilon, (-)^*)$, the mapping on objects $\mathbb{C}$ extends to a functor by setting

$$\mathbb{C}(f \colon A \to B) = (f \cdot \varepsilon_A)^*$$

and the comultiplication is defined componentwise by

$$\delta_A = (\mathbb{C}(A) \xrightarrow{\mathrm{id}} \mathbb{C}(A))^*.$$

Conversely, given data $(\mathbb{C}, \varepsilon, \delta)$, the coextension operation is given by

$$(\mathbb{C}(A) \xrightarrow{f} B)^* = (\mathbb{C}(A) \xrightarrow{\delta_A} \mathbb{C}^2(A) \xrightarrow{\mathbb{C}(f)} C(B).$$

In this text, when referring to 'comonads', we use both definitions, depending on which suits the situation better. Typically, when we need to construct a new comonad it is easier to use the coextension form, whereas when higher-order operations on comonads are required, the comonoid form is easier to work with.

Comonad and adjunctions

Adjunctions

To fix a notation, recall that by an adjunction $L \dashv R : \mathcal{A} \leftrightarrows \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$, as shown below

$$\mathcal{A} \underset{L}{\overset{R}{\curvearrowleft \top \curvearrowright}} \mathcal{B}$$

we mean a pair of functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ and a pair of mappings, called **conjugations**,

- $(\text{-})^\flat$ maps a morphism $g\colon A \to R(B)$ to $g^\flat\colon L(A) \to B$, and
- $(\text{-})^\#$ maps a morphism $f\colon L(A) \to B$ to $f^\#\colon A \to R(B)$

which are inverse to each other and which satisfy the following rules.[5]

5: Note that these rules simply express that the bijection $\hom(A, R(B)) \cong \hom(L(A), B)$ given by $(\text{-})^\flat$ and $(\text{-})^\#$ is natural in A and B.

- $(A' \xrightarrow{h} A \xrightarrow{g} R(B))^\flat = L(A') \xrightarrow{L(h)} L(A) \xrightarrow{g^\flat} B$
- $(L(A) \xrightarrow{f} B \xrightarrow{h} B')^\# = A \xrightarrow{f^\#} R(B) \xrightarrow{R(h)} R(B')$

Equivalently, instead of $(\text{-})^\flat$ and $(\text{-})^\#$ we can specify the same adjunction using a pair of natural transformations

$$\eta\colon \mathrm{Id} \Rightarrow RL \quad \text{and} \quad \varepsilon\colon LR \Rightarrow \mathrm{Id}$$

called the **unit** and **counit** of the adjunction, respectively. These are required to satisfy the following equalities.

$$\begin{array}{ccc} L & \xRightarrow{L\eta} & LRL \\ & \searrow^{\mathrm{id}} & \Downarrow \varepsilon L \\ & & L \end{array} \qquad \begin{array}{ccc} R & \xRightarrow{\eta R} & RLR \\ & \searrow^{\mathrm{id}} & \Downarrow R\varepsilon \\ & & R \end{array}$$

Note that the pair (η, ε) obtained from $(\text{-})^\flat$ and $(\text{-})^\#$ is defined componentwise as

$$\eta_A = (L(A) \xrightarrow{\mathrm{id}} L(A))^\# \quad \text{and} \quad \varepsilon_B = (R(B) \xrightarrow{\mathrm{id}} R(B))^\flat.$$

Conversely, given the unit and counit pair (η, ε), the corresponding pair of mappings $(\text{-})^\flat$ and $(\text{-})^\#$ is obtained by:

$$(A \xrightarrow{g} R(B))^\flat = L(A) \xrightarrow{L(g)} LR(B) \xrightarrow{\varepsilon_B} B$$

$$(L(A) \xrightarrow{f} B)^\# = A \xrightarrow{\eta_A} RL(A) \xrightarrow{R(f)} R(B)$$

Any adjunction $L \dashv R$ induces a comonad $\mathbb{C} = LR$ and, conversely, every comonad induces adjunctions from which it can be obtained back again. However, there is a whole category of adjunctions inducing the same comonad, with an initial object (the Kleisli adjunction) and a terminal object (the Eilenberg–Moore adjunction), which we describe below.

Kleisli category

Given a comonad $\mathbb{C}$ on a category $\mathscr{C}$, the **Kleisli category** [75] for $\mathbb{C}$ on $\mathscr{C}$ is the category $\mathsf{Kl}(\mathbb{C})$ where

- $\mathrm{Ob}(\mathsf{Kl}(\mathbb{C})) = \mathrm{Ob}(\mathscr{C})$, i.e. the objects are the same as in $\mathscr{C}$, and
- $\hom_{\mathsf{Kl}(\mathbb{C})}(A, B) = \hom_{\mathscr{C}}(\mathbb{C}(A), B)$.

To avoid ambiguity in this subsection, we shall write $f : A \rightsquigarrow B$ if we intend $f : \mathbb{C}(A) \to B$ to be understood as a **Kleisli morphism** from A to B rather than a morphism in $\mathscr{C}$. Furthermore, we use the distinct notation $g \bullet f$ for the Kleisli composite of morphisms $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ in $\mathsf{Kl}(\mathbb{C})$, defined below.

Using this notation, the rest of the category structure is given as follows:

- The identity morphism $A \rightsquigarrow A$ in $\mathsf{Kl}(\mathbb{C})$ is the counit component $\varepsilon_A : \mathbb{C}(A) \to A$ in $\mathscr{C}$.
- The composite of morphisms $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ in $\mathsf{Kl}(\mathbb{C})$ is given by
$$g \bullet f = \mathbb{C}(A) \xrightarrow{f^*} \mathbb{C}(B) \xrightarrow{g} C.$$

The axioms for a comonad in the coextension form ensure that this is a well-defined category.

Finally, the adjunction between $\mathscr{C}$ and $\mathsf{Kl}(\mathbb{C})$ has the following form

$$\begin{array}{c} \mathsf{Kl}(\mathbb{C}) \\ U_{\mathbb{C}} \left(\dashv \right) F_{\mathbb{C}} \\ \mathscr{C} \end{array}$$

where

- $F_{\mathbb{C}}(A) = A$ and $F_{\mathbb{C}}(A \xrightarrow{f} B) = \mathbb{C}(A) \xrightarrow{\varepsilon_A} A \xrightarrow{f} B$,
- $U_{\mathbb{C}}(A) = \mathbb{C}(A)$ and $U_{\mathbb{C}}(A \overset{f}{\rightsquigarrow} B) = \mathbb{C}(A) \xrightarrow{f^*} \mathbb{C}(B)$,
- $(U_{\mathbb{C}}(A) \xrightarrow{f} B)^\# = A \overset{f}{\rightsquigarrow} F_{\mathbb{C}}(B)$, and
  $(A \overset{f}{\rightsquigarrow} F_{\mathbb{C}}(B))^\flat = U_{\mathbb{C}}(A) \xrightarrow{f} B$.

Observe that the last item defining $(\text{-})^\#$ and $(\text{-})^\flat$ is only about interpreting the same morphism in a different category. For example, a morphism $U_{\mathbb{C}}(A) \to B$ is simply a morphism $\mathbb{C}(A) \to B$ in $\mathscr{C}$ which, in turn, is also morphism $A \rightsquigarrow F_{\mathbb{C}}(B)$.

Remark 2.2.1 Later in the text we prefer to work with $\mathsf{Kl}(\mathbb{C})$ as the category we map into with functor $F_{\mathbb{C}}$ and, therefore, no confusion should arise about morphisms belonging to $\mathscr{C}$ or $\mathsf{Kl}(\mathbb{C})$ anymore. For this reason we often simply write $F_{\mathbb{C}}(A) \to F_{\mathbb{C}}(B)$ instead of $F_{\mathbb{C}}(A) \rightsquigarrow F_{\mathbb{C}}(B)$.

Eilenberg–Moore category

On the other end of the spectrum is the **Eilenberg–Moore (EM) category** $\mathsf{EM}(\mathbb{C})$ for a comonad $(\mathbb{C}, \varepsilon, \delta)$ on $\mathscr{C}$, defined as follows.

Its objects are **$\mathbb{C}$-coalgebras** or simply just **coalgebras**, that is, pairs (A, α) where $A \in \mathrm{Ob}(\mathscr{C})$ and $\alpha \colon A \to \mathbb{C}(A)$ in $\mathscr{C}$ making the following two diagrams commute.

$$\begin{array}{ccc} A & & \\ {\scriptstyle\alpha}\downarrow & \searrow^{\mathrm{id}} & \\ \mathbb{C}(A) & \xrightarrow{\varepsilon_A} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha} & \mathbb{C}(A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta_A} \\ \mathbb{C}(A) & \xrightarrow{\mathbb{C}(\alpha)} & \mathbb{C}(\mathbb{C}(A)) \end{array} \tag{2.6}$$

Morphisms of coalgebras $f \colon (A, \alpha) \to (B, \beta)$ are morphisms $f \colon A \to B$ in $\mathscr{C}$ such that the following square commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbb{C}(A) & \xrightarrow{\mathbb{C}(f)} & \mathbb{C}(B) \end{array} \tag{2.7}$$

We write $\mathsf{EM}(\mathbb{C})$ for the Eilenberg–Moore category of coalgebras of $\mathbb{C}$ and their morphisms.

The induced adjunction

$$\begin{array}{c} \mathsf{EM}(\mathbb{C}) \\ U^{\mathbb{C}} \big(\dashv \big) F^{\mathbb{C}} \\ \mathscr{C} \end{array}$$

is given by the following data:

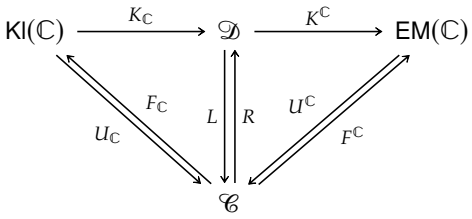
- $U^{\mathbb{C}}(A, \alpha) \to A$ and $U^{\mathbb{C}}((A, \alpha) \xrightarrow{f} (B, \beta) = A \xrightarrow{f} B$,
- $F^{\mathbb{C}}(A) = (\mathbb{C}(A),\ \mathbb{C}(A) \xrightarrow{\delta_A} \mathbb{C}^2(A))$ and
  $F^{\mathbb{C}}(A \xrightarrow{f} B) = (\mathbb{C}(A), \delta_A) \xrightarrow{\mathbb{C}(f)} (\mathbb{C}(B), \delta_B)$,
- $(U^{\mathbb{C}}(A, \alpha) \xrightarrow{f} B)^\# = (A, \alpha) \xrightarrow{\mathbb{C}(f)\cdot\alpha} (\mathbb{C}(B), \delta_B)$, and
  $((A, \alpha) \xrightarrow{h} F^{\mathbb{C}}(B))^\flat = A \xrightarrow{h} \mathbb{C}(B) \xrightarrow{\varepsilon_B} B$.

The coalgebra $F^{\mathbb{C}}(A) = (\mathbb{C}(A), \delta_A)$ is called the **cofree coalgebra on** A. The fact that δ_A is a correct coalgebra structure follows from (2.5). Cofree coalgebras are very important for the theory of game comonads, as we demonstrate in the following sections.

Comparison functors

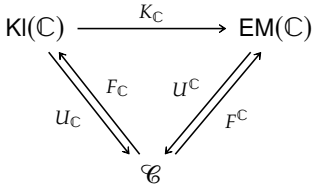
The Kleisli and EM adjunctions are the initial and terminal adjunctions inducing the same comonads. This means that for any adjunction $L \dashv R$ inducing $\mathbb{C}$[6] there are unique comparison functors $K_{\mathbb{C}}$ and $K^{\mathbb{C}}$ making the diagram below commute.

6: That is, if $\mathbb{C} = LR$, the counit $LR \Rightarrow \mathrm{Id}$ of $L \dashv R$ is the counit ε of $\mathbb{C}$ and $L\eta R$ is the comultiplication δ of $\mathbb{C}$ where $\eta\colon \mathrm{Id} \Rightarrow RL$ is unit of $L \dashv R$.



In case $K^{\mathbb{C}}$ is an equivalence of categories, the adjunction $L \dashv R$ or simply just the functor L is called **comonadic**.

In this text, we only need an explicit description of the **comparison functor** $K_{\mathbb{C}}$ between the Kleisli and EM categories:



(2.8)

The functor $K_{\mathbb{C}}$ is given as $K_{\mathbb{C}}(A) = (\mathbb{C}(A), \delta_A)$ and $K_{\mathbb{C}}(A \overset{f}{\rightsquigarrow} B) = f^*$. Furthermore, note that $K_{\mathbb{C}}$ is fully faithful.

2.3 Game comonads

Having reviewed the basics of the theory of comonads, we are now ready to define some examples of game comonads, based on the games discussed in Section 2.1. Then, in later section, we show how their Kleisli and Eilenberg–Moore categories show up when talking about the corresponding logic fragments.

To this end, given a relational signature σ, define

$$\mathcal{S}tr(\sigma)$$

to be the category of σ-structures and their homomorphisms where, for σ-structures A, B, a function $f\colon A \to B$ between their underlying sets is a **homomorphism** if

$$R^A(a_1, \ldots, a_n) \quad \text{implies} \quad R^B(f(a_1), \ldots, f(a_n)) \tag{2.9}$$

for any n-ary $R \in \sigma$ and $a_1, \ldots, a_n \in A$.

Similarly, we also define the category

$$\mathcal{S}tr_*(\sigma)$$

which is a category of pointed σ-structures and their homomorphisms which, for pointed σ-structures (A, a_0) and (B, b_0), are homomorphisms of σ-structures $f : A \to B$ such that $f(a_0) = b_0$.

Remark 2.3.1 When σ is a modal signature, $\mathcal{S}tr_*(\sigma)$ is the category of Kripke frames. Note that the morphisms of $\mathcal{S}tr_*(\sigma)$, as defined above, are not *p-morphisms* i.e. *bounded morphisms*, which are usually the chosen morphisms of Kripke frames.

Next we overview three main comonads, as defined in [14].

Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$

For any relational signature σ and for any integer k, there is a comonad[7] $\mathbb{E}_k$ on $\mathcal{S}tr(\sigma)$ defined as follows. The universe of $\mathbb{E}_k(A)$ consists of non-empty words/sequences

$$[a_1, \ldots, a_n]$$

with $a_1, \ldots, a_n$ from A and $n \leq k$. The counit

$$\varepsilon_A : \mathbb{E}_k(A) \to A$$

maps $[a_1, \ldots, a_n]$ to a_n.

For an m-ary relational symbol $R \in \sigma$ and $s_1, \ldots, s_m \in \mathbb{E}_k(A)$, set $R^{\mathbb{E}_k(A)}(s_1, \ldots, s_m)$ iff:

1. the s_i are *pairwise comparable in the prefix order* $\sqsubseteq$ *on words*, i.e. for every i, j either s_i is a prefix[8] of s_j or vice versa, and
2. $R^A(\varepsilon_A(s_1), \ldots, \varepsilon_A(s_m))$.

The coextension of $h : \mathbb{E}_k(A) \to B$ is

$$h^*([a_1, \ldots, a_n]) = [h([a_1]), h([a_1, a_2]), \ldots, h([a_1, \ldots, a_n])].$$

In other words, $h^*([a_1, \ldots, a_n])$ is the word $[h(s_1), \ldots, h(s_n)]$ of the same length as the input where $s_i = [a_1, \ldots, a_i]$.

Remark 2.3.2 Intuitively, $\mathbb{E}_k(A)$ encodes all positions on A during a k-round Ehrenfeucht–Fraïssé game. To be more precise, the elements of $\mathbb{E}_k(A)$ encode all *histories of moves* in the game; however, the difference between positions and histories does not matter in the case of the Ehrenfeucht–Fraïssé game. The distinction shows up later, e.g. when we describe the pebble comonads below.

In more detail, in a game between A and B, after n moves, the position is a pair $(\overline{a}, \overline{b})$, and hence the sequence $\overline{a} = [a_1, \ldots, a_n]$ corresponds precisely to one element of $\mathbb{E}_k A$. The counit $\varepsilon_A : \mathbb{E}_k A \to A$ extracts the last move out of a position and, finally, the winning condition for Duplicator is reflected in the relational structure of $\mathbb{E}_k A$. Namely, what matters at the position $\overline{a}$ for the Duplicator's winning condition at the end of the round, i.e. for the partial equivalence from (2.2), are the relations between elements $a_1, \ldots, a_n$ in A. Since every element encodes a position together with its last played move, we encode these relations between elements $a_1, \ldots, a_n$ in the positions leading up to the position $\overline{a}$, that is, in

$$[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_n]$$

7: In fact, we have a *graded comonad*, that is a collection of comonads $(\mathbb{C}_k)_k$ such that, for $k \leq l$, $\mathbb{C}_k$ is a sub-comonad of $\mathbb{C}_l$ since we have a monomorphic *comonad morphism* $\mathbb{C}_k \Rightarrow \mathbb{C}_l$, in sense of Subsection"Comonad morphisms" on page 156.

8: We say that $\overline{a} = [a_1, \ldots, a_n]$ is a **prefix** of $\overline{a}' = [a'_1, \ldots, a'_m]$, and write $\overline{a} \sqsubseteq \overline{a}'$, if $n \leq m$ and, for every $i \leq n$, we have $a_i = a'_i$.

so that this sequence is in partial correspondence with the substructure of A on the last played moves, that is, with

$$\varepsilon([a_1]) = a_1,\ \varepsilon([a_1, a_2]) = a_2,\ \ldots,\ \varepsilon([a_1, \ldots, a_n]) = a_n.$$

In other words, the relations in $\mathbb{E}_k(A)$ encode precisely the relations that matter for the winning condition in the same game. That is, only the positions comparable in the prefix order can appear in some tuple of some relation. The fact that $\overline{a}, \overline{a}' \in \mathbb{E}_k(A)$ where neither $\overline{a} \not\sqsubseteq \overline{a}'$ nor $\overline{a}' \not\sqsubseteq \overline{a}$ cannot be in a relation together corresponds to the fact that $\overline{a}$ and $\overline{a}'$ are positions of unrelated game plays.

Pebbling comonad $\mathbb{P}_k$

As with the Ehrenfeucht–Fraïssé comonads, the pebbling comonads are defined over the category $\mathcal{S}tr(\sigma)$. The universe of $\mathbb{P}_k(A)$ consists of non-empty words

$$[(p_1, a_1),\ \ldots,\ (p_n, a_n)]$$

of unrestricted (finite) length such that $a_1, \ldots, a_n \in A$ and $p_1, \ldots, p_n$ are *pebble indexes*, that is, $p_1, \ldots, p_n \in \{0, \ldots, k-1\}$.

As before, we define the counit $\varepsilon_A \colon \mathbb{P}_k A \to A$ as the map which extracts the last played position, that is as $[(p_1, a_1), \ldots, (p_n, a_n)] \mapsto a_n$.

For m-ary $R \in \sigma$, set $R^{\mathbb{P}_k(A)}(s_1, \ldots, s_m)$ iff:

1. The s_i are pairwise comparable in the prefix order $\sqsubseteq$ on words.
2. For s_i, s_j with $s_i \sqsubseteq s_j$, the *last played pebble of s_i did not move in the rest of s_j*, that is, assuming the length of s_i is n and $s_j = [(p_1, a_1), \ldots, (p_m, a_m)]$, we require that $p_n \notin \{p_{n+1}, \ldots, p_m\}$.
3. $R^A(\varepsilon_A(s_1), \ldots, \varepsilon_A(s_m))$.

For $h \colon \mathbb{P}_k(A) \to B$, the coextension h^* acts on words as:

$$h^*([(p_1, a_1), \ldots, (p_n, a_n)]) \ = \ [(p_1, h(s_1)), \ldots, (p_n, h(s_n))]$$

where $s_i = [(p_1, a_1), \ldots, (p_i, a_i)]$ for all $i \in \{1, \ldots, n\}$.

Remark 2.3.3 As in Remark 2.3.2, we understand elements of $\mathbb{P}_k A$ as histories of plays on A in the k-pebble game. In particular, a sequence $[(p_1, a_1),\ \ldots,\ (p_n, a_n)]$ encodes a history of pebble moves where in round 1 pebble p_1 is placed on a_1, in round 2 pebble p_2 is placed on a_2, and so on. Then, the relational structure of $\mathbb{P}_k A$ reflects the winning condition of Duplicator in round n, given the last position of each pebble in the game played according to history $[(p_1, a_1),\ \ldots,\ (p_n, a_n)]$.

Modal comonad $\mathbb{M}_k$

The last of our principal examples of game comonads are modal comonads $\mathbb{M}_k$, which are defined over the category of pointed Kripke structures $\mathcal{S}tr_*(\sigma)$, given a modal signature σ. The universe of $\mathbb{M}_k(A, a_0)$ consists of paths in A starting from a_0, i.e. sequences

$$a_0 \xrightarrow{R_1} a_1 \xrightarrow{R_2} a_2 \xrightarrow{R_3} \ldots \xrightarrow{R_n} a_n$$

such that $n \leq k$ and, for every i, $R_i \in \sigma$ and $R_i^A(a_{i-1}, a_i)$. In case we need a more compact description of these paths from a_0, we just write them as sequences $[a_0, R_1, a_1, \ldots, R_n, a_n]$.

The designated point of $\mathbb{M}_k(A, a_0)$ is $[a_0]$. The counit and coextension act as in the previous examples, that is, $\varepsilon_A([a_0, R_1, a_1, \ldots, R_n, a_n]) = a_n$ and, for $f : \mathbb{M}_k(A, a_0) \to (B, b_0)$,

$$f^*([a_0, \ldots, R_n, a_n]) = [f(s_0), R_1, f(s_1), \ldots, R_n, f(s_n)]$$

where $s_i = [b_0, \ldots, R_i, b_i]$. Namely, $f(s_0) = h([a_0]) = b_0$ since f is a pointed homomorphism.

For a binary $R \in \sigma$, the relation $R^{\mathbb{M}_k(A,a_0)}$ consists of the pairs (s', s) where s is the extension of s' by $a_{n-1} \xrightarrow{R} a_n$ in $\mathbb{M}_k(A, a_0)$. For a unary $P \in \sigma$, $P^{\mathbb{M}_k(A,a_0)}$ consists of paths ending in a position where P^A holds, i.e. $P^{\mathbb{M}_k(A,a_0)}(s)$ iff $P^A(a_n)$.

Remark 2.3.4 Observe that this time, the histories of moves corresponding to elements of $\mathbb{M}_k A$ also encode which relation $R \in \sigma$ is taken in which move, similarly to how $\mathbb{P}_k A$ encodes which pebble is moved in which move.

2.4 Comonads and logic, positive existential and counting fragments

We now use our comonads to express the existence of a winning strategy of the model comparison games (cf. Remarks 2.3.2, 2.3.3 and 2.3.4). This should come as no surprise since the rules of the games were used to motivate the definitions of our comonads.

Namely, we obtain the following properties automatically, just from the definitions.

Lemma 2.4.1 ([3, 14]) *Given structures A, B, we have the following:*

1. *$\mathbb{E}_k A \to B$ exists iff Duplicator has a winning strategy in the k-round forth-only Ehrenfeucht–Fraïssé game from A to B.*
2. *$\mathbb{P}_k A \to B$ exists iff Duplicator has a winning strategy in the k-pebble forth-only game from A to B.*

Given pointed Kripke frames (A, a_0) and (B, b_0), we have the following:

3. *$\mathbb{M}_k(A, a_0) \to (B, b_0)$ exists iff Duplicator has a winning strategy in the k-round forth-only simulation game from (A, a_0) to (B, b_0).*

Consequently, using the correspondence of games and logical refinements, we obtain the following reformulation.

Theorem 2.4.2 ([3, 14]) *Given structures A, B, we have the following:*

1. *$\mathbb{E}_k A \to B$ exists iff $A \Rrightarrow_{\exists^+ \mathrm{FO}_k} B$.*
2. *$\mathbb{P}_k A \to B$ exists iff $A \Rrightarrow_{\exists^+ \mathcal{L}^k_{\infty\omega}} B$.*

Given pointed Kripke frames (A, a_0) and (B, b_0), we have the following:

3. $\mathbb{M}_k(A, a_0) \to (B, b_0)$ *exists iff* $(A, a_0) \Rrightarrow_{\exists^+\mathtt{ML}_k} (B, b_0)$.

Proof. (1) follows from Proposition 2.1.7, (2) follows from Proposition 2.1.10, and (3) follows from Proposition 2.1.12. □

Recall that, for any comonad $\mathbb{C}$, a morphism $\mathbb{C}(A) \to B$ is the same as a morphism $F_{\mathbb{C}}(A) \to F_{\mathbb{C}}(B)$. Therefore, we see that the properties on the left-hand side of equivalences in Theorem 2.4.2 can be expressed relative to functor $F_{\mathbb{C}}$.

Is natural to ask what happens when we replace the existence of a morphism $F_{\mathbb{C}}(A) \to F_{\mathbb{C}}(B)$ in the above statement with a stronger relation, namely the isomorphism of the objects in the Kleisli category. This corresponds to the existence of a winning strategy for Duplicator in the bijective versions of these games, thereby yielding the corresponding logical equivalences.

Theorem 2.4.3 ([3, 14]) *Given finite structures A, B, we have the following:*

1. $F_{\mathbb{E}_k}(A) \cong F_{\mathbb{E}_k}(B)$ *iff* $A \equiv_{\mathsf{C}_k} B$.
2. $F_{\mathbb{P}_k}(A) \cong F_{\mathbb{P}_k}(B)$ *iff* $A \equiv_{\mathsf{C}^k} B$.

Given finite pointed Kripke frames (A, a_0) and (B, b_0), we have the following:

3. $F_{\mathbb{M}_k}(A, a_0) \cong F_{\mathbb{M}_k}(B, b_0)$ *iff* $(A, a_0) \equiv_{\#\mathtt{ML}_k} (B, b_0)$.

The next natural step is to capture logical equivalences with respect to **standard fragments** $\mathtt{FO}_k$, $\mathcal{L}^k_{\infty\omega}$, and $\mathtt{ML}_k$. Although this can be done in the same setting (see [3] for the $\mathbb{P}_k$ case), the Kleisli category $\mathsf{Kl}(\mathbb{C})$ for our game comonads $\mathbb{C}$ is not the most suitable framework for this. To this end, it is better to move to the more general setting of the Eilenberg–Moore category of coalgebras.

In fact, by moving from the Kleisli category to the EM category, we can still express the same properties, as in Theorems 2.4.2 and 2.4.3 since the left-hand sides of the said theorems can also be expressed entirely in terms of cofree objects, i.e. images of $F^{\mathbb{C}}$ in the Eilenberg–Moore category $\mathsf{EM}(\mathbb{C})$.

Lemma 2.4.4 *Given objects A, B in a category $\mathscr{C}$ with a comonad $\mathbb{C}$, the following statements are equivalent.*

- $\mathbb{C}(A) \to B$ *exists in* $\mathscr{C}$
- $F_{\mathbb{C}}(A) \to F_{\mathbb{C}}(B)$ *exists in* $\mathsf{Kl}(\mathbb{C})$
- $F^{\mathbb{C}}(A) \to F^{\mathbb{C}}(B)$ *exists in* $\mathsf{EM}(\mathbb{C})$

Similarly, we also have the equivalence of the following statements:

- $F_{\mathbb{C}}(A) \cong F_{\mathbb{C}}(B)$ *in* $\mathsf{Kl}(\mathbb{C})$
- $F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B)$ *in* $\mathsf{EM}(\mathbb{C})$

Proof. This folklore observation uses the fact that the comparison functor

$K_{\mathbb{C}}$ is fully faithful and commutes with cofree functors, cf. (2.8). Indeed,

$$\begin{aligned}\hom_{\mathscr{C}}(\mathbb{C}(A), B) &= \hom_{\mathsf{Kl}(\mathbb{C})}(F_{\mathbb{C}}(A), F_{\mathbb{C}}(B)) \\ &\cong \hom_{\mathsf{EM}(\mathbb{C})}(K_{\mathbb{C}}(F_{\mathbb{C}}(A)), K_{\mathbb{C}}(F_{\mathbb{C}}(B))) \\ &= \hom_{\mathsf{EM}(\mathbb{C})}(F^{\mathbb{C}}(A), F^{\mathbb{C}}(B)).\end{aligned}$$

Furthermore, $F_{\mathbb{C}}(A) \cong F_{\mathbb{C}}(B)$ and $F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B)$ are equivalent since they are both expressed in terms of morphisms in the image of $K_{\mathbb{C}}$. Indeed, for morphisms $f\colon F_{\mathbb{C}}(A) \rightsquigarrow F_{\mathbb{C}}(B)$ and $g\colon F_{\mathbb{C}}(B) \rightsquigarrow F_{\mathbb{C}}(A)$ we have that $g \bullet f = \mathrm{id}$ iff $K_{\mathbb{C}}(g) \cdot K_{\mathbb{C}}(f) = \mathrm{id}$. □

We show how to relate standard logic fragments with the EM categories in Section 2.6 below. However, before we can do so, we have to understand the EM categories of our three principal game comonads.

2.5 Coalgebras and combinatorial properties

Following [14], we give a concrete description of the categories of coalgebras for our principal examples of game comonads. To this end, we say that a poset $(F, \leq)$ is a **forest** if, for every $x \in F$, the set

$$\downarrow x \coloneqq \{y \in F \mid y \leq x\}$$

is a finite chain. The **height of an element** x of a forest F, denoted by $\mathrm{ht}(x)$, is the cardinality of the set $\downarrow x \setminus \{x\}$. The elements of height 0 are called **roots**. A **tree** is a forest with precisely one root. The **height of the forest** F is the supremum of the set $\{\mathrm{ht}(x) \mid x \in F\}$ in $\mathbb{N} \cup \{\infty\}$. Finally, given elements x, y of a poset, we write

$$x \prec y,$$

to mean that y **covers** x, i.e. $x < y$ and whenever $x \leq z \leq y$ then either $x = z$ or $y = z$.

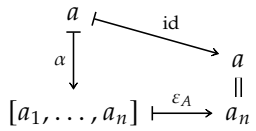
A function $f\colon F \to F'$ between forests is a **forest morphism** if it is monotone and preserves heights, i.e. the chains $\downarrow x$ and $\downarrow f(x)$ have the same length for each $x \in F$.[9] We often refer to forest morphisms between trees as **tree morphisms**.

9: Equivalently, a function $f\colon F \to F'$ is a forest morphism if and only if it preserves roots and the covering relation $\prec$.

Special case. To give an intuition for the description of coalgebras of our game comonads below, we look at a special case of $\mathbb{E}_k$ on $\mathcal{S}tr(\sigma)$ where the signature σ is empty. Then $\mathcal{S}tr(\sigma) = \mathbf{Set}$ and, in this case, we denote $\mathbb{E}_k$ by W_k, to emphasise that W_k is defined on the category of sets. In particular, the comonad W_k sends a set A to the set of words/sequences $[a_1, \ldots, a_n]$ where $a_1, \ldots, a_n \in A$ and $1 \leq n \leq k$.

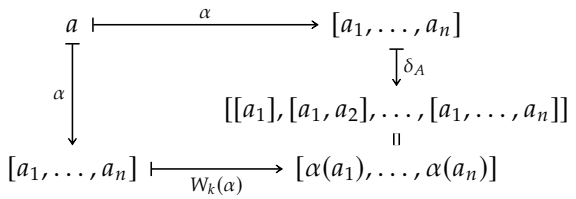
Lemma 2.5.1 *The Eilenberg–Moore category* $\mathsf{EM}(W_k)$ *of coalgebras of* W_k *is isomorphic to the category of forests of height* $\leq k$ *and forest morphisms.*

Proof. Given a coalgebra $(A, \alpha\colon A \to W_k(A))$, observe that the triangle law in (2.6) implies that



i.e. that, for $a \in A$, $\alpha(a) = [a_1, \ldots, a_n]$ satisfies that $a_n = a$.

On the other hand, the square law in (2.6) gives us that



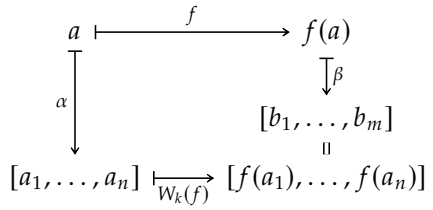
i.e. that, for every $i = 1, \ldots, n$, $\alpha(a_i) = [a_1, \ldots, a_i]$.

Next, define a partial order $\sqsubseteq_\alpha \subseteq A \times A$ by setting $a \sqsubseteq_\alpha b$ iff $\alpha(a) \sqsubseteq \alpha(b)$, i.e. the sequence $\alpha(a)$ *is a prefix of* $\alpha(b)$.

We see that the coalgebra laws (2.6) imply that $\downarrow a = \{a' \in A \mid a' \sqsubseteq_\alpha a\}$ is a finite chain since $\downarrow a = \{a_1, \ldots, a_n\}$ where $\alpha(a) = [a_1, \ldots, a_n]$. By definition, this means that $(A, \sqsubseteq_\alpha)$ is a forest of height $\leq k$.

Conversely, given a forest poset $(A, \leq)$ of height $\leq k$, define $\alpha_\leq\colon A \to W_k(A)$ by sending a to the list $[a_1, \ldots, a_n]$ where $\downarrow a$ is the chain $a_1 \prec a_2 \prec \cdots \prec a_n = a$. It is immediate that the coalgebra axioms (2.6) are satisfied. Furthermore, we also see that $(\sqsubseteq_{\alpha_\leq}) = (\leq)$ and $\alpha_{\sqsubseteq_\alpha} = \alpha$.

Next, we look at morphisms. Take a function $f\colon A \to B$ and coalgebra structures (A, α) and (B, β) on the sets A and B, respectively. Asking f to be a coalgebra morphism (2.7) amounts to requiring that, for every $a \in A$,



Therefore, we must have that $n = m$, i.e. that $\mathrm{ht}(a) = \mathrm{ht}(f(a))$. Also, if $a' \sqsubseteq_\alpha a$ i.e. if a' is equal to some a_i in $\alpha(a) = [a_1, \ldots, a_n]$, then $W_k(f)(\alpha(a')) = [f(a_1), \ldots, f(a_i)] = [b_1, \ldots, b_i] = f(a')$, i.e. $f(a') \sqsubseteq_\beta f(a)$. In other words, f is a forest morphism.

Conversely, assume that f is a forest morphism with respect to the forest orders $\sqsubseteq_\alpha$ and $\sqsubseteq_\beta$. Since f preserves heights, we have that $\alpha(a) = [a_1, \ldots, a_n]$ and $\beta(f(a)) = [b_1, \ldots, b_m]$ have the same lengths, i.e. that $n = m$. Next, let a' be equal to some a_i, i.e. $a' \sqsubseteq_\alpha a$. Since f is monotone, we have that $f(a') \sqsubseteq_\beta f(a)$ i.e. that $f(a')$ is equal to some b_j. In fact, $i = j$ because f preserves heights. Therefore, $[f(a_1), \ldots, f(a_n)] = [b_1, \ldots, b_n]$, as required. □

Coalgebras of game comonads. By example of the previous lemma, we can derive **concrete descriptions** of categories of coalgebras $\mathsf{EM}(\mathbb{C})$ of our game comonads $\mathbb{C}$ as categories of certain forest-ordered structures. To this end, for a comonad $\mathbb{C}$ among $\mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$ and a coalgebra $\alpha\colon A \to \mathbb{C}A$, set

$$a \sqsubseteq_\alpha b \iff \text{the sequence } \alpha(a) \text{ is a prefix of } \alpha(b). \tag{2.10}$$

Then, as in Lemma 2.5.1, the poset $(A, \sqsubseteq_\alpha)$ is a forest and coalgebra morphisms are forest morphisms. Next, following [14], we describe the remaining parts of the characterisation of the categories $\mathsf{EM}(\mathbb{C})$:

- The objects of $\mathsf{EM}(\mathbb{E}_k)$ are **forest-ordered σ-structures $(A, \leq)$ of height $\leq k$** such that if $R^A(x_1, \ldots, x_n)$, then $x_1, \ldots, x_n$ are pairwise comparable in the order $\leq$.
  Coalgebra morphisms correspond precisely to homomorphisms of the underlying relational structures that are also forest morphisms.
- The objects in $\mathsf{EM}(\mathbb{P}_k)$ can be described as **forest-ordered σ-structures** $(A, \leq, p)$ with a **pebble function** $p\colon A \to \{1, \ldots, k\}$ such that $R^A(x_1, \ldots, x_n)$ implies that (i) $x_1, \ldots, x_n$ are pairwise comparable and (ii) if $x_i < x_j$ then $p(x_i) \notin \{p(y) \mid x_i < y \leq x_j\}$.
  Morphisms in $\mathsf{EM}(\mathbb{P}_k)$ are the homomorphisms of relational structures that preserve the pebble functions and are also forest morphisms.
- Since $\mathbb{M}_k$ is an idempotent comonad,[10] the category $\mathsf{EM}(\mathbb{M}_k)$ is isomorphic to the full subcategory of $\mathcal{S}tr_*(\sigma)$ consisting of **synchronisation trees** [70] of height $\leq k$, i.e. tree-ordered σ-structures $(A, \leq)$ such that $a \lessdot b$ iff $R^A(a, b)$ for some relation R in σ.
  In particular, Kripke morphisms between synchronisation trees are automatically forest morphisms.

10: Recall that a comonad $\mathbb{C}$ is **idempotent** if the comultiplication $\delta_A\colon \mathbb{C}(A) \to \mathbb{C}(\mathbb{C}(A))$ is an isomorphism, for any A.

Remark 2.5.2 Given the concrete description $(A, \leq)$ of an object of $\mathsf{EM}(\mathbb{E}_k)$, as in Lemma 2.5.1 we define the corresponding coalgebra $\alpha\colon A \to \mathbb{E}_k A$, by setting $\alpha(a) = [a_1, \ldots, a_n]$ where $\downarrow a$ is the chain $a_1 \lessdot \cdots \lessdot a_n = a$. The requirement that $R^A(x_1, \ldots, x_n)$ implies that $x_1, \ldots, x_n$ are pairwise comparable is to ensure that the coalgebra map α is a homomorphism of σ-structures.

Similarly, for $(A, \leq, p)$ corresponding to an object of $\mathsf{EM}(\mathbb{P}_k)$, the induced coalgebra $\alpha\colon A \to \mathbb{P}_k A$ maps a to $[(p(a_1), a_1), \ldots, (p(a_n), a_n)]$ where, again, $\downarrow a$ is the chain $a_1 \lessdot \cdots \lessdot a_n = a$.

Remark 2.5.3 Note that, given any structure A, the induced forest order $\sqsubseteq_{\delta_A}$ for the cofree coalgebra $F^{\mathbb{E}_k}(A) = (\mathbb{E}_k(A), \delta_A)$ coincides with the prefix order on sequences, that is, for $\overline{a}, \overline{b} \in \mathbb{E}_k(A)$, $\overline{a} \sqsubseteq_{\delta_A} \overline{b}$ iff $\overline{a}$ is the prefix of $\overline{b}$.

A similar fact follows from the other comonads $\mathbb{P}_k$ and $\mathbb{M}_k$. Moreover, the pebble function $p\colon \mathbb{P}_k A \to [k]$ for the free coalgebra $F^{\mathbb{P}_k}(A)$ is the map that extracts the last used pebble, i.e. $p([(p_1, a_1), \ldots, (p_n, a_n)]) = p_n$.

In fact, given any $\mathbb{P}_k$-coalgebra (A, α), the corresponding pebble function $A \to [k]$ is obtained as the composition of $\alpha\colon A \to \mathbb{P}_k(A)$ with the pebble function for the cofree coalgebra $F^{\mathbb{P}_k}(A)$, defined above.

Coalgebras and graph parameters. It turns out that coalgebras provide an equivalent description of decompositions of relational structures, similar to familiar notions from graph theory. To this end, given an object A in a category $\mathscr{C}$ with a comonad $\mathbb{C}$, we say that A **admits a $\mathbb{C}$-coalgebra** if there exists an Eilenberg–Moore coalgebra $A \to \mathbb{C}(A)$, i.e. if $A \cong U^{\mathbb{C}}(X)$ for some $X \in \mathsf{EM}(\mathbb{C})$.

Proposition 2.5.4 ([14]) *Given a structure A and a pointed Kripke frame (B, b_0), we have the following:*

- *A admits a $\mathbb{E}_k$-coalgebra iff it has tree-depth $\leq k$, in sense of [98],*
- *A admits a $\mathbb{P}_k$-coalgebra iff it has tree-width $<k$, in sense of [98], and*
- *(B, b_0) admits a $\mathbb{M}_k$-coalgebra iff it is a synchronisation tree of height $\leq k$.*

In fact, many more combinatorial notions can be captured by comonads, see Table 2.1 and Section 6.2 below.

2.6 Comonads and standard logic fragments

Using the concrete description of the category $\mathsf{EM}(\mathbb{C})$, with $\mathbb{C}$ being one of the game comonads $\mathbb{M}_k$, $\mathbb{E}_k$, or $\mathbb{P}_k$, we can express logical equivalence with respect to the corresponding standard fragments. We do so by defining an abstract back-and-forth game for objects in $\mathsf{EM}(\mathbb{C})$.

The game is defined in terms of the forest order $\sqsubseteq_\alpha$ associated to any coalgebra (A, α), cf. (2.10). Namely, given coalgebras (A, α) and (B, β) in $\mathsf{EM}(\mathbb{C})$, where $\mathbb{C}$ is one of our game comonads, we define a back-and-forth Spoiler–Duplicator game as follows.

The positions are pairs $(a, b) \in A \times B$ of elements of the same height, with respect to the associate orders $\sqsubseteq_\alpha$ and $\sqsubseteq_\beta$, i.e. such that $\mathrm{ht}(a) = \mathrm{ht}(b)$. The initial position is the pair of roots. If the position in the game is (a, b), then either

- Spoiler chooses some a' such that $a \sqsubseteq_\alpha a'$ and Duplicator chooses some b' such that $b \sqsubseteq_\beta b'$; or
- Spoiler chooses some b' such that $b \sqsubseteq_\beta b'$ and Duplicator chooses some a' such that $a \sqsubseteq_\alpha a'$.

At every position during the game, Duplicator must maintain that the induced substructures $\downarrow a \subseteq A$ and $\downarrow b \subseteq B$ are isomorphic (including the forest order and pebble function). If Duplicator wins the round, the position at the next round is (a', b').

We say that coalgebras (A, α) and (B, β) are **back-and-forth equivalent**, and write

$$(A, \alpha) \rightleftharpoons (B, \beta),$$

if Duplicator has a winning strategy in this game.

Using this generic game, we obtain that the standard logic fragments can be described using the same game played between cofree coalgebras.

Theorem 2.6.1 ([14]) *Given structures A, B, we have the following:*

1. $F^{\mathbb{E}_k}(A) \rightleftharpoons F^{\mathbb{E}_k}(B)$ *iff* $A \equiv_{\mathtt{FO}_k} B$.
2. $F^{\mathbb{P}_k}(A) \rightleftharpoons F^{\mathbb{P}_k}(B)$ *iff* $A \equiv_{\mathscr{L}^k_{\infty\omega}} B$.

Given pointed Kripke frames (A, a_0) and (B, b_0), we have the following:

3. $F^{\mathbb{M}_k}(A, a_0) \rightleftharpoons F^{\mathbb{M}_k}(B, b_0)$ *iff* $(A, a_0) \equiv_{\mathtt{ML}_k} (B, b_0)$.

Proof. We implicitly use Propositions 2.1.4, 2.1.10 and 2.1.12 that link games and the corresponding logic fragments. Therefore, all that needs to be checked is that the available moves and winning conditions are exactly the same as in the corresponding game.

(1) Observe that a position in $F^{\mathbb{E}_k}(A) \times F^{\mathbb{E}_k}(B) = \mathbb{E}_k(A) \times \mathbb{E}_k(B)$ is a pair of words $(\overline{a}, \overline{b})$. Further, following Remark 2.5.3, observe that the downset $\downarrow\overline{a}$ in the $\sqsubseteq_{\delta_A}$ order, is the chain

$$[a_1], [a_1, a_2], \ldots, [a_1, \ldots, a_n]$$

of prefixes of $\overline{a}$. Therefore, requiring that $\mathrm{ht}(\overline{a}) = \mathrm{ht}(\overline{b})$ is the same as requiring that the sequences $\overline{a}$ and $\overline{b}$ have the same length and requiring that the induced substructure $\downarrow\overline{a} \subseteq A$ is isomorphic to $\downarrow\overline{b} \subseteq B$ is the same as requiring that we have a partial correspondence, as in (2.2).

Finally, in terms of the available moves, we see that choosing a position $\overline{a}'$ such that $\overline{a} \sqsubseteq_{\delta_A} \overline{a}'$ corresponds to finding an extension of $\overline{a}$. This corresponds to Spoiler announcing multiple next steps at once. Such modification of the back-and-forth Ehrenfeucht–Fraïssé game does not change its strength since Spoiler is still allowed to just do steps one-by-one.

(2) is proved following the same steps as in (1). The only difference is that now positions $(\overline{a}, \overline{b}) \in F^{\mathbb{P}_k}(A) \times F^{\mathbb{P}_k}(B)$ also encode the pebbles played. Therefore, for $\overline{a} = [(p_1, a_1), \ldots, (p_n, a_n)]$, the induced substructure $\downarrow\overline{a}$ of $(\mathbb{P}_k(A), \sqsubseteq_{\delta_A}, p)$, where $p \colon \mathbb{P}_k(A) \to [k]$ is the pebble function from Remark 2.5.3, contains the information which pebbles were played and when.

Consequently, for some position $(\overline{a}, \overline{b})$, the requirement that the induced substructures $\downarrow\overline{a}$ and $\downarrow\overline{b}$ are isomorphic corresponds to the requirement that Duplicator moved the same pebble as Spoiler.

(3) is proved similarly to (1) and (2). □

2.7 Comonads for other logic fragments

In Section 2.3 we described three original game comonads and in later sections we discussed their connection to logic fragments and combinatorial properties. However, the list of comonads does not stop there; many more comonads have been described since.

In fact, game comonads form an ever-evolving list, see Table 2.1 and also [9, 27, 50, 100, 112]. It is important to note, though, that not all correspondences have been fully proved for all comonads. For example, for the Hella

Table 2.1: Other examples of game comonads

Comonad	Ref.	Combinatorial property	Logic fragment
$\mathbb{E}_k$	[14]	tree-depth	quantifier rank
$\mathbb{P}_k$	[3]	tree-width	variable count
$\mathbb{M}_k$	[14]	sync. tree depth	modal depth
$\mathbb{PR}_k$	[92]	path-width	restricted conjunction & variable count
$\mathbb{He}_k$	[38]	k-ary generalised t.-w.	generalised quantifier k-variable extension
$\mathbb{G}_k$	[6]	guarded tree decomp.	quantifier-guarded
$\mathbb{LG}_k$	[116]	hypertree-width	k-conjunct guarded
$\mathbb{Hy}_k$	[8]	generated tree-depth	hybrid modal depth
$\mathbb{B}_k$	[7]	generated tree-depth	bounded quantifiers
...	...	...	...

comonad $\mathbb{He}_k$, we do not yet have a logical description of the relation induced by the back-and-forth equivalence $F^{\mathbb{He}_k}(A) \rightleftharpoons F^{\mathbb{He}_k}(B)$.

It is also important to note that our comonads often restrict to smaller subcategories, thanks to the following result.

Proposition 2.7.1 ([13]) *Let $\mathbb{C}$ be a comonad on $\mathcal{S}tr(\sigma)$ and let T be a first-order theory of* negative formulas*, i.e. formulas obtained by using negated atomic formulas, $\wedge, \vee, \exists$, and $\forall$.*
If the counit $\varepsilon : \mathbb{C} \to \mathrm{Id}$ of $\mathbb{C}$ is componentwise surjective then $\mathbb{C}$ restricts to the full subcategory $\mathbf{Mod}(T)$ of $\mathcal{S}tr(\sigma)$ consisting of structures A such that $A \models T$.

As an example of the use of Proposition 2.7.1, since $\mathbb{E}_k$ and $\mathbb{P}_k$ have componentwise surjective counits, we immediately obtain that $\mathbb{E}_k$ and $\mathbb{P}_k$ restrict to $\mathbf{Mod}(T)$, where $T = \{\varphi\}$ is a single first-order formula

$$\varphi = \forall x.\neg R(x, x)$$

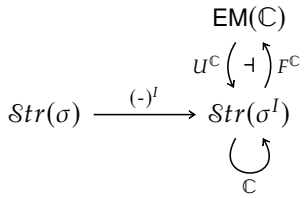
in the language of $\sigma = \langle R \rangle$ where R is binary. In other words, $\mathbf{Mod}(T)$ is the category of directed graphs without loops. In fact, one can directly see that $\mathbb{E}_k$ and $\mathbb{P}_k$ also restrict to the subcategory of simple graphs, i.e. structures with a binary relation which is symmetric and irreflexive.

Changing φ to $\forall xyz.\neg R(x, y) \vee \neg R(y, z) \vee \neg R(z, x)$ shows that $\mathbb{E}_k$ and $\mathbb{P}_k$ restrict to the subcategory of directed graphs without an oriented triangle. A similar technique can be used to show that these two comonads also restrict to any subcategory of graphs that does not admit a homomorphism from a given set of graphs.

2.8 Logic extensions using translations

As mentioned in the discussion following Proposition 2.1.4, the way to capture logic with equality is to extend the signature σ with an additional binary relation I interpreting equality.

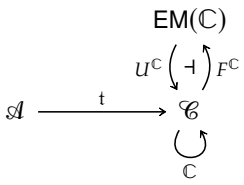
Since our game comonads are defined for any signature, they are also defined for $\mathcal{S}tr(\sigma^I)$, where σ^I is this extended signature. We have the following picture, where $(\text{-})^I$ is the functor mapping A to its extension A^I in signature σ^I where I is interpreted as equality.



Then, as in Proposition 2.1.4, for structures A, B, we obtain the following equivalences.

- $F^{\mathbb{E}_k}(A^I) \rightleftharpoons F^{\mathbb{E}_k}(A^I)$ iff $A \equiv_{\mathtt{FO}^=_k} B$
- $F^{\mathbb{P}_k}(A^I) \rightleftharpoons F^{\mathbb{P}_k}(A^I)$ iff $A \equiv_{\mathcal{L}^{=k}_{\infty\omega}} B$

In general, we use the same pattern, replacing $(\text{-})^I$ by an arbitrary functor $\mathsf{t}\colon \mathcal{A} \to \mathcal{C}$, to arrive at the following situation.



(2.11)

This allows us to express a number of logics by reusing the same comonads (or modifying them slightly). In particular, we have the following examples.

Example 2.8.1 ([64, 115]) Consider $\mathcal{A} = \mathcal{S}tr(\sigma)$ and $\mathcal{C} = \mathcal{S}tr(\sigma^+)$ where σ^+ is a signature of two-sorted structures. In particular, all relations in $\sigma = \langle R_1, \dots, R_n \rangle$ are kept in σ^+ for the first sort and we add an additional binary relation $e(\cdot,\cdot)$ between the two sorts. Then, we define $\mathbb{E}_k$ on two-sorted structures as $\mathbb{E}_k$ on single-sorted structures where any sequence $[a_1, \dots, a_n] \in \mathbb{E}_k(A)$, for a two-sorted structure A, is allowed to mix elements from both sorts, however, the sort of the sequence $[a_1, \dots, a_n]$ is the same as the sort of the last element a_n.

Then, we define $\mathsf{t}(A) = (A, \mathfrak{P}(A); R_1^A, \dots, R_n^A, e^A)$ where $\mathfrak{P}(A)$ denotes the powerset of the universe of A and we set $e^A(x, S)$ iff $x \in S$.

It follows that, for structures A, B,

$$F^{\mathbb{E}_k}(\mathsf{t}A) \rightleftharpoons F^{\mathbb{E}_k}(\mathsf{t}B) \quad \text{iff} \quad A \equiv_{\mathtt{MSO}^=_k} B$$

where $\mathtt{MSO}_k$ is the fragment of monadic second-order logic (without equality) consisting of formulas of quantifier rank $\leq k$. In case we want $\mathtt{MSO}_k$ with equality we define σ^+ as an extension of σ^I.

Example 2.8.2 ([66]) Let $\mathsf{t}\colon \mathcal{S}tr(\sigma) \to \mathcal{S}tr(\sigma^+)$ be a translation of σ-structures into the extension σ^+ of σ^I with an extra binary relation $\mathtt{Con}(\cdot,\cdot)$. The interpretation of $\mathtt{Con}$ in $\mathsf{t}(A)$ consists of all pairs (a, b) appearing in the same component of the **Gaifman graph** of A, that is, pairs (a, b) such that there is a path $a \leftrightsquigarrow x_1 \leftrightsquigarrow \cdots \leftrightsquigarrow x_n \leftrightsquigarrow b$ where

$x \leftrightsquigarrow y$ holds whenever x and y are in a common tuple in an R^A of some $R \in \sigma$.

It is easy to see that $F^{\mathbb{E}_k}(\mathfrak{t}A) \rightleftharpoons F^{\mathbb{E}_k}(\mathfrak{t}B)$ expresses logical equivalence in the bounded quantifier depth fragment extended with the connectivity predicate, akin to [28, 113]. While still being polynomial-time computable, the connectivity relation is not definable in bounded quantifier depth fragments of first-order logic, and so this is a proper extension.

Example 2.8.3 ([66]) Consider $\mathfrak{t}\colon \mathcal{S}tr_*(\sigma) \to \mathcal{S}tr_*(\sigma^+)$ where σ^+ is the extension of the modal signature σ with the *global relation* $G(a,b)$, true in $\mathfrak{t}(A, a_0)$ for any pair of elements. Then, it is immediate that the relation $F^{\mathbb{M}_k}(\mathfrak{t}(A, a_0)) \rightleftharpoons F^{\mathbb{M}_k}(\mathfrak{t}(B, b_0))$ expresses logical equivalence with respect to modal logic with global modalities.

Note that this list of logic extensions obtained this way is not in any way exhaustive; for example, this way we can also capture Description Logic [27], weak bisimulation [66], guarded fragments [6], etc.

2.9 Transfer theorems for the EM category

Since much of the theory of game comonads depends on the properties of the EM categories, it is essential to understand their structure in more detail. In this section, we focus on theorems that allow us to transfer categorical properties from the base category of the comonad to the corresponding EM category. To this end, let us fix a comonad $\mathbb{C}$ on a category $\mathcal{C}$.

It is standard that the comonadic functor $U^{\mathbb{C}}\colon \mathsf{EM}(\mathbb{C}) \to \mathcal{C}$ not only preserves but also **creates colimits** [18, Proposition 20.12], i.e. for every diagram D in $\mathsf{EM}(\mathbb{C})$ such that $U^{\mathbb{C}}(D)$ admits a colimit C in $\mathcal{C}$, there exists a colimit X of D in $\mathsf{EM}(\mathbb{C})$ such that $U^{\mathbb{C}}(X) \cong C$. Therefore, we have the following fact, which applies to all of our comonads since the categories of (pointed) relational structures are cocomplete.

Proposition 2.9.1 *If $\mathcal{C}$ is cocomplete then so is $\mathsf{EM}(\mathbb{C})$.*

Factorisation systems

Crucial for the theory of game comonads is the fact that our comonads transfer the factorisation system from the underlying category to the EM category of coalgebras. In this section, we first review some basic facts concerning factorisation systems[11] and then state a transfer theorem.

11: More details can be found, e.g. in [18, Chapter 14] and [106, §11.2].

A pair of classes of morphisms $(\mathbb{Q}, \mathcal{M})$ on a category $\mathcal{C}$ is an **orthogonal factorisation system** if

(a) every morphism f in $\mathbb{C}$ **factorises** as $f = m \cdot q$ for some $q \in \mathbb{Q}$ and $m \in \mathcal{M}$;

(b) every $q \in \mathcal{Q}$ is **orthogonal** to every $m \in \mathcal{M}$, written as $q \perp m$, i.e. for every commutative square as on the left-hand side below,

there exists a unique **diagonal filler**, i.e. an arrow d such that the right-hand diagram above commutes; and

(c) $\mathcal{Q}$ and $\mathcal{M}$ are **closed under composition with isomorphisms**.[12]

12: This means that a composition $i \cdot f \cdot j$, where i, j are isomorphisms, is in $\mathcal{Q}$ (resp. $\mathcal{M}$) whenever f is.

Let $(\mathcal{Q}, \mathcal{M})$ be an orthogonal factorisation system in a category. We refer to $\mathcal{Q}$-morphisms as **quotients** and denote them by $\twoheadrightarrow$, and to $\mathcal{M}$-morphisms as **embeddings** and denote them by $\rightarrowtail$. We say that $(\mathcal{Q}, \mathcal{M})$ is a **proper factorisation system** if every quotient is an epimorphism and every embedding is a monomorphism.

Example 2.9.2 The prototypical example of a proper factorisation system is the pair (surjective, injective) of surjective functions (as quotients) and injective functions (as embeddings) on the category **Set** of sets and functions.

Further, for the categories $\mathcal{S}tr(\sigma)$ and $\mathcal{S}tr_*(\sigma)$, unless said otherwise, in this text we automatically take the proper factorisation system

(surjective, strong injective)

of surjective homomorphisms and strong injective homomorphisms where a homomorphism of σ-structures $f : A \to B$ is **strong** if the implication in (2.9) is an equivalence. Therefore, the factorisation of a homomorphism of σ-structures $f : A \to B$ can be given as

$$A \xrightarrow{q} \mathrm{im}(f) \xrightarrow{e} B$$

where the image $\mathrm{im}(f) \subseteq B$ is taken as the induced substructure of B, e is the inclusion of the image $\mathrm{im}(f)$ into B and q is just the restriction of the range of f to $\mathrm{im}(f)$.

We use the following standard facts about proper factorisation systems $(\mathcal{Q}, \mathcal{M})$, often without further reference (see e.g. [52, 106]):

Lemma 2.9.3 *Let $(\mathcal{Q}, \mathcal{M})$ be an orthogonal factorisation system. The following statements hold:*

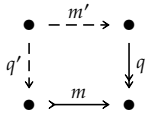
1. *$\mathcal{Q}$ and $\mathcal{M}$ are closed under composition.*
2. *$\mathcal{Q} \cap \mathcal{M} = \{\text{isomorphisms}\}$*
3. *The pullback of an $\mathcal{M}$-morphism along any morphism, if it exists, is again in $\mathcal{M}$.*
4. *$\mathcal{M}$ contains equalisers, i.e. whenever $\bullet \xrightarrow{f} \bullet \rightrightarrows \bullet$ is an equaliser then $f \in \mathcal{M}$.*

Moreover, if $(\mathcal{Q}, \mathcal{M})$ is proper, the following hold:

5. *$g \cdot f \in \mathcal{Q}$ implies $g \in \mathcal{Q}$.*
6. *$g \cdot f \in \mathcal{M}$ implies $f \in \mathcal{M}$.*

Proof. These properties are all proved in [18, Chapter 14]. Namely, (1) and (2) are in Proposition 14.6, (3) is in Proposition 14.15, (4) follows from Proposition 14.10 and (5) and (6) follow from Proposition 14.11 therein. □

Finally, an important property for a factorisation system $(\mathbb{Q}, \mathcal{M})$ is when it is **stable**, which means that, given a quotient q and embedding m, as shown in the diagram below,



their pullback exists and, moreover, q' is a quotient too.

Note that m' in the diagram above is automatically an embedding, by Lemma 2.9.3.

Example 2.9.4 Stability holds in **Set**, $\mathcal{S}tr(\sigma)$ and $\mathcal{S}tr_*(\sigma)$ for the examples of factorisation systems mentioned in Example 2.9.2. For example, in **Set** stability simply means that if we have a surjective map $q\colon A \twoheadrightarrow B$ and a subset $X \subseteq B$, inducing an injective map $m\colon X \rightarrowtail B$, then the restriction of q to the preimage $q^{-1}[X] \subseteq A$ is still surjective as a map $q^{-1}[X] \to X$.

Note that we use the definition of orthogonal systems from [18, Chapter 14], where it is called the 'factorization structure.' We prefer this definition since it is easy to verify and work with. On the other hand, it is very common in the literature to replace (b) and (c) with a single axiom stating that $\mathbb{Q} = {}^{\perp}\mathcal{M}$ and $\mathcal{M} = \mathbb{Q}^{\perp}$ (see e.g. [106]) where, for a class of morphisms $\mathcal{K}$,

$$ {}^{\perp}\mathcal{K} = \{f \mid \forall k \in \mathcal{K},\ f \perp k\} \quad \text{and} \quad \mathcal{K}^{\perp} = \{f \mid \forall k \in \mathcal{K},\ k \perp f\}. $$

It is folklore that these two definitions are equivalent, as we remark below. We provide a full proof of this equivalence, using references in published literature, since this equivalence is not so easy to find spelled out in full.[13]

13: There is also an excellent *unpublished* account of (most of) the equivalence and also of the relationship to *weak factorisation systems* by Emily Riehl [107] and on André Joyal's page titled "Factorisation systems" at https://ncatlab.org/joyalscatlab/.

Lemma 2.9.5 *Given a pair of classes of morphisms $(\mathbb{Q}, \mathcal{M})$ on a category, it is an orthogonal factorisation system if and only if*

(a') every morphism f can be factorised as $f = m \cdot q$, where $q \in \mathbb{Q}$ and $m \in \mathcal{M}$; and
(b') $\mathbb{Q} = {}^{\perp}\mathcal{M}$ and $\mathcal{M} = \mathbb{Q}^{\perp}$.

Proof. For the left-to-right implication it is enough to entail (b'). However, Lemma 2.9.3 implies that $\mathbb{Q} \cap \mathcal{M}$ contains identities and so Proposition 2.9 in [34] is exactly the intended implication. Conversely, (b') implies (b) and [52, Proposition 2.1.1] gives that both classes contain isomorphisms and are closed under composition, which implies (c). □

See also [52, Section 2.2] and [18, Proposition 14.7] for other equivalent reformulations.

Transfer theorems. We fix a comonad $\mathbb{C}$ on a category $\mathscr{C}$ with an orthogonal factorisation system $(\mathscr{Q}, \mathscr{M})$. We say that $\mathbb{C}$ is **regular**[14] if it maps embeddings to embeddings.

14: This property should be probably called 'coregular' since it is the dual of 'regular monads', which are required to preserve quotients. On the other hand, preservation of embeddings is, in our case, the same as preservation of 'regular monos,' since the latter are precisely strong injective homomorphisms in $\mathcal{S}tr(\sigma)$ or $\mathcal{S}tr_*(\sigma)$.

Lemma 2.9.6 *Let $\mathbb{C}$ be a comonad on a category $\mathscr{C}$ with an orthogonal factorisation system $(\mathscr{Q}, \mathscr{M})$.*

- *If $\mathbb{C}$ is regular, then $\mathsf{EM}(\mathbb{C})$ admits a proper factorisation system*

$$(\overline{\mathscr{Q}}, \overline{\mathscr{M}})$$

  where a morphism of coalgebras $h\colon (A,\alpha) \to (B,\beta)$ is in $\overline{\mathscr{Q}}$ (resp. $\overline{\mathscr{M}}$) if the underlying morphism $h\colon A \to B$ is in $\mathscr{Q}$ (resp. $\mathscr{M}$).
- *If, in addition, $(\mathscr{Q}, \mathscr{M})$ is proper then so is $(\overline{\mathscr{Q}}, \overline{\mathscr{M}})$.*
- *If, in addition, $(\mathscr{Q}, \mathscr{M})$ is stable and $\mathbb{C}$ sends pullbacks of quotients along embeddings to weak pullbacks,*[15] *then $(\overline{\mathscr{Q}}, \overline{\mathscr{M}})$ is also stable.*

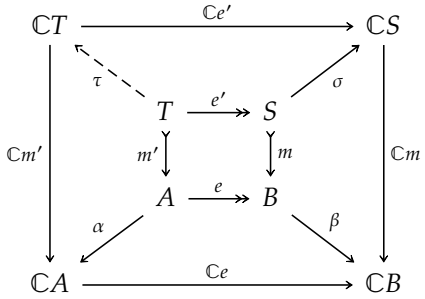
15: Weak pullback is defined the same way as pullback, except that uniqueness of the comparison map is not required.

Proof. The fact that $(\overline{\mathscr{Q}}, \overline{\mathscr{M}})$ is a (proper) factorisation system whenever $\mathbb{C}$ preserves embeddings is proved similarly to [82, Lemma 2].

For the last part, let $e\colon (A,\alpha) \twoheadrightarrow (B,\beta)$ and $m\colon (S,\sigma) \rightarrowtail (B,\beta)$ be, respectively, a quotient and an embedding in $\mathsf{EM}(\mathbb{C})$. We denote the pullback of the underlying morphisms in $\mathscr{C}$ as follows.

$$\begin{array}{ccc} T & \xrightarrow{e'} & S \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{e} & B \end{array}$$

As $\mathbb{C}$ sends the latter pullback to a weak pullback, there exists a morphism τ making the following diagram commute.



We claim that (T,τ) is an Eilenberg–Moore coalgebra. By naturality of ε we see that

$$m' \cdot \varepsilon_T \cdot \tau = \varepsilon_A \cdot \mathbb{C}m' \cdot \tau = \varepsilon_A \cdot \alpha \cdot m' = m',$$

where the last equality holds because (A,α) is a coalgebra. Hence, since m' is a monomorphism, we get that $\varepsilon_T \cdot \tau = \mathrm{id}$. Next, to check the equality $\delta_T \cdot \tau = \mathbb{C}\tau \cdot \tau$, we observe that

$$\mathbb{C}^2 m' \cdot \delta_T \cdot \tau = \delta_A \cdot \mathbb{C}m' \cdot \tau = \delta_A \cdot \alpha \cdot m' = \mathbb{C}\alpha \cdot \alpha \cdot m',$$

where the first equality holds by naturality of δ, the second by the

commutativity of the left face in the above diagram, and the third because (A, α) is a coalgebra. Then, using the commutativity of the left face twice, we have that

$$\mathbb{C}\alpha \cdot \alpha \cdot m' = \mathbb{C}\alpha \cdot \mathbb{C}m' \cdot \tau = \mathbb{C}^2 m' \cdot \mathbb{C}\tau \cdot \tau.$$

Therefore, we get $\mathbb{C}^2 m' \cdot \delta_T \cdot \tau = \mathbb{C}^2 m' \cdot \mathbb{C}\tau \cdot \tau$, which gives us the desired equality since $\mathbb{C}^2 m'$ is a monomorphism (just recall that $\mathbb{C}$ preserves embeddings).

It remains to show that (T, τ) is a pullback of m and e. Suppose that $h\colon (Z, \zeta) \to (A, \alpha)$ and $l\colon (Z, \zeta) \to (S, \sigma)$ satisfy $f \cdot h = g \cdot l$. Since the underlying morphisms commute in $\mathscr{C}$, there is a $d\colon Z \to T$ such that $h = m' \cdot d$ and $l = e' \cdot d$. To see that d is a morphism $(Z, \zeta) \to (T, \tau)$, observe that

$$\mathbb{C}m' \cdot \tau \cdot d = \alpha \cdot m' \cdot d = \mathbb{C}m' \cdot \mathbb{C}d \cdot \zeta$$

and, since $\mathbb{C}m'$ is a monomorphism, we get the required equality $\tau \cdot d = \mathbb{C}d \cdot \zeta$. Uniqueness of d follows from the fact that m' is a monomorphism. □

It is easy to see that our comonads are regular.

Lemma 2.9.7 *$\mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$ are regular comonads.*

The following lemma, together with Lemma 2.9.6, justifies that the transferred factorisation system (surjective, strong injective) to $\mathsf{EM}(\mathbb{C})$, for our game comonads $\mathbb{C}$, is stable.

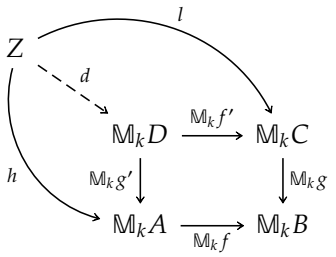
Lemma 2.9.8 *$\mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$ send pullbacks to weak pullbacks.*

Proof. Let $f\colon A \to B$ and $g\colon C \to B$ be homomorphisms in $\mathcal{S}tr_*(\sigma)$. We do not explicitly write their designated points, which are a_0, b_0, and c_0, respectively. Let the pullback of f and g be as shown below.

$$\begin{array}{ccc} D & \xrightarrow{f'} & C \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array}$$

We assume without loss of generality that D is the substructure of $A \times C$ consisting of the pairs (a, c) such that $f(a) = g(c)$, where (a_0, c_0) is the designated point.

Suppose there is a pointed Kripke structure Z, and homomorphisms $l\colon Z \to \mathbb{M}_k C$ and $h\colon Z \to \mathbb{M}_k A$, such that the outer diagram below commutes.



We define a map $d\colon Z \to \mathbb{M}_k D$. For any $z \in Z$, we have that

$$h(z) = [a_0, \tau_1, a_1, \ldots, \tau_n, a_n] \quad \text{and} \quad l(z) = [c_0, \phi_1, c_1, \ldots, \phi_m, c_m].$$

Since the outer diagram above commutes, we get $\mathbb{M}_k f(h(z)) = \mathbb{M}_k g(l(z))$. So, $m = n$ and, for every $i \in \{1, \ldots, n\}$, $\tau_i = \phi_i$ and $f(a_i) = g(c_i)$. Therefore, we can define $d(z) \coloneqq [(a_0, c_0), \tau_1, (a_1, c_1), \ldots, \tau_n, (a_n, c_n)]$. It is immediate from the definition of d that the above diagram commutes. Finally, since both h and l are homomorphisms, so is d.

The same argument immediately adapts for $\mathbb{E}_k$ and $\mathbb{P}_k$. □

Completeness. Note that the fact that our comonads are regular also automatically gives us that their EM categories are complete, by (the dual of) Linton's theorem [82].

The assumptions of the theorem are fairly minimal;[16] the only new requirement is that the category is $\mathcal{M}$**-well-powered**, that is, that every object A of $\mathcal{C}$ has, up to isomorphism, only a set of embeddings $X \rightarrowtail A$.

16: See [66] for a simplified proof, which uses exactly the assumptions as stated here.

Proposition 2.9.9 (Linton [82]) *Given a comonad $\mathbb{C}$ on a category $\mathcal{C}$ equipped with a proper factorisation system $(\mathbb{Q}, \mathcal{M})$ if*

1. *$\mathbb{C}$ is regular,*
2. *$\mathcal{C}$ admits coproducts and*
3. *$\mathcal{C}$ is $\mathcal{M}$-well-powered.*

then $\mathsf{EM}(\mathbb{C})$ *is complete.*

The fact that our EM categories are complete, in particular that they have equalisers, is important for liftings of functors, cf. Section 7.3.

Locally finitely presentable categories

Important for the categorical approach to model theory are the locally finitely presentable categories [20, 87], which have a rich categorical structure with many desirable properties. It turns out that if a comonad has *finite rank*, then the property of being locally finitely presentable transfers from the base category to the category of coalgebras.

Remark 2.9.10 Below we introduce the terminology necessary in order to define these terms. Note that understanding the theory of game comonads does not require understanding of the theory of locally finitely presentable categories. However, this theory serves as a useful intuition for our 'synthetic approach' to model-theoretic games, cf. Chapter 4.

Colimits of diagrams of the form $D \to \mathcal{C}$, where D is a directed poset, are called **directed colimits**. Then, an object C of a category $\mathcal{C}$ is **finitely presentable**[17] if the functor $\hom(C, \text{-})$ preserves directed colimits.[18]

17: This notion generalises the corresponding notions from algebra, such as finitely presentable groups, monoids, etc.

18: Equivalently, we might ask $\hom(C, \text{-})$ to preserve *filtered colimits*, which is a more general notion [20].

Example 2.9.11 The finitely presentable objects in **Set** are precisely the finite sets, while those in $\mathcal{S}tr(\sigma)$ or $\mathcal{S}tr_*(\sigma)$ are the finite structures.

We say that the comonad $\mathbb{C}$ has **finite rank** if

1. $\mathbb{C}$ is **finitary**, i.e. its underlying functor preserves directed colimits,

2. if every morphism of the form $f : C \to U^{\mathbb{C}}(X)$, from a finitely presentable C, admits a factorisation

$$f = C \xrightarrow{f_0} U^{\mathbb{C}}(X_0) \xrightarrow{U^{\mathbb{C}}(\gamma)} U^{\mathbb{C}}(X)$$

for some $\gamma : X_0 \to X$ such that $U^{\mathbb{C}}(X_0)$ is finitely presentable, and
3. this factorisation is *essentially unique*, i.e. if

$$f = C \xrightarrow{g} U^{\mathbb{C}}(X_0) \xrightarrow{U^{\mathbb{C}}(\gamma)} U^{\mathbb{C}}(X)$$

for some other $g : C \to U^{\mathbb{C}}(X_0)$ then for some factorisation of γ into $\lambda : X_0 \to X_0'$ and $\gamma' : X_0' \to X$ such that $U^{\mathbb{C}}(X_0')$ is finitely presentable, $U^{\mathbb{C}}(\lambda) \cdot f_0 = U^{\mathbb{C}}(\lambda) \cdot g$.

Finally, we say that a category $\mathscr{C}$ is **locally finitely presentable (lfp)** if

- it is cocomplete and
- it has a set[19] of finitely presentable objects such that every object of $\mathscr{C}$ is a directed colimit of objects in this set.

19: It is important that it is a *set* and not a *proper class*. Note that lfp categories typically have a proper class of finitely presentable objects. This requirement merely says that, up-to isomorphism, this class is 'small'.

Since our game comonads have finite rank [104] and since our base categories of (pointed) relational structures are locally finitely presentable [20], we obtain the following fact which automatically applies to our situation.

Proposition 2.9.12 ([44, Proposition 1.12.1]) *If $\mathbb{C}$ is of finite rank and if $\mathscr{C}$ is locally finitely presentable then so is* $\mathsf{EM}(\mathbb{C})$.

Note that the original proof contains a gap; see [104, Appendix B] for a correction.

As a consequence, when $\mathbb{C}$ has finite rank, we see that the functor $F^{\mathbb{C}} : \mathscr{C} \to \mathsf{EM}(\mathbb{C})$ is a *morphism of lfp categories*, i.e. it is a finitary limit preserving functor between lfp categories. Note that a functor is a morphism of lfp categories if and only if it has a left adjoint which preserves finitely presentable objects [20, Sec. 2.B]. In this case, this means that $U^{\mathbb{C}}$ has this additional property.

Logical reading. Every lfp category is equivalent to a category of models $\mathbf{Mod}(T)$ for some *cartesian theory* or, equivalently, *limit theory* [20], which is a theory with axioms of the form

$$\forall \overline{x}(\varphi(\overline{x}) \to \exists! \overline{y}.\psi(\overline{x}, \overline{y}))$$

where φ and ψ are conjunctions of atoms. In our case, since $\mathscr{C} \cong \mathbf{Mod}(T)$ and $\mathbb{C}$ has finite rank, we have that $\mathsf{EM}(\mathbb{C}) \cong \mathbf{Mod}(T^{\mathbb{C}})$ for some limit theory $T^{\mathbb{C}}$. Furthermore, morphisms of lfp categories correspond to 'interpretations' of theories, in the opposite direction. Therefore, $F^{\mathbb{C}}$ induces an interpretation of $T^{\mathbb{C}}$ in T.

2.10 Back-and-forth games abstractly

The explanation in Section 2.6 regarding how game comonads capture standard logic fragments relies too heavily on the concrete description

of our categories of coalgebras. To fully realize the comonadic approach to logic, definitions must abstract away from this concrete description. Consequently, back-and-forth equivalence $\rightleftharpoons$ should not be defined via the induced forest order $\sqsubseteq_\alpha$ arising from a coalgebra structure $\alpha\colon A \to \mathbb{C}(A)$, as this order depends explicitly on the concrete description of $\mathbb{C}(A)$. Instead, the $\rightleftharpoons$ relation should be characterized purely in terms of the categorical structure of $\mathsf{EM}(\mathbb{C})$.

Now that we know that our EM categories come equipped with a proper factorisation system (Lemmas 2.9.6 and 2.9.7), we are able to address this issue. The core idea is to define positions, winning condition, etc. in an abstract back-and-forth game using embeddings from the factorisation system. This approach is inspired by that of arboreal categories, introduced in [10], which we also discuss in Chapter 4.

Positions and winning conditions. We fix a class $\mathscr{P}$ of objects of $\mathsf{EM}(\mathbb{C})$ called **paths**. For our example comonads $\mathbb{C}$, the class $\mathscr{P}$ consists of the coalgebras (P, π) such that $(P, \sqsubseteq_\pi)$ is a finite chain. Then, a *winning position* in the back-and-forth game between objects X and Y of $\mathsf{EM}(\mathbb{C})$ is a span of embeddings

$$X \xhookleftarrow{} P \xhookrightarrow{} Y$$

for some path $P \in \mathscr{P}$.

Example 2.10.1 Following Section 2.5, we view objects of $\mathsf{EM}(\mathbb{E}_k)$ as forest ordered structures. We fix a path (P, π), i.e. a finite chain $P = (p_0 \lessdot_\pi \cdots \lessdot_\pi p_n)$. Observe that, since morphisms in $\mathsf{EM}(\mathbb{E}_k)$ are tree morphisms, a morphism $f\colon (P, \pi) \to F^{\mathbb{E}_k}(A)$ in $\mathsf{EM}(\mathbb{E}_k)$ sends the root p_0 of P to some root of $F^{\mathbb{E}_k}(A)$, i.e. a sequence $[a_0]$, the element p_1 of height 1 of P is sent to some immediate successor of $[a_0]$ in $\mathbb{E}_k(A)$, i.e. to some length 2 sequence $[a_0, a_1]$ and so on. Therefore, f picks out a sequence $[a_0, \ldots, a_n]$, i.e. a position in a game.[20]

20: Due to the adjunction $U^{\mathbb{E}_k} \dashv F^{\mathbb{E}_k}$, there is a bijection between morphisms $(B, \beta) \to F^{\mathbb{E}_k}(A)$ and $U^{\mathbb{E}_k}(B, \beta) \to A$. Hence, $f\colon (P, \pi) \to F^{\mathbb{E}_k}(A)$ corresponds to the morphism $\varepsilon \cdot f\colon U^{\mathbb{E}_k}(P) \to A$ of σ-structures mapping p_i to a_i.

Then, requiring that f is an embedding corresponds to the requirement that, for any choice $i_1, \ldots, i_m$,

$$R^P(p_{i_1}, \ldots, p_{i_n}) \text{ iff } R^{\mathbb{E}_k A}(f(x_{p_1}), \ldots, f(x_{p_n})) \text{ iff } R^A(a_{i_1}, \ldots, a_{i_n}). \tag{2.12}$$

This shows that the relational structure of P is derived from that of A if f is an embedding. In other words, the embedding $(P, \pi) \rightarrowtail F^{\mathbb{E}_k}(A)$ is fully determined by the choice of the sequence $[a_1, \ldots, a_n]$.

Next, consider that we have a span of embeddings

$$F^{\mathbb{E}_k}(A) \xhookleftarrow{e_1} (P, \pi) \xhookrightarrow{e_2} F^{\mathbb{E}_k}(B)$$

This means that e_1 picks out some $[a_1, \ldots, a_n]$ in $\mathbb{E}_k(A)$ and e_2 picks out some $[b_1, \ldots, b_n]$ in $\mathbb{E}_k(B)$. Then, for an m-ary relational symbol R in the signature and $i_1, \ldots, i_m \in \{1, \ldots, n\}$, since e and e' are embeddings i.e. strong injective homomorphisms, by (2.12)

$$R^A(a_{i_1}, \ldots, a_{i_m}) \iff R^P(p_{i_1}, \ldots, p_{i_m}) \iff R^B(b_{i_1}, \ldots, b_{i_m})$$

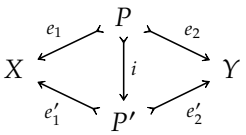
as required in the definition of partial correspondence (2.2), for the winning condition at round n.

Moves. When specifying an abstract (categorical) model comparison game we must define available moves of Duplicator or Spoiler at any given state of the game. This is achieved by considering *extensions* of embeddings.

The round, starting from position $X \xleftarrow{e_1} P \xrightarrow{e_2} Y$, has the following structure.

- First, Spoiler chooses an embedding $i\colon P \rightarrowtail P'$ where P' is a path and then either
- Spoiler chooses $e_1'\colon P' \rightarrowtail X$ such that $e_1 = e_1' \cdot i$ and then Duplicator chooses $e_2'\colon P' \rightarrowtail Y$; or
- Spoiler chooses $e_2'\colon P' \rightarrowtail Y$ such that $e_2 = e_2' \cdot i$ and then Duplicator chooses $e_1'\colon P' \rightarrowtail X$.

Duplicator wins the round if the following commutes.



In other words, the goal of Duplicator is to extend the previous winning position to a new winning position.

Example 2.10.2 Following Section 2.5, we view objects of $\mathsf{EM}(\mathbb{P}_k)$ as forest ordered structures. We fix a path (R, ρ), i.e. a finite chain $R = (r_0 \prec_\rho \cdots \prec_\rho r_n)$. Observe that a position $F^{\mathbb{P}_k}(A) \xleftarrow{e_1} (R, \rho) \xrightarrow{e_2} F^{\mathbb{P}_k}(B)$ determines sequences $[(p_1, a_1), \ldots, (p_n, a_n)]$ and $[(p_1, b_1), \ldots, (p_n, b_n)]$, i.e. the pebble choices $p_1, \ldots, p_n$ given by e_1 and e_2 are the same. Furthermore, any extension $i\colon R \rightarrowtail R'$ where $R' = (r_0 \prec_\rho \cdots \prec_\rho r_m)$ determines a choice of additional pebbles $p_{n+1}, \ldots, p_m$. Then, any time Spoiler picks e.g. e_1' such that $e_1 = e_1' \cdot i$, the choice of pebbles must be preserved. That is, Spoiler's choice of i and e_1' determines a choice of pebbles $p_{n+1}, \ldots, p_m$ followed by a choice of $a_{n+1}, \ldots, a_m$, giving a sequence $[(p_1, a_1), \ldots, (p_n, a_m)]$.

Since Duplicator must find e_2' extending e_2 along i, they do not affect the choice of pebbles and only choose $b_{n+1}, \ldots, b_m$ so that they obtain a sequence $[(p_1, b_1), \ldots, (p_n, b_m)]$.

Initial position. What remains is to define the initial position. This can be done in a number of ways, which depend on minor technical choices that we make in the definition of the abstract framework. We return to this question when the details of the abstract framework are discussed, in Subsection"Initial position" on page 53. For example, one might observe that, for every coalgebra $X \in \mathsf{EM}(\mathbb{C})$, there is a minimal subpath $0_X \in \mathcal{P}$ of X, that is, a 'least' embedding

$$0_X \xrightarrow{\perp_X} X.$$

Then, a game between X and Y starts by considering the pair $(\perp_X, \perp_Y)$ as an initial position. That is, Duplicator wins the initial round whenever $0_X \cong 0_Y$. Then we can assume, without loss of generality, that $0_X = 0_Y$ i.e. that the initial position is $X \xleftarrow{\perp_X} 0_X = 0_Y \xrightarrow{\perp_Y} Y$.

Synthetic model comparison games

Elementary path categories 3

An essential feature of the comonadic approach to model comparison games is the ability to abstract away the rules of individual games as simple categorical notions. This way, we describe and reason about similar types of games uniformly, using a simple common formalism. One can observe that the way we define abstract back-and-forth games in Section 2.10 works more abstractly. The same game can be specified in any category with a factorisation system, i.e. it does not have to be an EM category of some regular comonads. The abstract back-and-forth game we formulated in Section 2.10 can be defined in any category with a factorisation system.

Therefore, in this chapter we devise a simple categorical setting for formulating back-and-forth games. We show that requiring additional assumptions about this setting guarantees useful and desirable properties of back-and-forth games.

Although the approach taken in this chapter is inspired by that of Section 2.10 which, in turn, is inspired by that of arboreal categories [11], we work in a much more general setting. We assume only the minimal axioms necessary to state our definitions and support our proofs. This way we cover some examples which are beyond the scope of arboreal categories, such as the EM categories for comonads for the guarded and loosely-guarded fragments. Furthermore, by keeping our axiomatisation minimal we can simplify the reasoning in later chapters (e.g. Chapter 9), as we do not have to check all axioms of arboreal categories in our constructions; instead we just check the assumptions of our minimal setting. The disadvantage is that we sometimes have to be overly explicit about our assumptions. However, this explicitness has its advantages too, as it highlights the concepts which are behind our arguments.

3.1 Elementary path categories and logic 47
Logical equivalences 48
One-way logical refinement 48
3.2 Additional assumptions . . 49
Basic axioms 49
Axioms about quotients . . 50
Axioms about colimits of path embeddings 50
3.3 Back-and-forth systems . . 52
Initial position 53
Operations on b&f system . 54
3.4 Bisimilarity and behavioural equivalence . . . 55
Open pathwise embeddings 55
Composing with b&f systems 56
Bisimilarity and behavioural equivalence 57
Open morphisms à la Joyal–Nielsen–Winskel 58
Pullbacks of open morphisms 60

3.1 Elementary path categories and logic

Inspired by the fact that in order to define back-and-forth games in the categories $\mathsf{EM}(\mathbb{C})$ of our game comonads $\mathbb{C}$, only the embedding part of the lifted factorisation system is needed, we define an **elementary path category**[1] as a triple $\mathfrak{X} = (\mathfrak{X}, \mathscr{M}, \mathscr{P})$ where

- $\mathscr{M}$ is a collection of morphisms in $\mathfrak{X}$, called **embeddings** and denoted by $\rightarrowtail$, which satisfy that
  1. $\mathscr{M}$ contains isomorphisms and every morphism in $\mathscr{M}$ is a monomorphism,
  2. if $f, g \in \mathscr{M}$ then $fg \in \mathscr{M}$ (if the composition is defined),
  3. if $fg \in \mathscr{M}$ then $g \in \mathscr{M}$, and
- $\mathscr{P}$ is a class of objects in $\mathfrak{X}$, called **paths**.

Observe that the requirements (1)–(3) follow from properties of embeddings of a proper factorisation system, cf. Lemma 2.9.3.

1: In fact, Reggio and Abramsky originally defined a *path category* as a category satisfying all conditions of arboreal categories except for the path-generatedness condition. We coined the term "elementary path category", to avoid confusion with the earlier introduced terminology.

Elementary path categories too are a weakening of arboreal categories [11]. The adjective 'elementary' is supposed to mimic the notion of 'elementary toposes,' as the weakening of toposes which is still suitable to do most of topos logic. Note that some different weakenings of arboreal categories have been considered too, such as *wooded categories* [105].

Usually, we have a functor

$$\mathcal{A} \xrightarrow{F} \mathfrak{X} \tag{3.1}$$

where $\mathcal{A}$ is an arbitrary category, representing our objects of study, $\mathfrak{X}$ is an elementary path category, encoding some logic, and F translates objects of $\mathcal{A}$ to their logical representation in $\mathfrak{X}$.

Moreover, often F has a left adjoint U, in which case we call this scenario a **path adjunction**, that is, we have

$$\mathcal{A} \underset{F}{\overset{U}{\rightleftarrows}} \mathfrak{X} \tag{3.2}$$

where $U \dashv F$ and $\mathfrak{X}$ is an elementary path category.

Example 3.1.1 Our principal examples of path adjunctions are of the form where $\mathcal{A} = \mathcal{S}tr(\sigma)$ or $\mathcal{A} = \mathcal{S}tr_*(\sigma)$ and $\mathfrak{X} = \mathsf{EM}(\mathbb{C})$ for some game comonad $\mathbb{C}$ on $\mathcal{A}$. Then, the adjunction $U \dashv F$ is the comonadic adjunction $U^{\mathbb{C}} \dashv F^{\mathbb{C}}$. The rest of the elementary path category structure is implicitly derived from this setting. In particular, since our comonads are regular, the embeddings $\mathcal{M}$ in $\mathfrak{X}$ are derived from those of σ-structures, i.e. embeddings in $\mathfrak{X}$ are the coalgebra morphisms whose underlying morphisms is a strong injective homomorphisms (cf. Lemma 2.9.6). Finally, the paths in $\mathfrak{X}$ are the coalgebras which are finite chains in their induced forest order (cf. Section 2.5).

Logical equivalences

Inspired by the way we express logical equivalence in the standard fragment, positive existential fragment and the counting fragment of the logic captured by a game comonad, cf. Sections 2.4 and 2.10, we define the following abstract equivalences with respect to a functor F, as in (3.1). For objects A, B of $\mathcal{A}$, define

- $A \equiv_{\mathfrak{X}} B$ iff $FA \rightleftharpoons FB$
- $A \equiv_{\exists^+\mathfrak{X}} B$ iff $FA \leftrightarrows FB$
- $A \equiv_{\#\mathfrak{X}} B$ iff $FA \cong FB$

where, for X, Y in an elementary path category $\mathfrak{X}$, the b&f equivalence $X \rightleftharpoons Y$ is defined formally in Section 3.3 below, $X \leftrightarrows Y$ expresses that there exist morphisms $X \to Y$ and $Y \to X$ and, as usual, $\cong$ expresses isomorphism in a given category.

One-way logical refinement

The following one-sided relations are used to express correspondences with respect to other types of logic of the fragment captured by the adjunction:

- $A \Rrightarrow_{\exists^+\mathfrak{X}} B$ iff there is a morphism $FA \to FB$
- $A \Rrightarrow_{\exists\mathfrak{X}} B$ iff there is a *pathwise-embedding* $FA \to FB$

- $A \Rrightarrow_{+\mathfrak{X}} B$ iff there exist morphisms

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ FA & & FB \end{array}$$

  where f, g are open *pathwise-embeddings* and q is a *tree bijection*.

Observe that the relation $A \Rrightarrow_{\exists^+\mathfrak{X}} B$ expresses preservation of positive existential formulas in our examples, cf. Section 2.4. Similarly, $A \Rrightarrow_{\exists\mathfrak{X}} B$ and $A \Rrightarrow_{+\mathfrak{X}} B$ capture the same property with respect to the existential formulas and positive formulas, respectively [11].

We define 'pathwise-embeddings' and 'open pathwise-embeddings' in Section 3.4 below and we define 'tree bijections', later in Section 4.2.

On notation. Note that the notation and terminology we use here comes from first-order logic and Ehrenfeucht–Fraïssé games. However, the applicability of this approach, of encoding logics using a functor F into an elementary path category, as in (3.1), goes beyond these. See, for example, [9, Theorem 6.5] for the usage of the same techniques for expressing trace equivalence using linear arboreal categories.

Also, the notation assumes that the functor F is understood from the context. If we want to make it explicit, we flag the name of the functor in the superscript, for example, by writing e.g. $\equiv^F_{\mathfrak{X}}$ or just $\equiv_F$. Furthermore, if the functor F is induced by a comonad $\mathbb{C}$, i.e. if $F = F^{\mathbb{C}}$, we may write

$$\equiv_{\mathbb{C}}, \quad \equiv_{\exists^+\mathbb{C}}, \quad \equiv_{\#\mathbb{C}}, \quad \Rrightarrow_{\exists^+\mathbb{C}}, \quad \Rrightarrow_{\exists\mathbb{C}}, \text{ and } \Rrightarrow_{+\mathbb{C}}$$

instead of the more verbose $\equiv_{\mathsf{EM}(\mathbb{C})}$, $\equiv_{\exists^+\mathsf{EM}(\mathbb{C})}$, etc.

3.2 Additional assumptions

In this text we consider various convenient assumptions about our elementary path categories $(\mathfrak{X}, \mathscr{M}, \mathscr{P})$, which we know are sound with respect to the EM categories of our game comonads, from Chapter 2.

The axioms are stated in the way we use them. In fact, we do not aim for the minimal list of axioms as some combinations of axioms imply the others. However, the point of this text is not to study strengths of these axioms. Instead, we simply want to be explicit about our assumptions.

Basic axioms

First we list the most basic assumptions, expressible in the terms of just paths and embeddings.

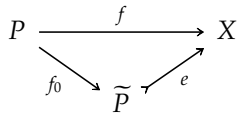
($\mathbf{P_0}$) $\mathfrak{X}$ has an **initial object 0** which is a path.

($\mathbf{P_{\rightarrowtail}}$) $\mathscr{P}$ is closed under subobjects, i.e. if $P \in \mathscr{P}$ and $S \rightarrowtail P$ then $S \in \mathscr{P}$.

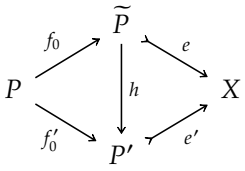
($\mathbf{M_{pb}}$) The pullback of an embedding along any morphism exists and is an embedding too[2].

2: Motivated by item 3 of Lemma 2.9.3.

(PE$_{\mathbf{mn}}$) Morphisms from paths factor through a **minimal path embedding**: Every morphism $f : P \to X$ where P is a path, factors through some **path embedding** $e : \widetilde{P} \rightarrowtail X$, i.e. an embedding where P is a path, that is, there is some f_0 such that $f = e f_0$:



Furthermore, e is *minimal*, that is, for any other factorisation $f = e' f_0'$ where $e' : P' \rightarrowtail X$ is a path embedding, there exists a necessarily unique $h : \widetilde{P} \to P'$ such that $e = e'h$. Consequently, the following diagram commutes.



Axioms about quotients

In our examples the embeddings in our elementary path categories are usually a part of a lifted factorisation system, Lemma 2.9.6. We often assume the following axioms which explain the relationship of paths and the quotient part of the factorisation system.

(QM) There is a class[3] of morphisms $\mathbb{Q}$, called *quotients*, such that $(\mathbb{Q}, \mathscr{M})$ forms a *proper* factorisation system.

(P$_{\twoheadrightarrow}$) (QM) holds and **paths are closed under quotients**, that is, if P is a path and $P \twoheadrightarrow X$ is a quotient then X is (isomorphic to) a path.

3: In fact, the choice of the class of quotients is unique, by Lemma 2.9.5.

Remark 3.2.1 The axiom (P$_{\twoheadrightarrow}$) is inspired by a property of quotients in arboreal categories, cf. Lemma 4.1.2. Observe that (P$_{\twoheadrightarrow}$) implies (PE$_{\mathrm{mn}}$) since the $(\mathbb{Q}, \mathscr{M})$-factorisation $f = eq$ of a path embedding is minimal by orthogonality of quotients and embeddings.

Axioms about colimits of path embeddings

Finally, we specify axioms which say when we can glue collections of path embeddings together, using colimits. We make use of these axioms when expressing with model-theoretic types in the language of elementary path categories, in Chapter 9. To this end, we first recall an important construction in category theory.

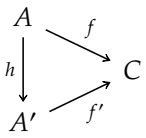
Comma categories. For an object C in a category $\mathscr{C}$, the **comma category**

$$\mathscr{C} \downarrow C$$

is the category whose objects are the pairs[4] (A, f) where $f : A \to C$ is

4: Often when the object part of (A, f) can be derived from the context, we simply refer to this object by f.

a morphism in $\mathscr{C}$ and morphisms $h\colon (A, f) \to (A', f')$ in $\mathscr{C} \downarrow C$ are the morphisms $h\colon A \to A'$ in $\mathscr{C}$ such that the arising triangle commutes.



Later on we also need the comma category defined dually, that is, the category $C \downarrow \mathscr{C}$ consisting of objects (A, f) where $f\colon C \to A$ and morphisms defined similarly to the above.

In the following we need to consider the restriction of comma categories to objects defined by embeddings. That is, for an object C in a category $\mathscr{C}$ which is endowed with an orthogonal factorisation system, define

$$\mathscr{C} \downarrowtail C$$

as the full subcategory of $\mathscr{C} \downarrow C$ on objects (A, f) where $f\colon A \rightarrowtail C$ is an embedding.

Finally, given a subcategory $\mathscr{D}$ of $\mathscr{C}$ and an object C of $\mathscr{C}$, define

$$\mathscr{D} \downarrow C, \quad C \downarrow \mathscr{D} \quad \text{and} \quad \mathscr{D} \downarrowtail C$$

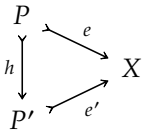
as the subcategory of $\mathscr{D} \downarrow C$, $C \downarrow \mathscr{D}$ and $\mathscr{D} \downarrowtail C$, respectively, restricted to objects (A, f) such that $A \in \mathscr{D}$ and morphisms $h\colon (A, f) \to (A', f')$ such that $h \in \mathscr{D}$.

The category of path embeddings. As a special case of the above, consider an elementary path category $\mathscr{X}$ and fix $X \in \mathscr{X}$. Then, the category

$$\mathscr{P} \downarrowtail X$$

is the category of **path embeddings** $P \rightarrowtail X$. Next, consider a morphism[5] $h\colon (P, e) \to (P, e')$ in $\mathscr{P} \downarrowtail X$, depicted below.

5: Note that h is automatically an embedding due to axioms for embeddings in elementary path categories.



In such situation we say that e' **extends** e **along** h.

Furthermore, since embeddings are monomorphisms, there is at most one such h along which the extension goes, i.e. the 'extends' relation is a pre-order. Formally, for path embeddings e, e' in $\mathscr{P} \downarrowtail X$, define $e \leq e'$ if there is a morphism $e \to e'$ in $\mathscr{P} \downarrowtail X$, that is:

$$e \leq e' \iff \exists h \quad \text{s.t.} \quad \begin{array}{ccc} S & & \\ {\scriptstyle h}\downarrow & \searrow^{e} & \\ S' & \nearrow_{e'} & X \end{array} \tag{3.3}$$

Remark 3.2.2 Observe that the objects of $\mathcal{P} \downarrow X$ correspond to positions in the abstract back-and-forth game and extensions of morphisms (i.e. morphisms in $\mathcal{P} \downarrow X$) correspond to taking moves in the game, cf. Section 2.10. We discuss this game in details in Section 3.3 below.

Cocones and the axioms. Every object X of an elementary path category induces the **canonical (path) diagram**[6], i.e. the forgetful functor

$$\mathsf{PDiag}_X : \mathcal{P} \downarrow X \to \mathcal{X}$$

which drops the morphism part of the comma category, i.e. it sends $(P, e : P \rightarrowtail X)$ to P and $h : (P, e) \to (P', e')$ to h.

6: More precisely it should be probably called the *canonical path diagram*, but that would be too long.

Remark 3.2.3 The canonical diagram PDiag_X encodes the structure of the moves played on X in the back-and-forth game, without reference to X itself. We can recover the full information about the positions encoded in PDiag_X by considering the induced cocone $\mathsf{PDiag}_X \to X$ in $\mathcal{X}$ where, for each $(P, e) \in \mathcal{P} \downarrow X$, we have the cocone component $\mathsf{PDiag}_X(P, e) \to X$ given by e. Therefore, if the colimit of PDiag_X exists then it induces a morphism (usually an embedding)

$$\operatorname{colim} \mathsf{PDiag}_X \to X \tag{3.4}$$

which, intuitively, selects out the part of X reachable during the game.

Finally, it is often useful to restrict PDiag_X to some subdiagram. However, we only permit forming certain subdiagrams, to ensure that the result still corresponds to a diagram of positions in a back-and-forth game. Formally, for a full inclusion functor $I : D \hookrightarrow \mathcal{P} \downarrow X$ we say that the composite

$$\mathcal{D} = \quad D \xhookrightarrow{I} \mathcal{P} \downarrow X \xrightarrow{\mathsf{PDiag}_X} \mathcal{X}$$

is a **path diagram** if D is *closed under subpaths*, i.e. if $e \leq e'$ and $e' \in D$ implies $e \in D$.

($\mathbf{PE}_\vee$) $\mathcal{X}$ has colimits of subpaths: For a path diagram $\mathcal{D} = \mathsf{PDiag}_X \circ I$,

- the colimit $\operatorname{colim} \mathcal{D}$ exists, and
- the induced morphism[7] $\operatorname{colim}(\mathsf{PDiag}_X \circ I) \to X$ is an embedding.

($\mathbf{PE}_{\mathbf{cp}}$) Path embeddings in $\mathcal{X}$ are compact: For a path diagram $\mathcal{D}$, every path embedding $P \rightarrowtail \operatorname{colim} \mathcal{D}$ factors through an inclusion morphism $d \to \operatorname{colim} \mathcal{D}$ for some $d \in D$

7: This morphism is obtained as the composition of the induced morphism

$$\operatorname{colim} \mathsf{PDiag}_X \to X,$$

obtained from the cocone $\mathsf{PDiag}_X \to X$, with the morphism

$$\operatorname{colim} \mathcal{D} \to \operatorname{colim} \mathsf{PDiag}_X,$$

induced by the inclusion I.

3.3 Back-and-forth systems

Following Section 2.10, we define the abstract back-and-forth equivalence for elementary path categories. However, this time we do not require that the game has an initial position because that would require additional assumptions on our elementary path categories, such as having an initial object and factorisations.

For objects X, Y in an elementary path category $\mathfrak{X}$, we say that X and Y are **back-and-forth equivalent (b&f equivalent)**, and write

$$X \rightleftharpoons Y,$$

if there is a **back-and-forth system (b&f system)** between X and Y, i.e. a collection $\mathcal{S}$ of spans[8] of path embeddings $X \xleftarrow{m} P \xrightarrow{n} Y$ satisfying the following axioms.

8: That is, the winning strategy of the Duplicator in the (abstract) back-and-forth game is given by a subclass of objects of $(\mathcal{P}\downarrow X) \times (\mathcal{P}\downarrow Y)$.

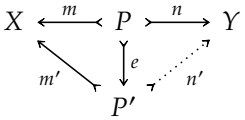
Initial positions:

- **(init-forth)** For every path embedding $P \xrightarrow{m} X$ there is some path embedding $P \xrightarrow{n} Y$ such that $X \xleftarrow{m} P \xrightarrow{n} Y$ is in $\mathcal{S}$.
- **(init-back)** For every path embedding $P \xrightarrow{n} Y$ there is some path embedding $P \xrightarrow{m} X$ such that $X \xleftarrow{m} P \xrightarrow{n} Y$ is in $\mathcal{S}$.

Extensions:

- **(forth)** Whenever $X \xleftarrow{m} P \xrightarrow{n} Y$ is in $\mathcal{S}$ and m' extends m along some $e : P \rightarrowtail P'$ then there is some n' extending n along e such that $X \xleftarrow{m'} P' \xrightarrow{n'} Y$ is in $\mathcal{S}$.
- **(back)** Whenever $X \xleftarrow{m} P \xrightarrow{n} Y$ is in $\mathcal{S}$ and $n' : P' \rightarrowtail X$ extends n along some e then there is m' extending m along e such that $X \xleftarrow{m'} P' \xrightarrow{n'} Y$ is in $\mathcal{S}$.

The first two conditions replace the requirement of having a fixed initial position e.g. the pair of empty sequences in the original Ehrenfeucht–Fraïssé game. Then, as before, the forth and back conditions express that Duplicator always has a response to any Spoiler move. In case of e.g. the (forth) condition, this can be pictured as follows.



Remark 3.3.1 Often we denote the elements of a b&f system $\mathcal{S}$ as pairs (m, n) or triples (P, m, n), to flag the common path object. Also, if the situation is completely symmetrical when checking the conditions, we sometimes only discuss the forth resp. back conditions whilst quietly omitting the other.

Initial position

In order to show that this modification of the game is equivalent to the game we defined in Section 2.10, we must show the equivalence of (init-forth) and (init-back) with having an initial position.

We show this equivalence in the presence of additional assumptions which hold true in $\mathfrak{X} = \mathsf{EM}(\mathbb{C})$ for our game comonads $\mathbb{C}$.

In particular, we need $(\mathrm{P_0})$, $(\mathrm{PE_{mn}})$ to be able to define the initial position. For $X \in \mathcal{X}$, denote by

$$\mathbf{0} \longrightarrow 0_X \overset{\perp_X}{\rightarrowtail} X$$

the factorisation of the unique morphism from $\mathbf{0}$ to X.

Instead of **(init-forth)** and **(init-back)** we consider the following:

- **(init-⊥)** The pair $(\perp_X \cdot i, \perp_Y \cdot i')$ is in $\mathcal{S}$, for some isomorphisms $i : P \to 0_X$ and $i' : P \to 0_Y$.

Next we show that, under mild additional assumptions, **(init-⊥)** is equivalent to **(init-forth)** and **(init-back)**.

Lemma 3.3.2 *Assume $\mathcal{X}$ is an elementary path category satisfying $(\mathrm{P_0})$, $(\mathrm{PE_{mn}})$ and*

- *paths of the form 0_X are **rigid**, i.e. $\hom_{\mathcal{X}}(0_X, 0_X) = \{\mathrm{id}\}$.*

*Let $\mathcal{S}$ satisfy **(back)** and **(forth)**, then $\mathcal{S}$ satisfies **(init-⊥)** if and only if it satisfies **(init-back)** and **(init-forth)**.*

Proof. From left-to-right, assume we are given a path embedding $m : P \rightarrowtail X$. By minimality of $\perp_X : 0_X \rightarrowtail X$, there is some $e : 0_X \rightarrowtail P$ such that m extends $\perp_X$ along e. By **(init-⊥)**, $(\perp_X \cdot i, \perp_Y \cdot i') \in \mathcal{S}$, for some isomorphisms $i : Q \to 0_X$ and $i' : Q \to 0_Y$. Then, by **(forth)**, there is some n extending $\perp_Y \cdot i' : Q \rightarrowtail Y$ along e such that $(m, n) \in \mathcal{S}$. **(init-back)** is proved dually.

Conversely, to show **(init-⊥)**, given $\perp_X : 0_X \rightarrowtail X$, by **(init-forth)** there is some $n : 0_X \rightarrowtail Y$ such that $X \overset{\perp_X}{\longleftarrow} 0_X \overset{n}{\longrightarrow} Y$ is in $\mathcal{S}$. By $(\mathrm{PE_{mn}})$, n extends the minimal path embedding $\perp_Y : 0_Y \rightarrowtail Y$ along some $f : 0_Y \rightarrowtail 0_X$, i.e. $\perp_Y = n \cdot f$. The converse embedding $0_X \rightarrowtail 0_Y$ is obtained symmetrically from $\perp_X \cdot f$ and minimality of $\perp_Y$. Therefore, since paths 0_X and 0_Y are rigid, f is an isomorphism and so $(\perp_X, n) = (\perp_X, \perp_Y \cdot f^{-1}) \in \mathcal{S}$, as required. □

Remark 3.3.3 Since 0_X and 0_Y are defined up to isomorphism, we can equivalently require that $(\perp_X, \perp_Y) \in \mathcal{S}$ (which implies that $0_X = 0_Y$) instead of **(init-⊥)**.

Operations on b&f system

Reasoning about games in terms of b&f system is quite flexible as it allows us to work with it using standard operations on relations. We summarise this fact in the following observation.

Lemma 3.3.4 *The following properties hold on any elementary path category.*

1. *The diagonal relation $\Delta_X = \{(e, e) \mid e : P \rightarrowtail X\}$ is a b&f system.*
2. *If $\mathcal{S}$ is a b&f system between X and Y then its inverse*

$$\mathcal{S}^{-1} = \{(n, m) \mid (m, n) \in \mathcal{S}\}$$

is a b&f system between Y and X.

3. *If $\mathcal{S}_1$ is a b&f system between X and Y and $\mathcal{S}_2$ is a b&f system between Y and Z, then their relational composite*

$$\mathcal{S}_1 ; \mathcal{S}_2 = \{(m, e) \mid (m, n) \in \mathcal{S}_1, \ \ (n, e) \in \mathcal{S}_2\}$$

is a b&f system between X and Z.

4. *If $\{\mathcal{S}_i\}_i$ is a collection of b&f systems between X and Y, then their union*

$$\bigcup_i \mathcal{S}_i = \{(m, n) \mid (m, n) \in \mathcal{S}_i \ \text{ for some } \ i\}$$

is a b&f system between X and Y.

Corollary 3.3.5 *The relation $\rightleftharpoons$ on $\mathfrak{X}$ is symmetric, reflexive and transitive.*

3.4 Bisimilarity and behavioural equivalence

It is the case in modal logic that back-and-forth equivalence is equivalent to the existence of a span of *bounded morphisms*, which are also called *p-morphisms*. It turns out that the same is true in the EM categories of coalgebras of our game comonads.

To make this claim precise we define the variant of bounded morphisms for elementary path categories, dubbed *open pathwise-embeddings*, which originates in the setting of arboreal categories [11, 14] and which, in turn, is inspired by the *open maps* formulation of bisimulation of concurrent systems [71].

It turns out that our simpler setting of elementary path categories suffices for implications in one direction (Proposition 3.4.4). For the converse, one needs the full power of arboreal categories (Proposition 4.8.2), which we introduce in Chapter 4. Furthermore, we show a correspondence with the related notion of *behavioural equivalence*, studied extensively in the Universal Coalgebra literature [19, 63, 111]. In fact, in the setting of arboreal categories all four notions coincide (Theorem 4.8.8).

Open pathwise embeddings

We say that a morphism $f : X \to Y$ in $\mathfrak{X}$ is a **pathwise-embedding** if for every path embedding $e : P \rightarrowtail X$ the composite

$$P \xrightarrow{e} X \xrightarrow{f} Y$$

is an embedding too. And we say that f is **embedding-open** if for any commutative diagram of path embeddings as displayed below

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle m} \\ P & \xrightarrow{n} & Q \end{array}$$

there exists a **diagonal filler**, that is, an embedding $d\colon Q \rightarrowtail X$ which makes the two arising triangles commute (i.e. $e = dn$ and $m = fd$).

We denote by $\multimap\!\!\!\rightarrow$ the morphisms which are **open pathwise-embeddings**, that is, which are both embedding-open and pathwise-embedding, which we often abbreviate by **OPE**.

Composing with b&f systems

Without proof we observe that (open) pathwise-embeddings behave well with respect to composition.

Lemma 3.4.1 *Pathwise-embeddings and open pathwise-embeddings are closed under composition.*

Next, we observe that each OPE induces a b&f system, as follows.

Lemma 3.4.2 *Assuming $(P_{\mathbf{0}})$ and (PE_{mn}), if $f\colon X \multimap\!\!\!\rightarrow Y$ is an open pathwise-embedding, then*

$$\mathcal{S}_f = \{(e, fe) \mid e\colon P \rightarrowtail X \text{ is a path-embedding}\}$$

is a b&f system between X and Y, i.e. $X \rightleftharpoons Y$.

Proof. **(init-forth)** Let $e\colon P \rightarrowtail X$ be a path embedding. Since f is a pathwise-embedding, fe is an embedding and so $(e, fe) \in \mathcal{S}_f$.

(init-back) Let $e\colon P \rightarrowtail Y$ be a path embedding. Take the factorisation $\mathbf{0} \xrightarrow{!} 0_Y \xrightarrow{\perp_Y} Y$ of the unique morphism $\mathbf{0} \xrightarrow{!} Y$. Due to initiality of $\mathbf{0}$, the composite $\mathbf{0} \xrightarrow{!} P \xrightarrow{e} Y$ is also equal to $\mathbf{0} \xrightarrow{!} Y$. Hence, by minimality of the factorisation, there is some $n\colon 0_Y \rightarrowtail Q$ such that $\perp_Y = en$.

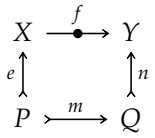
Similarly, we have a factorisation $\mathbf{0} \xrightarrow{!} 0_X \xrightarrow{\perp_X} X$ of the initial morphism $\mathbf{0} \xrightarrow{!} X$. However, since f is a pathwise-embedding, $f\perp_X$ is an embedding and so we obtain another factorisation of $\mathbf{0} \xrightarrow{!} Y$ as $\mathbf{0} \xrightarrow{!} 0_Y \xrightarrow{f\perp_X} X$ Therefore, by minimality of $\perp_Y$ there is some $m\colon 0_Y \rightarrowtail 0_X$ such that $\perp_Y = f\perp_X m$. Therefore, we have the following commutative square.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \perp_X \cdot m}\uparrow & & \uparrow{\scriptstyle e} \\ 0_Y & \xrightarrow{n} & Q \end{array}$$

Therefore, by openness of f there is a diagonal filler $d\colon Q \rightarrowtail X$. Therefore, we have that $e = fd$ and so $(e, d) = (e, fe) \in \mathcal{S}_f$.

(forth) Assume $(e, fe) \in \mathcal{S}_f$ and $e \leq e'$, i.e. $e = e'm$ for some m. Then, $fe \leq fe'$ since $fe = fe'm$ and $(e', fe') \in \mathcal{S}_f$.

(back) Assume $(e, fe) \in \mathcal{S}_f$ and $fe \leq n$, i.e. $fe = nm$ for some m. This means that we have the following commutative diagram.



Since f is open, there is a diagonal filler $d\colon Q \rightarrowtail X$ such that $n = fd$. Therefore, $(d, n) = (d, fd) \in \mathcal{S}_f$. □

Lemma 3.4.2 together with Lemma 3.3.4 yields the following useful fact about compositions of open pathwise-embeddings and b&f systems.

Lemma 3.4.3 *Assume (P_0) and (PE_{mn}) and that $f\colon X \rightarrow\!\!\!\!\bullet\; Y$ is an open pathwise-embedding. Then,*

- *if $\mathcal{S}$ witnesses $Y \rightleftharpoons Z$ then $\{(e, e') \mid (fe, e') \in \mathcal{S}\}$ witnesses $X \rightleftharpoons Z$, and*
- *if $\mathcal{S}$ witnesses $Z \rightleftharpoons X$ then $\{(e, fe') \mid (e, e') \in \mathcal{S}\}$ witnesses $Z \rightleftharpoons Y$.*

Bisimilarity and behavioural equivalence

This observation motivates two more definitions of bisimilarity akin to how bisimilarity is usually defined in the universal coalgebra literature. For X, Y in an elementary path category $\mathfrak{X}$,

- $X \sim Y$ denotes that X, Y are **bisimilar**, i.e. there exists a **SOPE** between X and Y, that is, a span of OPEC of the form $X \leftarrow\!\!\!\!\bullet\; Z \rightarrow\!\!\!\!\bullet\; Y$,
- $X \leftrightarrows Y$ denotes that X, Y are **behaviourally equivalent**, i.e. there exists a **coSOPE** between X and Y, that is, a cospan of OPEs of the form $X \rightarrow\!\!\!\!\bullet\; Z \leftarrow\!\!\!\!\bullet\; Y$.

Next we show that bisimilarity and behavioural equivalence are both stronger than b&f equivalence, defined earlier.

Proposition 3.4.4 *In an elementary path category satisfying (P_0) and (PE_{mn}),*

- *$X \sim Y$ implies $X \rightleftharpoons Y$, and*
- *$X \leftrightarrows Y$ implies $X \rightleftharpoons Y$.*

Proof. For the first item, consider a SOPE

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

witnessing $X \sim Y$. By Lemma 3.3.4 and Lemma 3.4.2, the relation

$$\{(fe, ge) \mid e : P \rightarrowtail Z,\ P \in \mathcal{P}\}.$$

obtained as $\mathcal{S}_f^{-1}\,;\mathcal{S}_g$ is a b&f system witnessing $X \rightleftharpoons Y$.

The second item is proved dually, i.e. a coSOPE $X \xrightarrow{f} Z \xleftarrow{g} Y$ yields the b&f system

$$\{(m : P \rightarrowtail X,\ n : P \rightarrowtail Y) \mid fm = gn\}$$

obtained as $\mathcal{S}_f \,;\, \mathcal{S}_g^{-1}$, witnessing $X \rightleftharpoons Y$. □

As we discuss in Section 4.8, in arboreal categories we have equivalences in both items of Proposition 3.4.4. However, the full power of arboreal categories is not needed in order to show that behavioural equivalence implies bisimilarity. Below we show that we only need pullbacks for that.

Open morphisms à la Joyal–Nielsen–Winskel

In [71] the notion of open morphisms is defined for an arbitrary category $\mathcal{X}$ with a collection of objects $\mathcal{P}$ called paths[9], i.e. the setting in *op. cit.* does not refer to embeddings in the definition. Following [71] we say that a morphism $f : X \to Y$ is **open** if for any diagram

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Q & \longrightarrow & Y \end{array}$$

where P and Q are paths, there exists a diagonal filler $Q \to X$.

9: To be more precise, the setting of [71] is parameterised by a subcategory $\mathcal{P}$ which need not be full. Then, in the square diagram below, the morphism $P \to Q$ is required to be from $\mathcal{P}$. In our setting we always assume that $\mathcal{P}$ is a full subcategory.

Lemma 4.1 in [11] shows that in the setting of arboreal categories this notion agrees with the previously defined notion of open pathwise-embeddings. We show that this is the case even in the much weaker setting of elementary path categories, i.e. unlike in *op. cit.* we do not need implicitly given paths, in the sense of Section 4.1.

We prove the equivalence in steps, going via an intermediate notion of **semi-open** morphisms, which are the morphisms $f : X \to Y$ such that any diagram

$$\begin{array}{ccc} P & \rightarrowtail & X \\ \downarrow & & \downarrow f \\ Q & \rightarrowtail & Y \end{array}$$

where the horizontal morphisms are path embeddings, there exists a diagonal filler $Q \to X$. Clearly, we have the following in any elementary path category.

$$\text{open} \ \Rightarrow\ \text{semi-open} \ \Rightarrow\ \text{embedding-open}.$$

First, we show that the first implication reverses under mild assumptions.

Lemma 3.4.5 *In an elementary path category satisfying $(P_{\twoheadrightarrow})$, every semi-open morphism is open.*

Proof. Assume $f : X \to Y$ is semi-open and we are given a commutative square

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Q & \longrightarrow & Y \end{array}$$

where P and Q are paths. We factorise the horizontal arrows as follows

$$\begin{array}{ccccc} P & \twoheadrightarrow & P' & \rightarrowtail & X \\ \downarrow & & \downarrow d & & \downarrow f \\ Q & \twoheadrightarrow & Q' & \rightarrowtail & Y \end{array}$$

where the diagonal filler d exists from the orthogonality of quotients and embeddings in factorisation systems. Observe that P' and Q' are paths by the assumption that paths are closed under quotients. Then, since f is semi-open, we have a diagonal filler $Q' \to X$ giving us the required diagonal filler of the original square as the composite $Q \twoheadrightarrow Q' \to X$. □

Therefore, to show the desired equivalence, we need to establish the converse of the second implication above. We do this in two steps.

Lemma 3.4.6 *In an elementary path category satisfying ($P_{\twoheadrightarrow}$), every semi-open morphism is a pathwise-embedding.*

Proof. Let $f : X \to Y$ be a semi-open morphism and let $e : P \rightarrowtail X$ be a path embedding. Let

$$\begin{array}{ccc} P & \overset{e}{\rightarrowtail} & X \\ {\scriptstyle q}\downarrow & & \downarrow f \\ P' & \overset{m}{\rightarrowtail} & Y \end{array}$$

be such that the arrows going down-right are the factorisation of fe. Since f is semi-open we have a diagonal filler $d : P' \to X$ and since $e = dq$ is an embedding we further get that q is an embedding too. We know that embeddings compose and so mq is an embedding too, as required. □

Finally, we check the fact that OPEs are semi-open.

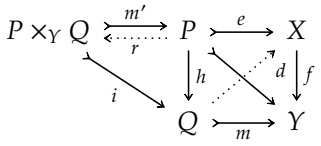
Lemma 3.4.7 *In an elementary path category with (M_{pb}), every open pathwise-embedding is semi-open.*

Proof. Let $f : X \to Y$ be an OPE and suppose we have the following commutative square of solid arrows

$$\begin{array}{ccc} P & \overset{e}{\rightarrowtail} & X \\ \downarrow{\scriptstyle h} & \nearrow_{fe} & \downarrow f \\ Q & \overset{m}{\rightarrowtail} & Y \end{array}$$

where P and Q are paths. Since f is a pathwise-embedding, fe is an embedding too. Let $P \times_Y Q$ be the pullback of m and fe, as displayed

below.



Recall that the pullback projections m' and i are embeddings since m and fe are so too, by (M_{pb}). Then, by the fact that f is embedding-open, there is a $d\colon Q \to X$ such that $em' = di$ and $m = fd$. Next, since the original square commutes, by the universal property of pullbacks, there is some $r\colon P \to P \times_Y Q$ such that $\mathrm{id} = m'r$ and $h = ir$. Then, we obtain that d diagonalises the original square since

$$dh = dir = em'r = e.$$

The other triangle commutes automatically by the defining property of d. □

Putting all these lemmas together gives us the promised fact. Note that we can further simplify the assumptions by requiring (QM). Then, just requiring pullbacks of embeddings to exist implies (M_{pb}) (cf. Remark 3.2.1).

Theorem 3.4.8 *In an elementary path category with (M_{pb}) and ($P_{\twoheadrightarrow}$), a morphism is open if and only if it is an open pathwise-embedding.*

This theorem allows us to obtain some results specific to OPEs in a much more uniform way, see e.g. Proposition 3.4.10 and Proposition 8.3.1.

Pullbacks of open morphisms

Next we recall that open morphisms are closed under pullbacks along any morphism, which holds true in any category with specified path objects.

Lemma 3.4.9 (Proposition 3 of [71]) *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

be a pullback square. If f is open then so is f'.

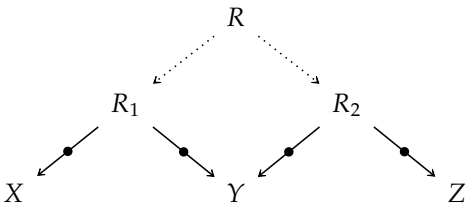
This allows us to entail that bisimilarity is a transitive relation and that behavioural equivalence implies bisimilarity.

Proposition 3.4.10 *Let $\mathfrak{X}$ be an elementary path category with pullbacks and satisfying ($P_{\twoheadrightarrow}$) and (M_{pb}). Then,*

1. *$X \leftrightarrows Y$ implies $X \sim Y$, and*
2. *$X \sim Y$ and $Y \sim Z$ implies $X \sim Z$.*

Proof. By Theorem 3.4.8 and Lemma 3.4.9 we know that OPEs are closed under pullbacks. Therefore, (1) follows from the fact that any cospan of OPEs $X \twoheadrightarrow S \twoheadleftarrow Y$ can be turned into a span of OPEs $X \twoheadleftarrow R \twoheadrightarrow Y$ by taking the pullback.

Next, for (2), consider a pair of spans of OPEs witnessing $X \sim Y$ and $Y \sim Z$:



The pullback of both OPEs with codomain Y is an OPE too. Furthermore, OPEs compose (Lemma 3.4.1), which gives us a span of OPEs $X \twoheadleftarrow R \twoheadrightarrow Z$ as required. □

Furthermore, the converse implication to those in Proposition 3.4.10 holds in the setting of arboreal categories, see Proposition 4.8.3.

Arboreal categories 4

Abramsky and Reggio [10] introduced arboreal categories in order to capture the features of EM categories of coalgebras which ensure good properties of the back-and-forth equivalence and bisimilarity. Consequently, arboreal categories provide a powerful framework for a general study of logical equivalences and a new synthetic approach to logical-meta theorems. From our point of view arboreal categories provide an important class of examples of elementary path categories. Furthermore, their rich structure ensures that they admit a representation theorem (Theorem 4.5.26).

The point of this chapter is not to provide a deep dive into the theory of arboreal categories, for that it is better to follow standard references, such as [12]. Instead, we use the theory of arboreal categories to justify the technical properties of elementary path categories, introduced in Section 3.2, which is important e.g. in Chapter 9.

4.1 Implicitly given paths . . . 63
4.2 Path functor 64
4.3 Arboreal categories 67
4.4 Concrete arboreal categories 69
4.5 The representation theorem 70
Preliminaries 71
The motivation 72
The functor of paths 72
From functors to categories 74
Factorisation system of $\mathsf{Ar}(\Phi)$ 75
Subobjects and paths in $\mathsf{Ar}(\Phi)$ 77
Tree colimits in $\mathsf{Ar}(\Phi)$ 78
The correspondence 80
4.6 Examples 85
Game Comonads 85
Models of concurrency . . . 85
Arboreal categories of types 86
4.7 Consequences of the representation theorem 86
4.8 Four ways of expressing bisimulations 92
Simplifications 93
Implications for logic 96

4.1 Implicitly given paths

Arboreal categories stem from the realisation that the open maps formulation of bisimulation from concurrent systems [71] applies directly to study of games in finite model theory, as we described extensively in Chapter 3. However, the second crucial insight is that, unlike with general elementary path categories where the class of paths has to be given explicitly, paths in arboreal categories are specified implicitly, based on the choice of the factorisation system.

Paths from embeddings. We say that an elementary path category $\mathcal{X}$ has **paths given implicitly** if the class of paths $\mathcal{P}$ consists of precisely the objects $X \in \mathcal{X}$ such that their embedding comma category $\mathcal{X} \downarrow X$ is equivalent to a finite chain poset (viewed as a category).

The reason to require 'equivalent to' rather than the simpler requirement that $\mathcal{X} \downarrow X$ 'is' a finite chain is because we might have two subobject embeddings $e\colon S \rightarrowtail X$ and $e'\colon S' \rightarrowtail X$ such that $e \leq e'$ and $e' \leq e$, giving $S \cong S'$ but not $S = S'$.

For this reason it is often more convenient to work with $\mathsf{Sub}(X)$, which is the quotient of $\mathcal{X} \downarrow X$ by the equivalence relation $\approx$ induced by $\leq$ relation. Formally, define

$$\mathsf{Sub}(X) = \{[e] \mid e\colon S \rightarrowtail X\}$$

as the class of equivalence classes $[e] = \{e' \mid e \approx e'\}$ with respect to the relation $\approx$ where

$$e \approx e' \quad \text{iff} \quad e \leq e' \text{ and } e' \leq e.$$

Then, the preorder $\leq$ lifts to the $\approx$-equivalence classes by setting $[e] \leq [e']$ iff $e \leq e'$.

Observe that an object X of $\mathcal{X}$ is an implicitly given path if and only if $\mathsf{Sub}(X)$ *is* a finite chain.

To avoid set-theoretic issues in the following we require that $\mathcal{X}$ is $\mathcal{M}$**-well-powered**, i.e. that, for every X, $\mathsf{Sub}(X)$ is a set and not a proper class of equivalence classes.

Example 4.1.1 Following Example 3.1.1, we take the categories $\mathsf{EM}(\mathbb{E}_k)$, $\mathsf{EM}(\mathbb{P}_k)$ and $\mathsf{EM}(\mathbb{M}_k)$ as examples of elementary path categories where paths are the objects which are linear chains in the implicit coalgebra order $\sqsubseteq_\alpha$ and embeddings $\overline{\mathcal{M}}$ are lifted using Lemma 2.9.6. Importantly, these categories have paths given implicitly. This is due to the fact that embeddings $S \rightarrowtail (A, \alpha)$ in $\mathsf{EM}(\mathbb{C})$, for our game comonads $\mathbb{C}$ are uniquely given by their image, i.e. as a subset $S \subseteq A$. Furthermore, since morphisms in $\mathsf{EM}(\mathbb{C})$ are tree morphisms, such subset $S \subseteq A$ must be a downset with respect to the implicit order $\sqsubseteq_\alpha$.

The following is an immediate observation following just from the fact that paths are given implicitly.

Lemma 4.1.2 *If $\mathcal{X}$ has implicitly given paths then the following hold.*

1. *$(P_{\rightarrowtail})$ holds, i.e. paths are closed under subobjects.*
2. *Paths are **linearly ordered**, i.e. for e and e' in $\mathcal{P} \downarrow X$ if X is a path then either $e \leq e'$ or $e' \leq e$.*
3. *If $\mathcal{X}$ has an initial object $\mathbf{0}$ then it is a path.*
4. *if (QM) holds and the required factorisation system is stable, then $(P_{\twoheadrightarrow})$ holds, i.e. paths are closed under quotients.*

Proof. (1) and (2) are immediate. (3) To see that $\mathbf{0}$ is a path, consider its subobject $S \rightarrowtail \mathbf{0}$. By initiality of $\mathbf{0}$, we obtain that $\mathbf{0} \xrightarrow{!} S \rightarrowtail \mathbf{0}$ is the identity and so, by Lemma 2.9.3, $S \rightarrowtail \mathbf{0}$ is both a quotient and an embedding, i.e. an isomorphism. Therefore, $\mathsf{Sub}(\mathbf{0})$ is a singleton poset.

For (4) follow the proof of [11, Lemma 3.5], which only relies on the assumptions listed here. □

For a category $\mathcal{X}$ with a proper factorisation system $(\mathcal{Q}, \mathcal{M})$, observe that, by Lemma 2.9.3, the triple $(\mathcal{X}, \mathcal{P}, \mathcal{M})$, with paths $\mathcal{P}$ given implicitly, is an elementary path category.

Assumption 4.1.3 In the following, when given a category with a proper factorisation system, unless said otherwise, we automatically assume that it as an elementary path with paths given implicitly.

4.2 Path functor

The path functor $\mathbb{P}$ is a central construction in the theory of arboreal categories. However, we define it in the weaker setting of elementary path categories because it also serves us as a motivation for the axioms of arboreal categories.

Given an object X of an elementary path category $\mathcal{X}$, observe that the domain $\mathcal{P} \downarrow X$ of the functor $\mathsf{PDiag}_X : \mathcal{P} \downarrow X \to \mathcal{X}$ corresponds to a subclass of $\mathsf{Sub}(X)$. In particular, define

$$\mathbb{P}X = (\mathcal{P} \downarrow X)/_{\approx}$$

as the poset of $\approx$-equivalence classes of path embeddings $\mathcal{P} \downarrow X$.

Lemma 4.2.1 *For any X in a $\mathcal{M}$-well-powered $\mathcal{X}$ with a proper factorisation system $(\mathbb{Q}, \mathcal{M})$, the poset $\mathbb{P}X$ ordered by $\leq$ is a forest, in sense of Section 2.5. Furthermore, if $\mathcal{X}$ has an initial object $\mathbf{0}$, then $\mathbb{P}X$ is a tree.*

Proof sketch (following [11, Lemma 3.12]). To see that $\mathbb{P}X$ is a forest, we need to check that, for each path embedding $e : P \rightarrowtail X$, the downset $\downarrow[e]$ in $\mathbb{P}X$ is a finite chain. To this end, consider $[e_1], [e_2] \leq [e]$ in $\mathbb{P}X$. By definition, this means that $e_1 \leq e$ and $e_2 \leq e$. Then, since P is a path given implicitly, by Lemma 4.1.2, either $e_1 \leq e_2$ or $e_2 \leq e_1$, in which case we get $[e_1] \leq [e_2]$ or $[e_2] \leq [e_1]$, respectively.

If $\mathcal{X}$ has $\mathbf{0}$ then, by Lemma 4.1.2, $\mathbf{0}$ is a path. Hence, the factorisation $\mathbf{0} \twoheadrightarrow 0_X \rightarrowtail X$ of the initial morphism $! : \mathbf{0} \to X$ is the least element of $\mathbb{P}X$. □

It turns out that under two additional assumptions the mapping $X \mapsto \mathbb{P}X$ extends to a functor into the **category of trees** $\mathcal{T}$, that is, the category whose objects are trees and whose morphisms are forest morphisms, in sense of Section 2.5.

Given a morphism $f : X \to Y$ in $\mathcal{X}$ with an proper factorisation system, we define $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$ by

$$[e] \mapsto [\exists_f e]$$

where $[e]$ denotes the $\approx$-equivalence class of $e : P \rightarrowtail X$ and $\exists_f e$ is the embedding part of the factorisation of fe as $P \twoheadrightarrow \cdot \rightarrowtail Y$.

The mapping $\mathbb{P}f$ is clearly monotone. To make sure that it is a tree morphism, we need additional assumptions. In particular, we need the following, which can be viewed as a standalone axiom of elementary path categories $\mathcal{X}$ satisfying (QM).

($\mathbf{Q}_{2\text{-}3}$) Quotients satisfy the 2-out-of-3 condition: For any paths P, Q, R in $\mathcal{X}$, if a composite $P \to Q \to R$ is a quotient then so is $P \to Q$.

Following [105], we gather our current assumptions under one definition.

Definition 4.2.2 *A category $\mathcal{X}$ is a **wooded path category** if*

- *it is equipped with a stable proper factorisation system $(\mathbb{Q}, \mathcal{M})$,*
- *it is $\mathcal{M}$-well-powered,*
- *has an initial object $\mathbf{0}$,*
- *its paths[1] satisfy the 2-out-of-3 property ($Q_{2\text{-}3}$).*

1: Following Assumption 4.1.3, paths are given implicitly.

We obtain the following.

Theorem 4.2.3 *If $\mathfrak{X}$ is a category with a stable proper factorisation system and satisfying the 2-out-of-3 condition, then $\mathbb{P}$ is a functor*

$$\mathbb{P}\colon \mathfrak{X} \to \mathfrak{F}$$

where $\mathfrak{F}$ is the category of forests and their morphisms. If, furthermore, $\mathfrak{X}$ has an initial object, i.e. it is wooded, *then $\mathbb{P}$ further restricts to*

$$\mathbb{P}\colon \mathfrak{X} \to \mathfrak{T}$$

where $\mathfrak{T}$ is the category of trees and tree morphisms.

Proof. The proof follows the same step as the proof of Theorem 3.11 in [11], while noting that no more than the assumptions of this theorem are invoked in the proof. □

The path functor is useful for detecting properties of morphisms.[2]

2: Item 1 has been proved before, using stronger assumptions, see e.g. [5, Lemma 6.11]. The proof of item 3 is standard, see [11, Lemma 3.15], however, the statement is slightly misstated therein.

Proposition 4.2.4 *Let $\mathfrak{X}$ be a wooded path category. Then:*

1. *A morphism f between paths is a quotient iff $\mathbb{P}f$ is surjective.*
2. *If a morphism f is an embedding then $\mathbb{P}f$ is injective.*
3. *Between any two paths exists at most one embedding, up to isomorphism.*

Proof. (1) Let $f\colon P \to Q$ be a morphism between paths. The left-to-right implication follows from stability of the factorisation system as, for any subpath $e\colon Q' \rightarrowtail Q$, the pullback

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle e'}\uparrow & & \uparrow{\scriptstyle e} \\ P' & \xrightarrow{f'} & Q' \end{array}$$

exists and, moreover, q' and e' are quotient and embedding, respectively, by stability and Lemma 2.9.3. Since factorisations are unique, up to isomorphism, $\mathbb{P}(f)(e') = e$.

Conversely, if $\mathbb{P}(f)$ is surjective then there is some $e\colon P' \rightarrowtail P$ such that $\mathbb{P}(f)(e) = \mathrm{id}_Q$. By definition of $\mathbb{P}(f)$, we have the following commutative square.

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_Q} \\ P' & \longrightarrow & Q \end{array}$$

Then, f is a quotient, by Lemma 2.9.3, since both $P' \twoheadrightarrow Q$ and id_Q are too.

(2) Follows from the fact that $\mathbb{P}f$ for f an embedding is computed by $e \mapsto f \cdot e$. Then, if $\mathbb{P}f([e]) = [f \cdot e] = [f \cdot e'] = \mathbb{P}f([e'])$, we see that $f \cdot e = f \cdot e' \cdot i$ for some isomorphism. Therefore, since embeddings are monos $e = e' \cdot i$, i.e. $[e] = [e']$.

(3) Whenever $m, n\colon P \rightarrowtail Q$ are two embeddings between paths we have that $\mathbb{P}(m) = \mathbb{P}(n)$ since there is at most one tree morphisms

between any two finite chains in $\mathcal{T}$. Consider the trivial path embedding $\mathrm{id}_P \colon P \rightarrowtail P$. Since path embeddings are also pathwise-embeddings, $\mathbb{P}(m)([\mathrm{id}_P]) = [m \cdot \mathrm{id}_P] = [m]$ and similarly $\mathbb{P}(n)([\mathrm{id}_P]) = [n]$. Therefore, assuming $\mathbb{P}(m) = \mathbb{P}(n)$ implies that $[m] = [n]$ i.e. that $m \approx n$. □

Example 4.2.5 By Lemma 2.9.8 we see that our principal examples of EM categories $\mathcal{X} = \mathsf{EM}(\mathbb{C})$ for a game comonad $\mathbb{C}$ (cf. Example 4.1.1 and Example 3.1.1), satisfy that the lifted factorisation system is stable and proper. Moreover, it is immediate from the concrete description of our EM categories (Section 2.5) that the paths in $\mathsf{EM}(\mathbb{C})$ satisfy the 2-out-of-3 condition.

Tree bijection. Fix a morphism $f \colon X \to Y$ in a wooded path category. We say that f is a **tree bijection** if $\mathbb{P}f \colon \mathbb{P}X \to \mathbb{P}Y$ is an isomorphism of trees. This now completes the definition of the $A \Rrightarrow_{+\mathcal{X}} B$ relation from Section 3.1.

4.3 Arboreal categories

We saw in the previous sections, that, under mild additional assumptions on our elementary path categories, their objects of such categories resemble trees. This matches our intuition from the concrete description of the EM categories of our game comonads, cf. Section 2.5.

Next, we encode the fact that the paths[3] are the basic building blocks of arboreal categories. This means that all other objects are uniquely encoded using just paths and, moreover, we are always able to decompose every object back to its encoding using paths.

3: With their 'linear structure', cf. Lemma 4.1.2.

To this end, we fix an elementary path category $\mathcal{X}$. The first property is formally represented as follows.

- The objects of $\mathcal{X}$ are **path generated** if, for every $X \in \mathcal{X}$, the induced morphism $\operatorname{colim} \mathsf{PDiag}_X \to X$ from (3.4) is an isomorphism.

For the latter property, we need the notion of tree colimits. Recall from Section 4.2 that the canonical path diagram functor PDiag_X, for a fixed X, is equivalent to a **tree diagram**, that is, a functor

$$D \colon T \to \mathcal{X}$$

where T is a tree in the sense of Section 2.5, and for every $x \leq y$ in T, $D(x \leq y)$ is an embedding between paths $P \rightarrowtail P'$. In particular, for every $x \in T$, $D(x) \in \mathcal{P}$. We call colimits of tree diagrams **tree colimits**.

Consequently, to express that objects can be 'decomposed' back into their encoding using paths, we require that paths 'decompose tree colimits', which is expressed as follows.

- A path P is **tree-connected** if, given a tree diagram $D \colon T \to \mathcal{X}$ and an embedding $e \colon P \rightarrowtail \operatorname{colim} D$, there exists the least $t \in T$ such that e factors via the inclusion map $D(t) \to \operatorname{colim} D$.

We define arboreal categories as the wooded path categories which are path generated, admit tree colimits and have tree-connected paths. In full, this amounts to the following.

Definition 4.3.1 *A category $\mathcal{X}$ is **arboreal** if*

- *it is equipped with a stable proper factorisation system $(\mathcal{Q}, \mathcal{M})$,*
- *it is $\mathcal{M}$-well-powered,*
- *it admits tree colimits and has an initial object $\mathbf{0}$,*
- *its paths[4] are tree-connected and*
- *satisfy the 2-out-of-3 property ($Q_{2\text{-}3}$), and*
- *its objects are path generated.*

4: Following Assumption 4.1.3, paths are given implicitly.

Next we show that most of our axioms for elementary path categories are automatically validated in the arboreal setting.

Theorem 4.3.2 *Let $\mathcal{X}$ be an arboreal category, then as an elementary path category it satisfies*

$$(P_{\mathbf{0}}),\ (P_{\rightarrowtail}),\ (PE_{mn}),\ (QM),\ \text{and}\ (P_{\twoheadrightarrow}).$$

If $\mathcal{X}$ has pullbacks of embeddings then (M_{pb}) holds.

Proof. We check the axioms one by one:

- (QM) appears explicitly in the axiomatisation of arboreal categories.
- ($P_{\mathbf{0}}$), ($P_{\rightarrowtail}$), and ($P_{\twoheadrightarrow}$) follow from Lemma 4.1.2.
- (M_{pb}) is a standard property of orthogonal factorisation systems, cf. Lemma 2.9.3.
- (PE_{mn}) follows from ($P_{\twoheadrightarrow}$) and (QM), cf. Remark 3.2.1. □

We prove the missing ($PE_{\vee}$) and (PE_{cp}) in Theorem 4.7.8, using the representation theorem. Most likely these properties already follow from weaker assumptions, however, the representation makes verifying ($PE_{\vee}$) and (PE_{cp}) straightforward.

Example 4.3.3 We show that the Eilenberg–Moore category of coalgebras $\mathsf{EM}(\mathbb{C})$ is arboreal for comonads $\mathbb{C} = \mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$. Using Lemmas 2.9.6, 2.9.7 and 2.9.8 we see that $\mathsf{EM}(\mathbb{C})$ has a stable proper factorisation system, obtained by lifting the (surjective, strong injective) factorisation system on the underlying category $\mathcal{S}tr(\sigma)$ or $\mathcal{S}tr(\sigma^+)$.

The fact that $\mathsf{EM}(\mathbb{C})$ is $\mathcal{M}$-well-powered follows from the fact that the underlying category is. Next, we see that $\mathsf{EM}(\mathbb{C})$ has tree colimits and the initial object from the fact that it is cocomplete, by Proposition 2.9.1. Example 4.2.5 gives us that $\mathsf{EM}(\mathbb{C})$ satisfies the 2-out-of-3 property for paths given implicitly. What is left to show is that paths are tree-connected and dense.

For the latter, observe that the concrete description of our categories of coalgebras from Section 2.5 gives us that each $\mathsf{EM}(\mathbb{C})$ can be identified with a category of structures with a tree order $\sqsubseteq$ where $\mathsf{EM}(\mathbb{E}_k)$ and $\mathsf{EM}(\mathbb{P}_k)$ instead of being forest-ordered we artificially make into tree-ordered by adding the empty sequence [] as a new root.

Then, for example in case when $\mathbb{C} = \mathbb{E}_k$, it is easy to see that every object $(A, R_1, \ldots, R_n, \sqsubseteq)$ of $\mathsf{EM}(\mathbb{E}_k)$ is a tree-colimit of $D\colon (A, \sqsubseteq) \to \mathsf{EM}(\mathbb{C})$

where, for $a \in A$, $D(a)$ is the subpath of A on the universe $\downarrow a$. The morphisms $D(a \leq b) \colon D(a) \to D(b)$ are the inclusions of subpaths in A, i.e. embeddings. Hence paths are dense in $\mathsf{EM}(\mathbb{E}_k)$.

Finally, to see why paths in $\mathsf{EM}(\mathbb{E}_k)$ are tree-connected, consider a tree diagram $D \colon T \to \mathsf{EM}(\mathbb{E}_k)$ and a path embedding $f \colon P \rightarrowtail \operatorname{colim} D$. Since $\operatorname{colim} D$ is computed as the quotient of the disjoint union $\sqcup_{t \in T} D(t)$ by the least equivalence relation containing all pairs (x, y) of the form $x = D(t \leq t')(y)$. Since T is a tree, there is a minimal $t \in T$ such that $f(\top)$ is in the equivalence class of elements of $D(t) \subseteq \operatorname{colim} D$, where $\top$ is the top element of P. Then, we see that f factors via the colimit inclusion $D(t) \hookrightarrow \operatorname{colim} D$.

Checking that paths are dense and tree-connected in $\mathsf{EM}(\mathbb{P}_k)$ and $\mathsf{EM}(\mathbb{M}_k)$ goes along the same lines. On the other hand, Elias Percy has recently shown[5] that $\mathsf{EM}(\mathbb{C})$, where $\mathbb{C}$ is the guarded or loosely-guarded comonad, is not arboreal since it does not satisfy the 2-out-of-3 property.

5: Private communication, in summer 2026.

Remark 4.3.4 Note that our definition of arboreal categories originates in Jakl and Reggio [68], which is an adaptation of the axiomatisation from the original paper on arboreal categories due to Abramsky and Reggio [10, 11]. The change is motivated by the limitations uncovered in [68]. However, as shown in Section 6 of [68], the fundamental properties of arboreal categories in the 'old sense' still apply in this 'new sense'. Therefore, we can often directly invoke properties of arboreal categories from [10, 11]. Note that some proofs therein accidentally rely on the assumption that there is at most one embedding between any two paths. In many instances this assumption can be avoided, however, we typically work with 'concrete' arboreal categories, introduced below, which is a stronger assumption, cf. Remark 4.4.3.

Finally, we rephrase the last condition of arboreal categories in standard categorical terms.

Lemma 4.3.5 ([11, Lemma 5.1]) *Let $\mathcal{X}$ be a category with a stable proper factorisation system. Then, its objects are path generated if and only if the inclusion of paths $\mathcal{P} \hookrightarrow \mathcal{X}$ is **dense**, i.e. for every object $X \in \mathcal{X}$,*

$$X \cong \operatorname*{colim}_{P \in \mathcal{P},\ P \to X} P.$$

4.4 Concrete arboreal categories

For the representation theorem we need the following additional assumption for arboreal categories, which is validated in our example models, see Example 4.2.5.

Definition 4.4.1 *Let $\mathcal{X}$ be an elementary path category. We say that it is **concrete** if between any two paths P, Q there is at most one morphism $P \to Q$.*

The name is justified by the following fact, which shows that concrete arboreal categories are those arboreal categories which are 'concrete over trees'.

Lemma 4.4.2 (Lemma 6.12 in [5]) *An arboreal category*[6] $\mathfrak{X}$ *is concrete if and only if if the path functor* $\mathbb{P}\colon \mathfrak{X} \to \mathcal{T}$ *is faithful.*

6: Note that here we use different axioms of arboreal categories from those in *op. cit.* but the result still applies, cf. Remark 4.3.4.

Remark 4.4.3 There are additional weaker notions to consider. Namely, for paths P, Q, we can require that

1. there is at most one morphism $P \to Q$,
2. there is at most one quotient $P \twoheadrightarrow Q$, or
3. there is at most one embedding $P \rightarrowtail Q$.

It follows that (1) is equivalent to (2) and they both imply (3).

The implications (1) $\to$ (2) and (1) $\to$ (3) are immediate. Hence, what is left to show is that (2) implies (1). Following [5, Lemma 6.12], consider two morphisms $f_1, f_2\colon P \to Q$ and their factorisations $f_i = e_i \cdot q_i$. Since e_1, e_2 are embeddings into Q, we have that $e_1 = e_2 \cdot i$ for some isomorphism i since, either $e_1 \leq e_2$ or $e_2 \leq e_1$ and in both cases the comparison morphism e.g. $i\colon P_1 \to P_2$ such that $e_1 = e_2 \cdot i$ must be a quotient and hence iso by Proposition 4.2.4. Then, by (2), we have that $q_2 = i \cdot q_1$. Hence, $f_2 = e_2 \cdot q_2 = e_2 \cdot i \cdot q_1 = e_1 \cdot q_1 = f_1$.

Note that (3) is equivalent to fact that the automorphism group $\mathsf{Aut}(P)$ is trivial. The left-to-right direction of this claim follows from the fact that automorphisms are embeddings. Conversely, consider embeddings $m, n\colon P \rightarrowtail Q$. By Proposition 4.2.4, we know that $m = n \cdot i$ for some isomorphism $i\colon P \to P$. Hence, i is the identity if we know that $\mathsf{Aut}(P)$ is trivial.

This equivalent rephrasing of (3) gives us another proof of the implication (2) $\to$ (3). Indeed, we have that $\mathsf{Aut}(P)$ is a submonoid of $\mathsf{End}(P)$. Moreover, (2) implies that the endomorphism monoid $\mathsf{End}(P)$ is trivial, i.e. paths are 'rigid' (as in Lemma 3.3.2) since, by Proposition 4.2.4, endomorphisms of paths $P \to P$ in $\mathcal{T}$ are always quotients.

4.5 The representation theorem

In this section we prove a representation theorem for concrete arboreal categories with only a **set of paths**, that is, which *up to isomorphism* only have a set of paths. This additional 'smallness' condition is not strictly necessary but allows us to avoid certain set-theoretic questions.

Note that the requirement to having only a set of paths exactly mirrors the smallness condition of locally finitely presentable categories, which are required to only have a set of compact objects, cf. Subsection"Locally finitely presentable categories" on page 40.

The representation theorem allows us to verify many desirable properties of arboreal categories quickly and construct new examples, see Sections 4.6, 4.7 and 4.8.

Preliminaries

Representable arboreal categories. Our representability theorem works for a class of arboreal categories which we call 'representable.' We define this notion for more general elementary path categories, since some of the constructions we describe on the way are interesting on their own.

To this end, given a wooded path category[7] $\mathcal{X}$ we say that it is **representable** if it is concrete[8] and has a set of paths.

7: Following Assumption 4.1.3, its paths are given implicitly.

8: 'Concrete' in sense of Definition 4.4.1.

Paths and their height. In the following it is convenient to notationally distinguish between different classes of paths $\mathcal{P}$ of different elementary path categories. Therefore, to emphasise the origin of a given class, we denote by

$$\mathcal{P}_{\mathcal{X}}$$

the *class* of all paths of $\mathcal{X}$. Furthermore, in case $\mathcal{X}$ has a set of paths, we denote by

$$\widetilde{\mathcal{P}}_{\mathcal{X}}$$

a *set* of chosen representatives[9] of isomorphism classes of paths in $\mathcal{X}$.

9: For most constructions, it would be enough to treat $\widetilde{\mathcal{P}}_{\mathcal{X}}$ as the set of equivalence classes of paths $\mathcal{P}_{\mathcal{X}}$. However, having a choice of representatives is necessary when defining 'canonical tree diagrams,' defined below.

In particular, we can represent each equivalence class $[e\colon P \rightarrowtail X]$ in $\mathbb{P}X$ by its unique representative $\widetilde{e}\colon \widetilde{P} \rightarrowtail X$ where $\widetilde{P}$ is from $\widetilde{\mathcal{P}}_X$.

Given a path P in an elementary path category with paths given implicitly (e.g. in an arboreal category), the **height** $\mathrm{ht}(P)$ of P is the height $\mathrm{ht}(\mathbb{P}P)$ of the path $\mathbb{P}P \in \mathcal{T}$, in sense of Section 2.5.

Lemma 4.5.1 *Let $\mathcal{X}$ be a wooded path category. If there is a quotient $P \twoheadrightarrow Q$ between paths P, Q then* $\mathrm{ht}(P) = \mathrm{ht}(Q)$.

Proof. By Proposition 4.2.4(1), having $P \twoheadrightarrow Q$ implies that the induced $\mathbb{P}P \to \mathbb{P}Q$ is a surjective tree map between finite chains, i.e. $\mathbb{P}P \cong \mathbb{P}Q$. Therefore, $\mathrm{ht}(P) = \mathrm{ht}(\mathbb{P}P) = \mathrm{ht}(\mathbb{P}Q) = \mathrm{ht}(Q)$. □

Canonical tree diagram. Let $\mathcal{X}$ be a representable wooded path category and let $\widetilde{\mathcal{P}}$ be a chosen *set* of representatives of isomorphism classes of paths in $\mathcal{X}$. Then, for X in $\mathcal{X}$, the canonical path diagram

$$\mathsf{PDiag}_X\colon \mathcal{P} \downarrow X \to \mathcal{X}$$

is equivalent to the **canonical tree diagram**

$$\mathsf{TDiag}_X\colon \mathbb{P}X \to \mathcal{X}$$

which sends the equivalence class $[e\colon P \rightarrowtail X]$ to the unique $\widetilde{P} \in \widetilde{\mathcal{P}}$ such that $P \cong \widetilde{P}$. Next, given $[e] \leq [e']$ there is a unique $m\colon P \rightarrowtail P'$ such that $e = e' \cdot m$. We set $\mathbb{P}([e] \leq [e'])$ to be an embedding $\widetilde{P} \rightarrowtail \widetilde{P}'$, which uniquely given by the fact that $\mathcal{X}$ is concrete.[10]

10: Note that concreteness is crucial here. Without it, for a different $e'' \in [e]$ there might be some non-trivial automorphism $i\colon P \to P$ such that $e = e'' \cdot i$ and, then, the choice of $\mathbb{P}([e] \leq [e'])\colon \widetilde{P} \to \widetilde{P}'$ would not be unique.

Trees as functors. In what follows it is convenient to view trees and forests as functors $\mathbb{N}^{op} \to \mathbf{Set}$ where $\mathbb{N}$ is the poset of natural numbers, with standard ordering, viewed as a category. The functor $F\colon \mathbb{N}^{op} \to \mathbf{Set}$ associated to a forest $(F, \leq)$ sends a natural number n to the set of elements of height n. On morphisms, $F(n \leq m)\colon F(m) \to F(n)$ sends an element x of height m to its unique predecessor of height n. Then, natural transformations between forests, when viewed as functors, are precisely forest morphisms, when viewed as posets. Consequently, the category $\mathcal{F}$ of forests and forest morphisms is equivalent to the functor category $[\mathbb{N}^{op}, \mathbf{Set}]$ of functors $\mathbb{N}^{op} \to \mathbf{Set}$ and natural transformations between them, i.e. it is the category of presheaves $\widehat{\mathbb{N}}$ over $\mathbb{N}$. Similarly, $\mathcal{T}$ is equivalent to the subcategory of $[\mathbb{N}^{op}, \mathbf{Set}]$, consisting of functors $F\colon \mathbb{N}^{op} \to \mathbf{Set}$ where $F(0)$ is a singleton set.

The motivation

Let $\mathcal{X}$ be a representable arboreal category with a chosen set of paths $\widetilde{\mathcal{P}}_{\mathcal{X}}$. Recall that, since $\mathcal{X}$ is path generated, every object X is uniquely determined by the canonical diagram $\mathsf{PDiag}_X\colon \mathcal{P}\downarrow X \to \mathcal{X}$. In fact, X is already determined by its canonical tree diagram $\mathsf{TDiag}_X\colon \mathbb{P}X \to \mathcal{X}$.

We see that, since TDiag_X is a tree diagram, only embeddings between paths matter for a representation of X, i.e. TDiag_X factors via the small subcategory $\widetilde{\mathcal{P}}^e_{\mathcal{X}}$ of $\mathcal{X}$ on paths in $\widetilde{\mathcal{P}}_{\mathcal{X}}$ and embeddings between them. However, since $\mathcal{X}$ is concrete, $\widetilde{\mathcal{P}}^e_{\mathcal{X}}$ is equivalent to a thin category i.e. a poset which, furthermore, is a forest (cf. Section 2.5).

In other words, the functor TDiag_X determining X gives a labelling to every node of the tree $\mathbb{P}X$ by a path in $\widetilde{\mathcal{P}}^e_{\mathcal{X}}$. It is easy to see that this labelling is actually a forest morphism $\mathbb{P}X \to \widetilde{\mathcal{P}}^e_{\mathcal{X}}$.

The representation theorem (Theorem 4.5.26) says that objects in arboreal categories are nothing more than trees and their labelling by paths and morphisms in arboreal categories are the tree morphisms which respect their labelling.

The functor of paths

To make the development of the theory of representation more systematic, we prefer to view the tree $\mathbb{P}X$ and the forest $\widetilde{\mathcal{P}}^e_{\mathcal{X}}$ as a functors $\mathbb{N}^{op} \to \mathbf{Set}$. In particular, the latter is represented by the functor

$$\Phi_{\mathcal{X}}\colon \mathbb{N}^{op} \to \mathbf{Set}$$

which sends a natural number n to the set of paths of height n. Furthermore, for $n \leq m$, the function $\Phi_{\mathcal{X}}(n \leq m)\colon \Phi_{\mathcal{X}}(m) \to \Phi_{\mathcal{X}}(n)$ sends a path of height m to its unique prefix of height n. In fact, for $P \in \Phi_{\mathcal{X}}(m)$ and $Q = \Phi_{\mathcal{X}}(n \leq m)(P)$, by being a unique prefix, there is a unique $Q \rightarrowtail P$ which, moreover, is in $\widetilde{\mathcal{P}}^e_{\mathcal{X}}$. Note that the uniqueness follows from the fact that $\mathcal{X}$ is concrete.

Observe that $\Phi_{\mathcal{X}}$ contains all information about objects of $\mathcal{X}$. Indeed, every object X of $\mathcal{X}$ is a colimit of some tree diagram $D\colon T \to \mathcal{X}$ (in fact, of $\mathsf{TDiag}_X\colon \mathbb{P}X \to \mathcal{X}$) and, by definition, codomain of tree diagrams can

be restricted to just the category $\widetilde{\mathcal{P}}^{e}_{\mathcal{X}}$ of the chosen paths and embeddings between them.

For representing morphisms in $\mathcal{X}$ we make use of the rest of the structure of $\widetilde{\mathcal{P}}_{\mathcal{X}}$, i.e. the morphsims not included in $\widetilde{\mathcal{P}}^{e}_{\mathcal{X}}$. In fact, the only information that is missing when going from $\widetilde{\mathcal{P}}_{\mathcal{X}}$ to $\widetilde{\mathcal{P}}^{e}_{\mathcal{X}}$ is the information about quotients between paths. Indeed, every morphism between paths $P \to Q$ factors as $P \twoheadrightarrow R \rightarrowtail Q$ where, by Lemma 4.1.2, R is again a path. Furthermore, since $P \twoheadrightarrow R$ is a quotient between paths, P and R are of the same height (Proposition 4.2.4). Therefore, the missing information about quotients can be encoded within the sets $\Phi_{\mathcal{X}}(n)$ of paths of the same height.

Therefore, instead of mapping $\Phi_{\mathcal{X}}$ to the category of sets, we map it to the category **Pos** of posets and monotone functions. In summary, for a representable arboreal category $\mathcal{X}$, we set

$$\Phi_{\mathcal{X}} : \mathbb{N}^{op} \to \mathbf{Pos}$$

to be the mapping which sends a natural number n to the (full) subcategory $\Phi_{\mathcal{X}}(n)$ of $\mathcal{X}$ consisting of paths in $\widetilde{\mathcal{P}}_{\mathcal{X}}$ of height n and quotients between them. Note that $\Phi_{\mathcal{X}}(n)$ is a poset by concreteness of $\mathcal{X}$, i.e. $P \leq Q$ iff there exists a morphism[11] $P \to Q$. For any $n \leq m$, the mapping $\Phi_{\mathcal{X}}(n \leq m)$ is the prefix map, as above.

11: Since P and Q are of the same height, we see by Proposition 4.2.4 that the morphism $P \to Q$ must be a quotient.

Next we verify the correctness of this construction.

Lemma 4.5.2 *The mapping $\Phi_{\mathcal{X}}$ is a well-defined functor.*

Proof. It is immediate to see that the action of $\Phi_{\mathcal{X}}$ preserves identity morphism and that $\Phi_{\mathcal{X}}(n \leq m) \cdot \Phi_{\mathcal{X}}(m \leq l) = \Phi_{\mathcal{X}}(n \leq l)$. What is left to show is that, for any $n \leq m$, the mapping $\Phi_{\mathcal{X}}(n \leq m)$ is monotone $\Phi_{\mathcal{X}}(m) \to \Phi_{\mathcal{X}}(n)$.

To this end, let $P \twoheadrightarrow Q$ be a quotient between paths of length m and let $P' \rightarrowtail P$ be the inclusion of the height n as a subpath of P, i.e. $\Phi_{\mathcal{X}}(n \leq m)(P) = P'$. Then, the factorisation of the composite of these two morphisms gives us the square

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \uparrow & & \uparrow \\ P' & \xrightarrow{f'} & Q' \end{array}$$

where Q' is a path since $\mathcal{X}$ has paths given implicitly and Q' is a subobject of Q. Without loss of generality assume that Q' is in $\widetilde{\mathcal{P}}_{\mathcal{X}}$. Moreover, by Lemma 4.5.1, Q' has the same height as P'. Since path prefixes are unique up to isomorphism, we can assume that $\Phi_{\mathcal{X}}(n \leq m)(Q) = Q'$. We have shown that $P \leq Q$ (in the quotient order) implies that $P' \leq Q'$, as required. □

Observe that Lemma 4.5.2 holds, more generally, for any category satisfying the conditions of Lemma 4.2.1. Also, existence of an initial object in $\mathcal{X}$ is reflected in the structure of $\Phi_{\mathcal{X}}(0)$.

Proposition 4.5.3 *Let $\mathfrak{X}$ be a representable wooded path category. Then, the mapping $\Phi_{\mathfrak{X}} : \mathbb{N}^{op} \to \mathbf{Pos}$ is functorial. Moreover, $\Phi_{\mathfrak{X}}(0)$ has the least element.*

Proof. Recall that $\mathfrak{X}$ has an initial object $\mathbf{0}$ which is a path of height 0 (Lemma 4.1.2). Therefore, for any path P of height 0, there is a unique morphism $\mathbf{0} \to P$, the factorisation of the initial map into P yields $\mathbf{0} \twoheadrightarrow \cdot \rightarrowtail P$. However, the embedding is an isomorphism, otherwise P would not be of height 0, i.e. $\mathbf{0}$ is the least element of $\Phi_{\mathfrak{X}}(0)$. □

From functors to categories

Given a functor $\Phi\colon \mathbb{N}^{op} \to \mathbf{Pos}$ where $\Phi(0)$ has a least element, we define the corresponding arboreal category $\mathsf{Ar}(\Phi)$ following the intuition we built above.

Recall that an object X of an arboreal category $\mathfrak{X}$ is fully described by a tree diagram $\mathsf{TDiag}_X : \mathbb{P}X \to \mathfrak{X}$. These, in turn, can be fully encoded as a tree labelling, i.e. a natural transformation $\mathbb{P}X \to |\Phi_{\mathfrak{X}}|$ mapping $e\colon P \rightarrowtail X$ to P where $|\Phi_{\mathfrak{X}}|\colon \mathbb{N}^{op} \to \mathbf{Set}$ is the description of $\Phi_{\mathfrak{X}}$ without the information about quotients.[12]

12: Using that $\mathfrak{X}$ is representable, recall from Subsection"Preliminaries" on page 71 that $\mathbb{P}X$ can be viewed as a poset of path embeddings $P \rightarrowtail X$ where $P \in \widetilde{\mathcal{P}}_{\mathfrak{X}}$.

Dually, we encode the same information as a natural transformation $d(\mathbb{P}X) \to \Phi_{\mathfrak{X}}$ where $d(\mathbb{P}X)\colon \mathbb{N}^{op} \to \mathbf{Pos}$ has a trivial order structure. Formally, $d(\text{-})$ is a functor

$$d\colon \mathcal{T} \to [\mathbb{N}^{op}, \mathbf{Pos}]$$

which takes a tree T, viewed as a poset, turns it into a functor $T : \mathbb{N}^{op} \to \mathbf{Set}$ (as on Page 72) and then adds the *discrete order* to the fibres, i.e. for any $n \in \mathbb{N}$ and $x, y \in d(T)(n)$ we have that $x \leq y$ iff $x = y$.

The arboreal category arising from Φ is defined as the category of labelled trees $d(T) \to \Phi$. Formally, define $\mathsf{Ar}(\Phi)$ to be the lax comma category

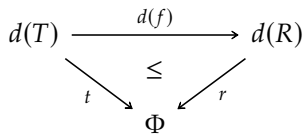
$$\mathsf{Ar}(\Phi) \;=\; d \downarrow_{lax} \Phi.$$

In more detail, objects of $\mathsf{Ar}(\Phi)$ are pairs (T, t) where T is a tree and t is a **labelling of T in Φ**, i.e. a natural transformation $t\colon d(T) \to \Phi$. Morphisms of $\mathsf{Ar}(\Phi)$ are **lax morphisms** $(T, t) \to (R, r)$ given by a tree morphisms $f : T \to R$ such that $t \leq r \cdot d$ pointwise, that is, for every $n \in \mathbb{N}$ and $x \in T$ of height n, [13]

13: In other words, $x \in d(T)(n)$.

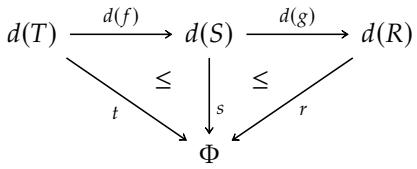
$$t_n(x) \leq r_n(d(f)_n(x)) \tag{4.1}$$

which, in the language of diagrams, we depict as follows.



The identity morphisms $(T, t) \to (T, t)$ in $\mathsf{Ar}(\Phi)$ are given by the tree identity $T \to T$. Furthermore, the composition of tree morphisms f and

g yields a composition in $\mathsf{Ar}(\Phi)$, since the order relation $\leq$ is transitive, as depicted below.



Remark 4.5.4 Due to the fact that $d(T)$ is a collection of discrete posets, in order to define a labelling $t\colon d(T) \to \Phi$, it is enough to ensure naturality of t. In other words, the requirement that each component map $t_n\colon d(T)(n) \to \Phi(n)$ of the labelling t is a morphism in **Pos** (i.e. a monotone function) is true automatically.

Remark 4.5.5 In the following we leave the application of the functor d implicit, that is, we only write T instead of $d(T)$ if we need to view the tree T as a functor $\mathbb{N}^{op} \to$ **Pos** and similarly for tree morphisms. Furthermore, we often omit writing the height index for natural transformations that correspond to labelling functions or tree morphisms. For example, the inequality (4.1) above is simplified to $t(x) \leq r(f(x))$.

Factorisation system of $\mathsf{Ar}(\Phi)$

An essential part of the structure of an arboreal category is its stable proper factorisation system. We consider the pair of

(surjective, strict injective)

classes of morphisms of $\mathsf{Ar}(\Phi)$ to take its role, where $f\colon (T,t) \to (R,r)$ is surjective (resp. injective) if the underlying tree morphism $f\colon T \to S$ is. Moreover, f is **strict** if $t = r \cdot f$, instead of just mere inequality $t \leq r \cdot f$.

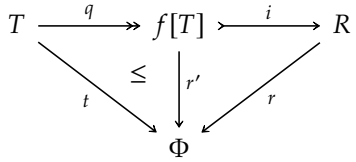
The following characterisation of isomorphisms justifies the intuition that embeddings (i.e. strict injective morphisms) capture the idea of inclusions which reflect the structure. In fact, as we show in Lemma 4.7.1 below, strict morphisms correspond precisely to pathwise-embeddings.

Lemma 4.5.6 *Isomorphisms are precisely the strict morphisms $f\colon (T,t) \to (S,s)$ such that f is an isomorphism of trees.*

Proof. Assume $f\colon (T,t) \to (S,s)$ and $g\colon (S,s) \to (T,t)$ are such that $f \cdot g = \mathrm{id}$ and $g \cdot f = \mathrm{id}$. To see that f is strict observe that, for any $x \in T$, $s(f(x)) \leq t(g(f(x))) = t(x)$ where the first inequality is the fact that g is a morphism $(S,s) \to (T,t)$. The converse direction is immediate. □

Proposition 4.5.7 *The pair (surjective, strict injective) is a proper factorisation system on $\mathsf{Ar}(\Phi)$.*

Proof. We observe that every morphism $f : (T, t) \to (R, r)$ factorises as



where $i \cdot q$ is the (surjective, injective) factorisation of f in the category $\mathcal{T}$ of trees and r' is the restriction of r to $f[T]$, that is, $i : (f[T], r') \to (R, r)$ is strict since $r' = r \cdot i$.

Next we check that strict injective morphisms are orthogonal to surjective morphisms. Consider a commutative diagram in $\mathsf{Ar}(\Phi)$

$$\begin{array}{ccc} (T,t) & \xrightarrow{q} & (R,r) \\ {\scriptstyle f}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle g} \\ (S,s) & \xrightarrow[m]{} & (V,v) \end{array} \tag{4.2}$$

where q is surjective and m is iso-injective. Since (surjective, injective) is a factorisation system in trees, we have a diagonal filler $d : R \to S$ for the square $g \cdot q = m \cdot f$.

To check that d is a morphism $(R, r) \to (S, s)$ we observe that

$$r \leq v \cdot g = v \cdot m \cdot d = s \cdot d$$

where the last inequality follows from the fact that m is strict.

Lastly, both surjective maps and strict injective morphisms are closed under composition. Therefore, (surjective, strict injective) is an orthogonal factorisation system. Furthermore, the factorisation system is proper since surjective and injective morphisms of trees are epis and monos in $\mathcal{T}$, respectively. □

Proposition 4.5.8 *The factorisation system (surjective, strict injective) is stable.*

Proof. Given a surjective morphism $q : (S, s) \twoheadrightarrow (U, u)$ and a strict injective morphism $m : (T, t) \rightarrowtail (U, u)$, we construct a pullback

$$\begin{array}{ccc} (W,w) & \overset{q'}{\dashrightarrow} & (T,t) \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ (S,s) & \xrightarrow{q} & (U,u) \end{array}$$

such that q' is surjective. Let W be the pullback of m and q in $\mathcal{T}$ with $m' : W \to S$ and $q' : W \to T$ being its projections. Since m is injective so is m' and since q is surjective and (surjective, injective) is a stable factorisation in $\mathcal{T}$, q' is surjective too. Next, define the labelling $w : W \to \Phi$ as the composite $s \cdot m'$, yielding that m' is a strict injective morphism $(W, w) \to (S, s)$.

Next we observe that q' is a morphism $(W, w) \to (T, t)$, however, this follows by a simple diagram chasing:

$$w = s \cdot m' \leq u \cdot q \cdot m' = u \cdot m \cdot q' = t \cdot q'$$

What is left to show is that this is a pullback diagram. Let $f\colon (V, v) \to (T, t)$ and $g\colon (V, v) \to (S, s)$ be another cone over q and m. Let $d\colon V \to W$ be the comparison morphism in $\mathcal{T}$. In particular, this means that $g = m' \cdot d$ and $f = q' \cdot d$.

Furthermore, observe that d is a morphism $(V, v) \to (W, w)$ since

$$v \leq s \cdot g = s \cdot m' \cdot d = w \cdot d$$

where the last equality holds because m' is strict. Finally, unicity of d follows from the fact that m' is a monomorphism. □

Subobjects and paths in $\mathsf{Ar}(\Phi)$

With the factorisation system for $\mathsf{Ar}(\Phi)$ in place we can identify what the implicitly given paths look like, in sense of Section 4.1.

For an object (T, t) of $\mathsf{Ar}(\Phi)$, observe that any embedding $e\colon (S, s) \rightarrowtail (T, t)$ is uniquely determined by the tree embedding part $e\colon S \rightarrowtail T$ since we must have that $s = t \cdot e$. In other words, we have an order isomorphism $\mathsf{Sub}(T, t) \cong \mathsf{Sub}(T)$.[14] Consequently, we have a simple characterisation of path objects.

14: Recall that $\mathsf{Sub}(\text{-})$ is the poset of equivalence classes of subobjects, cf. Section 4.1.

Lemma 4.5.9 *Path objects in* $\mathsf{Ar}(\Phi)$ *are of the form* (P, p) *where* P *is a finite path.*

Another direct consequence of the fact that $\mathsf{Sub}(T, t) \cong \mathsf{Sub}(T)$, for every $(T, t) \in \mathsf{Ar}(\Phi)$, is that $\mathsf{Ar}(\Phi)$ is well-powered since the category of trees $\mathcal{T}$ is too.

Proposition 4.5.10 *The category* $\mathsf{Ar}(\Phi)$ *is* $\mathscr{M}$*-well-powered, for the class* $\mathscr{M} = \{$*strict injective*$\}$.

Our concrete representation also allows to directly verify another requirement of arboreal categories.

Proposition 4.5.11 *Paths in* $\mathsf{Ar}(\Phi)$ *satisfy the 2-out-of-3 condition.*

Proof. Consider morphisms $f\colon (P, p) \to (Q, q)$ and $g\colon (Q, q) \to (R, r)$ between paths. If the composite is a quotient then that means that gf is surjective. Then, by the 2-out-of-3 condition for the category $\mathcal{T}$, we have that f is surjective, i.e. $f\colon (P, p) \to (Q, q)$ is a quotient in $\mathsf{Ar}(\Phi)$. □

Next we observe that paths (P, p) in $\mathsf{Ar}(\Phi)$ are determined by the image of their top element $\top_P$.

Lemma 4.5.12 *Let P be a path of height n in $\mathcal{T}$. Then, paths (P, p) in $\mathsf{Ar}(\Phi)$ are in a bijective correspondence with the elements of $\Phi(n)$.*

In particular, for every $a \in \Phi(n)$, the corresponding path (P, p_a) of height n has the labelling defined by $p_a(x) = \Phi(m \leq n)(a)$ where $x \in P$ is of height $m \leq n$.

Proof. Let $\top_P$ be the top element of P. Since labelling function $p \colon P \to \Phi$ needs to be natural, any choice of $p(\top_P) = a$ determines $p(x)$ for any other $x \leq \top_P$. Indeed, if x is of height m, then we must have that $p(x) = p(P(m \leq n)(\top_P)) = \Phi(m \leq n)(p(\top_P))$. □

Consequently, we see that, up to isomorphism, there is at most a set of paths. Let

$$\widetilde{\mathcal{P}}_{\mathsf{Ar}(\Phi)}$$

be the *set* of paths of the form (P_n, p_x) where P_n is a fixed path $(x_0 < x_1 < \ldots < x_n)$ in $\mathcal{T}$ of height n and $p_x \colon P_n \to \Phi$ is determined by $p_x(x_n) = x$ for some $x \in \Phi(n)$.

It is immediate that every path $(P, p) \in \mathsf{Ar}(\Phi)$ is isomorphic to the path (P_n, p_x) in $\widetilde{\mathcal{P}}_{\mathsf{Ar}(\Phi)}$ where n is the height of P and $p(\top_P) = x$.

Proposition 4.5.13 *For any functor $\Phi \colon \mathbb{N}^{op} \to \mathbf{Pos}$, the category $\mathsf{Ar}(\Phi)$ has a set of paths and is concrete.*

Proof. Above we have justified that $\mathsf{Ar}(\Phi)$ has a set of paths. Furthermore, it is also concrete because morphisms between paths $(P, p) \to (Q, q)$ are uniquely determined by morphisms between the underlying paths $P \to Q$ and there is at most one such morphism. □

Tree colimits in $\mathsf{Ar}(\Phi)$

As the next necessary ingredient we describe tree colimits in $\mathsf{Ar}(\Phi)$. Consider a tree-indexed diagram $T \to \mathsf{Ar}(\Phi)$ of paths and embeddings between them, where T is non-empty. Denote by (P_t, p_t) the image of $t \in T$ under this functor and, whenever $t \leq t'$, denote by $e_{t,t'} \colon (P_t, p_t) \rightarrowtail (P_{t'}, p_{t'})$ the corresponding embedding.

Since $\mathcal{T}$ is cocomplete, the diagram $\{e_{t,t'}\}_{t \leq t'}$ has a colimit in $\mathcal{T}$, which we denote by L and the inclusion maps by $e_t \colon P_t \to L$. We equip L with a labelling $l \colon L \to \Phi$ as follows. For an $x \in L$ of height n, set $l(x) \in \Phi(n)$ to be $p_t(y)$ for any t such that there is $y \in P_t$ such that $e_t(y) = x$. Note that the choice of t does not matter since all morphisms in the diagram are strict.

Lemma 4.5.14 *$l \colon L \to \Phi$ is a labelling, i.e. it is a natural transformation.*

Proof. For any $n \leq m$ we have to check that the following square commutes.

$$\begin{array}{ccc} L(m) & \xrightarrow{l_m} & \Phi(m) \\ {\scriptstyle L(n\leq m)}\downarrow & & \downarrow{\scriptstyle \Phi(n\leq m)} \\ L(n) & \xrightarrow{l_n} & \Phi(n) \end{array}$$

Let $x \in L(m)$ and let t be any $t \in T$ such there is a $y \in P_t(m)$ with $e_t(y) = x$. Since $e_t : P_t \to L$ is a natural transformation, we see that

$$\begin{aligned} \Phi(n \leq m)(l(x)) &= \Phi(n \leq m)(l(e_t(y))) \\ &= \Phi(n \leq m)(p_t(y))) \\ &= p_t(P_t(n \leq m)(y)) \end{aligned}$$

which is equal to $l(L(n \leq m)(x))$ since the latter is defined as $p_{t'}(y')$ for any t' and $y' \in P_{t'}$ such that $e_{t'}(y') = L(n \leq m)(x)$. □

Lemma 4.5.15 *The family* $\{e_t : (P_t, p_t) \to (L, l)\}$ *is a colimit cocone for* $\{e_{t,t'}\}_{t \leq t'}$.

Proof. We first observe that each $e_t : P_t \to L$ is not only a morphism $(P_t, p_t) \to (L, l)$ but it is in fact a strict injective morphism, i.e. an embedding. Consequently, the cocone is well-defined.

To show that this cocone is initial, consider a cocone $\{f_t : (P_t, p_t) \to (S, s)\}_{t \in T}$ in $\mathsf{Ar}(\Phi)$. Since the reduct $\{f_t : P_t \to S\}_{t \in T}$ is a cocone in trees, there is a unique comparison morphism $f : L \to S$ such that $f_t = f \cdot e_t$ for every $t \in T$.

We verify that f is a morphism $(L, l) \to (S, s)$ in $\mathsf{Ar}(\Phi)$, i.e. that for every $x \in L$, $l(x) \leq s(f(x))$. Let $y \in P_t$ be such that $e_t(y) = x$. Then, $p_t(y) \leq s(f_t(y))$. However, by definition, $l(x) = p_t(y)$ and also $s(f_t(y)) = s(f(e_t(y))) = s(f(x))$.

The unicity of this comparison morphism follows from the fact that the comparison map $L \to S$ in the category of trees is unique. □

In order to have an initial object in $\mathsf{Ar}(\Phi)$ we need an appropriate labelling $o : \mathbf{0} \to \Phi$ for the initial tree $\mathbf{0}$. Since $\mathbf{0}$ is a path of height 0, the labelling is fully determined by its choice of an element of $\Phi(0)$ where the root of $\mathbf{0}$ is sent. It is immediate that, if the chosen element is the least element in $\Phi(0)$ then, for any other $(T, t) \in \mathsf{Ar}(\Phi)$, there is a unique morphism $(\mathbf{0}, o) \to (T, t)$.

To conclude, we have shown the following.

Proposition 4.5.16 $\mathsf{Ar}(\Phi)$ *has non-empty tree colimits. If, furthermore,* $\Phi(0)$ *has a least element, then* $\mathsf{Ar}(\Phi)$ *has initial object.*

Finally, we verify the remaining two properties of arboreal categories.

Proposition 4.5.17 *Paths in* $\mathsf{Ar}(\Phi)$ *are tree-connected and every object of* $\mathsf{Ar}(\Phi)$ *is path-generated.*

Proof. First we check that paths in $\mathsf{Ar}(\Phi)$ are tree-connected. Let (Q, q) be a path and $m\colon (Q, q) \rightarrowtail \mathrm{colim}_{t\in T}(P_t, p_t)$ an embedding into a tree colimit, with the diagram morphisms denoted by $e_{t,t'}\colon P_t \rightarrowtail P_{t'}$ and colimit inclusion maps denoted by $e_t\colon (P_t, p_t) \to \mathrm{colim}_{t\in T}(P_t, p_t)$. Since Q is a path, the embedding m is determined by its image $m(\top_Q)$ of the top element $\top_Q$ of Q. Let P_t be the least path (in the embedding order) such that $m(\top_Q)$ is in the image of e_t in $\mathrm{colim}_t P_t$.

Let $m\colon Q \to P_t$ be the map that sends $\top_Q$ to the (unique) element of the same height in P_t. Then, we have that $m\colon Q \to \mathrm{colim}_t P_t$ decomposes as m followed by $e_t\colon P_t \to \mathrm{colim}_{t\in T} P_t$. Furthermore, since both m and e_t are strict, we have that

$$q = l \cdot m = l \cdot e_t \cdot m = p_t \cdot m$$

where l is the labelling of $\mathrm{colim}_{t\in T}(P_t, p_t)$. Consequently, m is a (strict) morphism $(Q, q) \to (P_t, p_t)$. Unicity follows from the fact that e_t is an embedding, cf. the proof of Lemma 4.5.15.

To check that $\mathsf{Ar}(\Phi)$ is path-generated, let (T, t) be any object of $\mathsf{Ar}(\Phi)$. We define a diagram $\{(P_t, p_t)\}_t$ of shape T in $\mathsf{Ar}(\Phi)$. For $t \in T$, the path P_t is the downset $\downarrow t$ of T. Let $e_t\colon P_t \hookrightarrow T$ be the inclusion map corresponding to the inclusion $\downarrow t \subseteq T$ and let p_t be the labelling function given by $t \cdot e_t$. We see that the collection $\{(P_t, p_t)\}_t$ together with the morphisms $e_{t,t'}\colon P_t \hookrightarrow P_{t'}$ induced by the inclusions $P_t \subseteq P_{t'}$, for $t \leq t'$, is a tree-shaped diagram of paths and embeddings. Therefore, by Lemma 4.5.15, since T is the colimit of $\{e_{t,t'}\colon P_t \to P_{t'}\}_{t,t'}$ in $\mathcal{T}$ so is (T, t) the colimit of $\{e_{t,t'}\colon (P_t, p_t) \to (P_{t'}, p_{t'})\}_{t,t'}$ in $\mathsf{Ar}(\Phi)$. □

The results above, i.e. Propositions 4.5.8, 4.5.10, 4.5.11, 4.5.13, 4.5.16 and 4.5.17, give us the desired result.

Theorem 4.5.18 *For any functor $\Phi\colon \mathbb{N}^{op} \to$ **Pos** where $\Phi(0)$ has a least element, the category $\mathsf{Ar}(\Phi)$ is a representable arboreal category.*

Remark 4.5.19 By carefully inspecting the proofs, one can see that the construction works for any functor $\Phi\colon \mathbb{N}^{op} \to$ **Pos**, that is, without requiring that the least element of $\Phi(0)$ exists. The obtained category $\mathsf{Ar}(\Phi)$ would satisfy all axioms of arboreal categories except that it would not have the initial object. However, recall from Theorem 4.2.3 that without the initial object the path functor only maps to forests $\mathbb{P}\colon \mathcal{X} \to \mathcal{F}$. On the other hand, one can check that the path functor of $\mathsf{Ar}(\Phi)$ produces a tree for every Φ. Therefore, for the representation theorem of these more general categories we would have to slightly strengthen the requirement that paths are dense to requiring that every object is a colimit of some tree diagram. As we know, the latter is equivalent to paths being dense if the category has an initial object.

The correspondence

Given a functor $\Phi\colon \mathbb{N}^{op} \to$ **Pos** where $\Phi(0)$ has a least element, by Theorem 4.5.18, the category $\mathsf{Ar}(\Phi)$ is a representable arboreal category. This means that, by Proposition 4.5.3, we also have a functor

$\Phi_{\mathsf{Ar}(\Phi)} \colon \mathbb{N}^{op} \to \mathbf{Pos}$. We show the two functors in question are naturally isomorphic.

Proposition 4.5.20 *For any functor* $\Phi \colon \mathbb{N}^{op} \to \mathbf{Pos}$,[15] *we have* $\Phi \cong \Phi_{\mathsf{Ar}(\Phi)}$.

15: We dropped the assumption that $\Phi(0)$ has a least element here because one can observe that this assumption is not needed for the definition of $\mathsf{Ar}(\Phi)$ and that $\Phi_{\mathfrak{X}}$ is also well-defined for wooded path categories without **0**. Cf. Remark 4.5.19.

Proof. Recall from Subsection"Subobjects and paths in $\mathsf{Ar}(\Phi)$" on page 77 that, for a fixed n, we have a bijection

$$\Phi(n) \to \Phi_{\mathsf{Ar}(\Phi)}(n) \tag{4.3}$$

sending $a \in \Phi(n)$ to (P_n, p_a) where P_n is a fixed chain $(x_0 < \ldots < x_n)$ and the label $p_a \colon P_n \to \Phi$ is determined by $p_a(x_n) = a$.

Furthermore, the mapping (4.3) is an isomorphism of posets. In other words, we have that $a \leq b$ in $\Phi(n)$ if and only if

$$\begin{array}{ccc} P_n & \xrightarrow{\quad id \quad} & P_n \\ & {\scriptstyle p_a} \searrow \quad \leq \quad \swarrow {\scriptstyle p_b} & \\ & \Phi & \end{array}$$

is a morphism in $\Phi_{\mathfrak{X}}$. This follows from the fact that $p_a(x_n) = a$ and $p_b(x_n) = b$ and the fact that, for $m \leq n$, $\Phi(m \leq n)$ is a monotone function $\Phi(n) \to \Phi(m)$. Indeed, $a \leq b$ if and only if, for every $m \leq n$, $p_a(x_m) = \Phi(m \leq n)(a) \leq \Phi(m \leq n)(b) = p_b(x_m)$.

What is left to check is that the collection of the equivalences (4.3) is natural in n, that is that, for $m \leq n$, the following diagram commutes.

$$\begin{array}{ccc} \Phi(n) & \xrightarrow{\cong} & \Phi_{\mathsf{Ar}(\Phi)}(n) \\ {\scriptstyle \Phi(m \leq n)} \downarrow & & \downarrow {\scriptstyle \Phi_{\mathsf{Ar}(\Phi)}(m \leq n)} \\ \Phi(m) & \xrightarrow{\cong} & \Phi_{\mathsf{Ar}(\Phi)}(m) \end{array}$$

To see why, observe that, for an $a \in \Phi(n)$, the right-to-bottom composition sends a to (P_m, p) where $p(x_i) = \Phi(i \leq n)(a)$. On the other hand, the bottom-to-right composition yields (P_m, p') where $p'(x_i) = \Phi(i \leq m)(\Phi(m \leq n)(a))$ which is equal to $p(x_i)$ by functoriality of Φ. □

In some sense Proposition 4.5.20 shows that, for a functor Φ, the category $\mathsf{Ar}(\Phi)$ encodes all information about Φ in its subcategory of paths. In the following we show also the converse, that is, that whenever we have a representable arboreal category $\mathfrak{X}$ then its subcategory of paths, and consequently also $\Phi_{\mathfrak{X}}$, encodes all information about $\mathfrak{X}$.

$\mathsf{Ar}(\Phi_{\mathfrak{X}})$ versus tree diagrams. Based on the minimal assumptions of Proposition 4.5.3, in the following we fix a representable wooded path category $\mathfrak{X}$ and a chosen set of paths $\widetilde{\mathcal{P}}_{\mathfrak{X}}$. Therefore, we have a functor $\Phi_{\mathfrak{X}}$ which, in turn, by Theorem 4.5.18, induces an arboreal category $\mathsf{Ar}(\Phi_{\mathfrak{X}})$. In case that $\mathfrak{X}$ is arboreal we show that we have an isomorphism $\mathfrak{X} \cong \mathsf{Ar}(\Phi_{\mathfrak{X}})$ in Theorem 4.5.26 below. However, first we show a useful technical fact about the structure of $\mathsf{Ar}(\Phi_{\mathfrak{X}})$ when $\mathfrak{X}$ is not required to be arboreal yet.

Let (T, t) be an object of $\mathsf{Ar}(\Phi_{\mathfrak{X}})$. Recall that, for every n, $\Phi_{\mathfrak{X}}(n)$ is a subset of $\widetilde{\mathcal{P}}_{\mathfrak{X}}$ and a full subcategory of $\mathfrak{X}$. Therefore, the labelling (i.e. the natural transformation)

$$t : T \to \Phi_{\mathfrak{X}}$$

determines a functor

$$D_t : T \to \mathfrak{X}.$$

where T is a tree, viewed as a poset. On objects, for $x \in T$, we set $D_t(x) = t(x)$, which is a path in $\widetilde{\mathcal{P}}_{\mathfrak{X}}$ of height $\mathrm{ht}(x)$. To define D_t on morphisms, observe that, by naturality of t, the inequality $x \leq y$ in T of heights $n \leq m$ gives us that

$$t(x) = t(T(n \leq m)(y)) = \Phi_{\mathfrak{X}}(n \leq m)(t(y)).$$

In other words, $D_t(x) = t(x)$ is a prefix of $D_t(y) = t(y)$, i.e. there is a (unique) embedding $D_t(x) \rightarrowtail D_t(y)$. We set $D_t(x \leq y)$ to be this embedding.

Conversely, let $D : T \to \mathfrak{X}$ be a tree diagram which **preserves heights**, i.e. for every x, $\mathrm{ht}(x) = \mathrm{ht}(D(x))$. Observe that D induces the corresponding labelling function $t_D : T \to \Phi_{\mathfrak{X}}$, by sending x to the path $D(x)$. Note that t_D is well-defined since D preserves heights and since prefixes of paths are unique in $\widetilde{\mathcal{P}}_{\mathfrak{X}}$. The latter follows from the fact that $\mathfrak{X}$ is concrete, i.e. between any two paths there is at most one morphism. Furthermore, it is immediate that the two constructions are converse to each other.

Lemma 4.5.21 *Given a tree T the assignments $t \mapsto D_t$ and $D \mapsto t_D$ establish a bijection between labellings $T \to \Phi_{\mathfrak{X}}$ and height-preserving tree diagrams $T \to \mathfrak{X}$.*

Next we show how to extend the bijection from Lemma 4.5.21 to a correspondence between morphisms of labelled trees and *morphisms of tree diagrams*.

Proposition 4.5.22 *For trees objects (T, t) and (T', t') of $\mathsf{Ar}(\Phi_{\mathfrak{X}})$, there is a one-to-one correspondence between*

1. *morphisms $(T, t) \to (T', t')$ in $\mathsf{Ar}(\Phi_{\mathfrak{X}})$, and*
2. *tree morphisms $f : T \to T'$ admitting a natural transformation $D_t \to D_{t'} \cdot f$.*

Furthermore, if $D_{t'} \cong \mathsf{TDiag}_Y$ i.e. $D_{t'}$ is the canonical tree diagram of some $Y \in \mathfrak{X}$, then the above are also in a bijection with

3. *cocones $\omega : D_t \to Y$.*

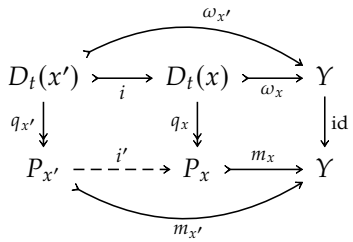
Proof. The equivalence of items 1 and 2 is immediate since morphisms $(T, t) \to (T', t')$ in $\mathsf{Ar}(\Phi_{\mathfrak{X}})$ are given by a tree morphism $f : T \to T'$ such that, pointwise, $t \leq t' \cdot f$ and, at the same time, since each $\Phi_{\mathfrak{X}}(n)$ is only a poset ordered by the quotient order $\twoheadrightarrow$, naturality of $D_t \to D_{t'} \cdot f$ is the same as pointwise inequality $D_t \leq D_{t'} \cdot f$.

Next, for the bijection between items 2 and 3, recall that the requirement $D_{t'} \cong \mathsf{TDiag}_Y$ makes sense since $\mathsf{TDiag}_Y : \mathbb{P}Y \to \mathfrak{X}$ is a height preserving tree diagram. Let ω be a cocone $D_t \to Y$. For each $x \in T$, factorise ω_x as a quotient $q_x : D_t(x) \twoheadrightarrow P_x$ followed by an embedding $m_x : P_x \rightarrowtail Y$. We

assume $P_x \in \widetilde{\mathcal{P}}_{\mathfrak{X}}$, which makes the choice of q_x and m_x unique since $\mathfrak{X}$ is concrete.

In the following we verify that the mapping $h\colon x \mapsto m_x$, defined by $T \to \mathbb{P}Y$, is a tree morphisms. First, we see that h preserves height by the fact that $\mathrm{ht}(x) = \mathrm{ht}(D_t(x))$, since D_t preserves height, and furthermore P_x is a quotient of $D_t(x)$ and so, by Lemma 4.5.1, they have the same height too.

Next, to show that h is monotone, let $x' \leq x$ in T. Since D_t is a tree diagram, we have an embedding $i = D_t(x' \leq x)\colon D_t(x') \rightarrowtail D_t(x)$. Then, we have commutative square of solid arrows



where the top triangle commutes because ω is a compatible cocone. The (unique) embedding $i'\colon P_{x'} \to P_x$ is obtained from orthogonality of $q_{x'}$ with m_x. Therefore, $m_{x'} \leq m_x$ in $\mathbb{P}Y$ and so h is monotone.

We have verified that $h\colon T \to \mathbb{P}Y$ is a tree morphism. The required natural transformation $D_t \to \mathsf{TDiag}_Y \cdot h$ is formed, pointwise, by the quotients q_x.

Conversely, starting from a natural transformation $\alpha\colon D_t \to \mathsf{TDiag}_Y \cdot f$ for some tree morphism $f\colon T \to \mathbb{P}Y$, we form a compatible cocone $\omega\colon D_t \to Y$ whose components are the following compositions.

$$\omega_x = \; D_t(x) \xrightarrow{\alpha_x} \mathsf{TDiag}_Y(f(x)) \xrightarrow{f(x)} Y$$

To see that $\omega = \{\omega_x\}_x$ is compatible, let $x' \leq x$ in T and let $i = D_t(x' \leq x)\colon D_t(x') \rightarrowtail D_t(x)$. Since f is monotone, $f(x') \leq f(x)$ in $\mathbb{P}Y$ and, therefore, there is an embedding $i'\colon \mathsf{TDiag}_Y(f(x')) \rightarrowtail \mathsf{TDiag}_Y(f(x))$ such that $f(x') = f(x) \cdot i'$. Furthermore, since $\mathfrak{X}$ is representable, i' is equal to the embedding $\mathsf{TDiag}_Y(f(x') \leq f(x))$. Next, we verify the required equality $\omega_{e'} = \omega_e \cdot i$ by a direct computation:

$$\omega_{x'} = f(x') \cdot \alpha_{x'} = f(x) \cdot i' \cdot \alpha_{x'} = f(x) \cdot \alpha_x \cdot i = \omega_x \cdot i$$

where the third equality follows from naturality α.

Finally, from the fact that factorisations are unique up to isomorphism, we see that the processes of going from cocones to natural transformations and converse are inverse to each other. □

The converse correspondence. Next, we construct a functor

$$H\colon \mathfrak{X} \to \mathsf{Ar}(\Phi_{\mathfrak{X}})$$

for our fixed representable wooded path category $\mathfrak{X}$. For an object $X \in \mathfrak{X}$, set $H(X)$ to be the pair $(\mathbb{P}X, l_X)$ where $l_X\colon \mathbb{P}X \to \Phi_{\mathfrak{X}}$ sends a path

embedding $e : P \rightarrowtail X$ of height n to $P \in \Phi_{\mathfrak{X}}(n)$. By the correspondence from Lemma 4.5.21, $H(X)$ is equivalently given as the canonical tree diagram $\mathsf{TDiag}_X : \mathbb{P}X \to \mathfrak{X}$.

Next, for a morphism $h : X \to Y$ in $\mathfrak{X}$, define $H(f)$ as $\mathbb{P}(f) : (\mathbb{P}X, l_X) \to (\mathbb{P}Y, l_Y)$. To see that this is correct, let $e : P \rightarrowtail X$ be a path embedding with $P \in \widetilde{\mathcal{P}}_{\mathfrak{X}}$ of height n. Recall that $\mathbb{P}(f)(e)$ is computed as the following factorisation

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e'=\mathbb{P}(f)(e)} \\ P & \xrightarrow[q_{f,e}]{} & P' \end{array}$$

where, without loss of generality, we assume that $P' \in \widetilde{\mathcal{P}}_{\mathfrak{X}}$. Since $\mathbb{P}(f)$ is a tree morphism P' is of the same height as P, i.e. $P, P' \in \Phi_{\mathfrak{X}}(n)$. Furthermore, $q_{f,e}$ witnesses that $P \leq P'$ in $\Phi_{\mathfrak{X}}(n)$. In other words, we have that $l_X(e) = P \leq P' = l_Y(\mathbb{P}(f)(e))$.

Alternatively, by the correspondence from Proposition 4.5.22, $H(f)$ is a natural transformation $\alpha_f : \mathsf{TDiag}_X \Rightarrow \mathsf{TDiag}_Y \cdot \mathbb{P}(f)$ where, for a path embedding $e : P \rightarrowtail X$, where $\alpha_{f,e}$ is precisely $q_{f,e}$ above.

The following is immediate from functoriality of $\mathbb{P}$ (Theorem 4.2.3).

Proposition 4.5.23 *For a representable wooded path category $\mathfrak{X}$, the mapping $H : \mathfrak{X} \to \mathsf{Ar}(\Phi_{\mathfrak{X}})$ is a functor.*

What remains to be checked is that H is an equivalence of categories in case when $\mathfrak{X}$ is arboreal. We do so in two steps, in the following lemmas.

Lemma 4.5.24 *If $\mathfrak{X}$ has tree colimits and if its paths are tree-connected then H is essentially surjective.*

Proof. Let $D_t : T \to \mathfrak{X}$ be the diagram corresponding to an object (T, t) of $\mathsf{Ar}(\Phi_{\mathfrak{X}})$, following Lemma 4.5.21. Since $\mathfrak{X}$ has tree colimits, the colimit of D_t exists. Denote by $\{\iota_x : D_t(x) \to L\}_{x \in T}$ the colimiting cocone.

We show that the canonical tree diagram TDiag_L of L is isomorphic to D_t. First, observe that for e in $\mathbb{P}L$, i.e. for some path embedding $e : P \rightarrowtail L$, since paths in $\mathfrak{X}$ are tree-connected, e factors as

$$P \xrightarrow{f} D_t(x) \xrightarrow{\iota_x} L$$

via the least $x \in T$. Since e is embedding, so is f, by Lemma 2.9.3. Furthermore, since x is the least such and since D_t preserves heights, we have that P and $D_t(x)$ are of the same height and so f is an epimorphism by Proposition 4.2.4. Therefore, f is an isomorphism by Lemma 2.9.3. Conversely, given $x \in T$, the inclusion ι_x is a path embedding (see Section 14), and so $\iota_x \in \mathbb{P}L$. Therefore, $\mathbb{P}L$ and T are isomorphic, as trees. Furthermore, $\mathsf{TDiag}_L \cong D_t$ since the isomorphism established above maps a path to a path isomorphic to it. □

Lemma 4.5.25 *If $\mathfrak{X}$ is path-generated then the functor H is fully faithful.*

Proof. Since $\mathfrak{X}$ is path-generated, morphisms $X \to Y$ in $\mathfrak{X}$ are in a bijective correspondence with compatible cocones $\mathsf{TDiag}_X \to Y$, for the canonical tree diagram TDiag_X of X. These are, in turn, in a bijection with the morphisms from $H(X) = (\mathbb{P}X, l_X)$ to $H(Y) = (\mathbb{P}Y, l_Y)$ in $\mathsf{Ar}(\Phi_{\mathfrak{X}})$, by Proposition 4.5.22. □

To conclude, we have shown that arboreal categories and functors $\mathbb{N}^{op} \to$ **Pos** are in a one-to-one correspondence via the constructions $\Phi \mapsto \Phi_{\mathfrak{X}}$ and $\mathfrak{X} \mapsto \Phi_{\mathfrak{X}}$.

Theorem 4.5.26 *For any (pre-)arboreal category $\mathfrak{X}$, we have $\mathfrak{X} \cong \mathsf{Ar}(\Phi_{\mathfrak{X}})$ and, for any functor $\Phi\colon \mathbb{N}^{op} \to$ **Pos**, we have $\Phi \cong \Phi_{\mathsf{Ar}(\Phi)}$.*

4.6 Examples

Game Comonads

Given the concrete description of $\mathsf{EM}(\mathbb{E}_k)$ from Section 2.5, it is immediate to see that $\mathsf{EM}(\mathbb{E}_k) \cong \mathsf{Ar}(\Phi_{\mathbb{E}_k})$ where $\Phi_{\mathbb{E}_k}(0)$ is the singleton poset and for $1 \leq n \leq k$, the poset $\Phi_{\mathbb{E}_k}(n)$ consists of all structures A on the linearly ordered universe $[n] = \{1, \ldots, n\}$ in the signature. Then, we set $A \leq A'$ iff the identity function $[n] \to [n]$ is a homomorphism A to A'.

Similarly, $\mathsf{EM}(\mathbb{P}_k) \cong \mathsf{Ar}(\Phi_{\mathbb{P}_k})$ has $\Phi_{\mathbb{P}_k}(0) = \{\star\}$ and, for $1 \leq n$, the poset $\Phi_{\mathbb{E}_k}(n)$ consists of structures A on the linearly ordered universe $[n]$ equipped with a pebble function $p\colon [n] \to [k]$ such that the $R^A(x_1, \ldots, x_n)$ implies that, if $x_i \leq x_j$, then $p(x_i) \notin \{p(y) \mid x_i < y \leq x_j\}$, as in the characterisation of $\mathbb{P}_k$-coalgebras in Section 2.5.

Finally, $\mathsf{EM}(\mathbb{M}_k) \cong \mathsf{Ar}(\Phi_{\mathbb{M}_k})$ has $\Phi_{\mathbb{M}_k}(n)$ equal to all possible structures $\star \xrightarrow{R_1} x_1 \xrightarrow{R_2} \ldots \xrightarrow{R_n} x_n$ with arbitrary interpretation of unary predicates on $\star, x_1, \ldots, x_n$.

Models of concurrency

Let M be a *(Mazurkiewicz) trace language*, that is, a triple $M = (M, L, I)$ where L is a set of letters, I an *independence relation* i.e. $I \subseteq L \times L$ which is symmetric and irreflexive, and M is a non-empty subset of sequences of strings, with letters from L, such that:

- (prefix closed) $[\overline{a}, b] \in M$ implies $[\overline{a}] \in M$,
- (I-closed) $[\overline{a}, i, j, \overline{b}] \in M$ and $(i, j) \in I$ then $[\overline{a}, j, i, \overline{b}] \in M$, and
- (coherent) $[\overline{a}, i] \in M$ and $(i, j) \in I$ then $[\overline{a}, i, j] \in M$.

We furthermore assume that

- $\forall i \in L\ \exists\, \overline{a}$ such that $[\overline{a}, i] \in M$.

Define Φ_M by setting $\Phi_M(0) = \{\star\}$ and

$$\Phi_M(n) = \{[\overline{a}] \in M \mid \text{length of } \overline{a} \text{ is } n\}.$$

Also, $\Phi_M(n' \leq n)$ sends $[\overline{a}]$ to its prefix of length n'.

Then, $\mathsf{Ar}(\Phi_M)$ contains all unravellings of *asynchronous transition systems* with the same independence relation [120, Theorem 10.1.6], i.e. we can define a functor

$$F\colon \mathbf{Atf} \longrightarrow \mathsf{Ar}(\Phi_M)$$

where **Atf** is a category of asynchronous transition systems and F computes the trace language (see [120, Theorem 51] for details). Then, $FA \rightleftharpoons FB$ represents bisimilarity of A and B with respect to their inherent trace languages.

It is likely that similar constructions can be used to represent unravellings of other types of models of concurrency [91, 120].

Arboreal categories of types

Let $\mathscr{H}\colon \mathbf{FinSet} \longrightarrow \mathbf{DL}$ be a *coherent hyperdoctrine* [53, 76], where **FinSet** is the category of finite sets and functions and **DL** is the category of bounded distributive lattices. Consider the following composition

$$\widehat{\mathscr{H}}\colon \mathbb{N}^{op} \longrightarrow \mathbf{FinSet}^{op} \xrightarrow{\mathscr{H}^{op}} \mathbf{DL}^{op} \longrightarrow \mathbf{Prie} \longrightarrow \mathbf{Pos}^{op}$$

where the first functor is the inclusion of natural numbers into **FinSet** via $n \mapsto [n]$, the middle right functor is the contravariant Priestley duality functor (establishing a duality between distributive lattices and Priestley spaces) and the last functor sends a Priestley spaces to its underlying posets.

By definition, $\widehat{\mathscr{H}}(n)$ consists of n-types of a given coherent logic, specified by $\mathscr{H}$ [53, 104], and $\mathsf{Ar}(\widehat{\mathscr{H}})$ represents consistent theories in such logic.

4.7 Consequences of the representation theorem

Internal properties of morphisms. The following statements classify important classes of morphisms of arboreal categories, using the internal representation of $\mathsf{Ar}(\Phi)$, for some fixed functor $\Phi\colon \mathbb{N}^{op} \longrightarrow \mathbf{Pos}$ where $\Phi(0)$ has a least element.

Lemma 4.7.1 *A morphism $f\colon (T,t) \longrightarrow (S,s)$ in $\mathsf{Ar}(\Phi)$ is a pathwise-embedding if and only if it is strict.*

Proof. The right-to-left direction is immediate since, given a path embedding $e\colon (P,p) \rightarrowtail (T,t)$ in $\mathsf{Ar}(\Phi)$, the composition of strict morphisms is strict and, furthermore, morphisms from paths are always injective. Therefore, $f \cdot e$ is a strict injective morphism i.e. an embedding.

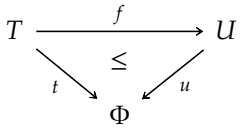
Conversely, let $x \in T$ we wish to show that the inequality $t(x) \leq s(f(x))$ is an equality. Define a path embedding $e\colon (P,p) \rightarrowtail (T,t)$ such that $\mathrm{ht}(P) = \mathrm{ht}(x)$ and $e(\top) = x$ where $\top$ is the top element of P. In order for e to be an embedding we must define p as $t \cdot e$. Then, using that f is a pathwise-embedding, we have that $f \cdot e$ is an embedding too, i.e. $p = s \cdot f \cdot e$. Consequently, we have that

$$t(x) = t(e(\top)) = p(\top) = s(f(e(\top))) = s(f(x)).$$

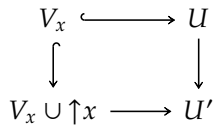
Since x was chosen arbitrarily, we see that f is strict. □

Lemma 4.7.2 *A morphism $f : (T, t) \to (U, u)$ in $\mathsf{Ar}(\Phi)$ is surjective if and only if it is an epimorphism.*

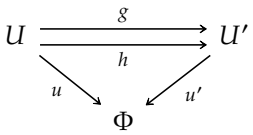
Proof. The left-to-right implication is immediate. For converse consider an epimorphism f in $\mathsf{Ar}(\Phi)$ as depicted below.



If $\mathrm{dom} f$ is not surjective, there is $x \in U$ that is not the image of f. Write $V_x := \{y \in U \mid y < x\}$ and let U' be defined by the following pushout in $\mathcal{T}$.



That is, U' is obtained by gluing to U a new copy of $\uparrow x$ with same parent as x. We use an asterisk to refer to elements in this new copy of $\uparrow x$. Consider the morphisms of trees $g, h : U \rightrightarrows U'$ that are the identity on $U \setminus \uparrow x$ and satisfy, for all $y \in \uparrow x$, $g(y) = y$ and $h(y) = y^*$. Define $u' : U' \to \Phi$ as $u : U \to \Phi$ on U and $u'(x^*) = u(x)$. Then, we can regard g, h as (strict) morphisms in $\mathsf{Ar}(\Phi)$, that is:



However, $g \cdot f = h \cdot f$ and $g \neq h$, which is a contradiction with the fact that f is an epimorphism. □

Lemma 4.7.3 *A pathwise-embedding $f : (T, t) \to (U, u)$ in $\mathsf{Ar}(\Phi)$ is injective (i.e. an embedding) if and only if it is a monomorphism.*

Proof. The left-to-right implication is immediate. For the converse, let $x, y \in T$ be such that $f(x) = f(y)$. Consider the sub-path inclusions $e_x : \downarrow x \hookrightarrow T$ and $e_y : \downarrow y \hookrightarrow T$ of T in $\mathcal{T}$. Since f is a tree morphism and $f(x) = f(y)$ we have that $\downarrow x$ and $\downarrow y$ are of the same height, i.e. $\downarrow x \cong \downarrow y$.

Next, we define the labelling $l_x : \downarrow x \to \Phi$ as $t \cdot e_x$ and similarly $l_y = t \cdot e_y$. Using the fact that f is a pathwise embedding, it is strict (Lemma 4.7.1) and so for $x' \in \downarrow x$ and $y' \in \downarrow y$ of the same height, i.e. f maps them to the same element, we have that $l_x(x') = t(e_x(x')) = u(f(e_x(x')) = u(f(e_y(y')) = t(e_y(y')) = l_y(y')$. In other words, $(\downarrow x, l_x)$ and $(\downarrow y, l_y)$ are isomorphic, as objects in $\mathsf{Ar}(\Phi)$.

As a result, we have a pair of morphisms $g, h : (\downarrow x, l_x) \rightrightarrows (T, t)$ in $\mathsf{Ar}(\Phi)$ where $g = e_x$ and h is e_y composed with the isomorphism $(\downarrow x, l_x) \cong (\downarrow y, l_y)$. Since f is mono and since $f \cdot g = f \cdot h$, we must have that $g = h$, i.e. that $x = y$. □

Lifting internal properties along equivalence. Observe that, for an object $(T, t) \in \mathsf{Ar}(\Phi)$ for some $\Phi\colon \mathbb{N}^{op} \to \mathbf{Pos}$, we have that $\mathbb{P}(T, t) \cong T$.[16] Therefore, the domain projection functor

16: cf. Subsection"Subobjects and paths in Ar(Φ)" on page 77

$$\mathrm{dom}\colon \mathsf{Ar}(\Phi) \to \mathcal{T}$$

defined by $(T, t) \mapsto T$ is equivalent to the path functor $\mathbb{P}$ for $\mathsf{Ar}(\Phi)$.

However, when $\Phi = \Phi_{\mathcal{X}}$ for some representable arboreal category $\mathcal{X}$, we have the following situation induced by the forgetful functors

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{H} & \mathsf{Ar}(\Phi_{\mathcal{X}}) \\ & {\scriptstyle\mathbb{P}}\searrow \quad \swarrow{\scriptstyle \mathrm{dom}} & \\ & \mathcal{T} & \end{array} \tag{4.4}$$

where H is the isomorphism from Lemmas 4.5.24 and 4.5.25. The diagram commutes by definition of H.

We use fact to show that there is actually no freedom when selecting the factorisation system for a representable arboreal category since it must always be (epi, strong mono).

Lemma 4.7.4 *The functor H preserves and reflects quotients and embeddings. Moreover, the factorisation system giving the structure of arboreal category on $\mathcal{X}$ must be (epi, strong mono).*

Proof. First we observe that H preserves and reflects quotients. Since H is an equivalence of categories, it preserves and reflects epimorphisms. However, then, we see that epimorphisms in $\mathcal{X}$ are precisely quotients, using Lemma 4.7.2 and [5, Lemma 6.11] which says that f is a quotient in $\mathcal{X}$ iff $\mathbb{P}f$ is surjective.

Next we show that H above preserves and reflects embeddings. We use the fact that quotients determine embeddings in any orthogonal factorisation system, since $\mathscr{M} = \mathbb{Q}^{\perp} = \{f \mid \forall q \in \mathbb{Q}.\ q \perp f\}$ by Lemma 2.9.5. Furthermore, since $\mathbb{Q} = \{\text{epis}\}$ it follows, by definition, that $\mathscr{M} = \{\text{strong monos}\}$. □

We can now prove a strengthening of Proposition 4.2.4, which shows that the path functor can detect properties of morphisms even between objects which are not required to be paths.

Proposition 4.7.5 *Let $f : X \to Y$ be a morphism in a representable arboreal category $\mathcal{X}$. Then*

1. *f is a quotient iff $\mathbb{P}f$ is surjective iff f is an epimorphism.*

If, furthermore, f is a pathwise-embedding then we also have the following:

2. *f is an embedding iff $\mathbb{P}f$ is injective iff f is a monomorphism.*
3. *f is open iff $\mathbb{P}f$ is a* p-morphism*, that is, $\mathbb{P}(\uparrow m) = \uparrow \mathbb{P}f(m)$ for all $m \in \mathbb{P}X$*

Proof. By Lemma 4.7.4, we see that H defined above preserves and reflects the factorisation system. Consequently, as H is an equivalence of categories (Theorem 4.5.26), since paths are given implicitly in both

categories, H also preserves and reflects paths and, consequently, also pathwise-embeddings.

Consequently, (1) follows Lemma 4.7.2 since f is a quotient iff $H(f)$ is a quotient iff $\mathbb{P}(f) = \mathbb{P}(H(f))$ is surjective. Similarly, for (2), assuming that f is a pathwise-embedding, we see that f is an embedding iff $H(f)$ is an embedding iff $\mathbb{P}f = \mathbb{P}(H(f))$ is injective iff f is mono (by Lemma 4.7.3). Finally (3) is [11, Proposition 4.2].[17] □

17: Recall Remark 4.3.4, justifying that we can use this reference despite the changed assumptions.

Remark 4.7.6 Note that first equivalences in item 1 and item 3 in Proposition 4.7.5 holds more generally, without assuming representability, see [5, Lemma 6.11] and [11, Proposition 4.2].

Proposition 4.7.7 *In every representable arboreal category we have the following.*

1. *Colimits of all tree-shaped diagrams of embeddings exist.*
2. *The inclusion maps $A_t \longrightarrow \operatorname{colim}_t A_t$ into tree-shaped diagrams of embeddings are embeddings too.*
3. *Let $\{A_t\}_{t\in T}$ be a tree-shaped diagram of embeddings and let $\{A_t \longrightarrow B\}_t$ be a cocone of embeddings. Then, the comparison morphism $\operatorname{colim}_t A_t \longrightarrow B$ is an embedding too.*

Proof. (1) Observe that in the proof that tree colimits exist (Lemma 4.5.15) we did not use the fact that the objects are paths. The same proof would work for any collection of objects in $\Phi_{\mathfrak{X}}$.

(2) and (3) are evident by analysing the proof Proposition 4.5.16. The generalisation from tree colimits to tree-shaped colimits of embeddings is similar to (1). □

Finally, we can finish asserting the validity of the remaining axioms of elementary path categories, on top of those we already verified in Theorem 4.3.2.

Theorem 4.7.8 *Let $\mathfrak{X}$ be a representable arboreal category, then as an elementary path category it satisfies $(PE_\vee)$ and (PE_{cp}).*

Proof. For $(\mathrm{PE}_\vee)$, observe that by Theorem 4.2.3, any path diagram $\mathsf{PDiag}_X \circ I$ is equivalent to a tree diagram, hence by axioms of arboreal categories, it has a colimit. The fact that the induced morphism $\operatorname{colim}(\mathsf{PDiag}_X \circ I) \longrightarrow X$ is an embedding follows from Proposition 4.7.7.

For $(\mathrm{PE}_{\mathrm{cp}})$, as above, we know that path diagrams are special types of tree diagrams and hence, compactness of paths follows from tree-connectedness. □

Special pushouts. In the following we describe that the tree part of pushouts in $\mathsf{Ar}(\Phi)$, when they exist, is computed in $\mathcal{T}$. This is useful in the following section, where we discuss consequences for logic.

Lemma 4.7.9 *Let there be a commutative square in* $\mathsf{Ar}(\Phi)$*, as shown below.*

$$\begin{array}{ccc} (R,r) & \xrightarrow{f_2} & (S_2,s_2) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \overline{f_2}} \\ (S_1,s_1) & \xrightarrow{\overline{f_1}} & (T,t) \end{array}$$

If this is a pushout square then T *is a pushout of* f_1, f_2 *in* $\mathcal{T}$*. Furthermore, if* f_1 *and* f_2 *are strict then so are* $\overline{f_1}$ *and* $\overline{f_2}$*.*

Proof. Let Q be the pushout of f_1, f_2 in $\mathcal{T}$, as shown below.

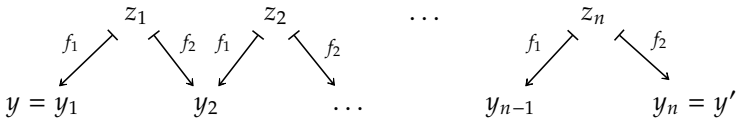
$$\begin{array}{ccc} R & \xrightarrow{f_2} & S_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ S_1 & \xrightarrow{g_1} & Q \end{array}$$

Recall that we can identify the elements of Q with the equivalence classes $[x]_\sim$ where $\sim$ is the least equivalence relation on $S_1 + S_2$ containing $f_1(r) \sim f_2(r)$ for all $r \in R$.

In the following, we check the following claim:

($\star$) For $[x]_\sim \in Q$ and for any $y, y' \in [x]_\sim$ we have that $t(\overline{f_i}(y)) = t(\overline{f_i}(y'))$ where $i \in \{1,2\}$ is such that y is from S_i and j is defined similarly with respect to y'.

To this end, assume that there is a sequence $y = y_1, z_1, y_2, \ldots, z_n, y_n = y'$ witnessing $y \sim y'$, e.g. in case when $y \in S_1$ and $y' \in S_2$, we have:



Then, we see that $t(\overline{f_1}(y)) = t(\overline{f_1}(f_1(z_1))) = t(\overline{f_2}(f_2(z_1))) = t(\overline{f_2}(y_2)) = \cdots = t(\overline{f_2}(y'))$, hence ($\star$) holds.

Therefore, we can define a function

$$q \colon Q \to \Phi,$$

by setting $q([x]_\sim)$ to be $t(\overline{f_i}(x))$ where, as above, i is such that $x \in S_i$. By ($\star$) we see that q is well-defined, i.e. if $[x]_\sim = [y]_\sim$ then $t(\overline{f_i}(x)) = t(\overline{f_j}(y))$.

Next, we check that (Q,q) is a valid object of $\mathsf{Ar}(\Phi)$, i.e. that $q \colon Q \to \Phi$ is a valid labelling. Let $[x'] \leq [x]$ in Q. Without loss of generality, we can assume[18] that both x and x' are from S_i and that $x' \leq x$. Then, for $n' = \mathrm{ht}(x')$ and $n = \mathrm{ht}(x)$, $\Phi(n' \leq n)(q([x])) = \Phi(n' \leq n)(t(\overline{f_i}(x))) = t(\overline{f_i}(x')) = q([x'])$.

Next, observe that each g_i is a morphism $(S_i, s_i) \to (Q,q)$ in $\mathsf{Ar}(\Phi)$. For $x \in S_i$, we have that $s(x) \leq t(\overline{f_i}(x)) = q([x]) = q(g_i(x))$. Therefore, since (T,t) is a pushout in $\mathsf{Ar}(\Phi)$, there is a unique $(T,t) \to (Q,q)$ morphism in $\mathsf{Ar}(\Phi)$, making the required diagrams commute. Conversely, since $\overline{f_1} \cdot f_2 = \overline{f_2} \cdot f_1$ in $\mathcal{T}$, we also have a unique $Q \to T$ morphism in

18: Observe that, if $y \sim y'$ in $S_1 + S_2$, then also $x \sim x'$ for predecessors $x \leq y$ and $x' \leq y'$ of the same height. Indeed, the sequence $y = y_1, z_1, y_2, \ldots, z_n, y_n = y'$ witnessing $y \sim y'$ when mapped to the height $\mathrm{ht}(x) = \mathrm{ht}(x')$ witnesses $x \sim x'$.

$\mathcal{T}$, making the required diagrams commute. In other words, we have that $T \cong Q$.

Finally, for the 'furthermore' part, assuming f_1 and f_2 are strict, we can define a different labelling $\widehat{q}\colon Q \to \Phi$, by setting $\widehat{q}([x]) = s_i(x)$ where i is chosen so that $x \in S_i$. To see that the choice of a representative from $[x]_\sim$ does not matter, let $y, y' \in [x]_\sim$, i.e. let $y \sim y'$. This is witnessed by a sequence $y = y_1, z_1, y_2, \ldots, z_n, y_n = y'$, as above. Then, since f_1 and f_2 are strict, we have that $s_1(y) = s_1(f(z_1)) = r(z_1) = s_2(f_2(z_1)) = s_2(y_2) = \cdots = s_2(y')$. Next, we check that $\widehat{q}$ is a labelling. For $[x'] \leq [x]$ where $x' \leq x$ are from S_i, $n' = \mathrm{ht}(x')$ and $n = \mathrm{ht}(x)$, we have that $\Phi(n' \leq n)(q([x])) = \Phi(n' \leq n)(s_i(x)) = s_i(x') = q([x'])$.

Next, observe that g_1, g_2 are strict morphisms $(S_i, s_i) \to (Q, \widehat{q})$. Hence, since (T, t) is a pushout, we have a morphism $\widehat{g}\colon (T, t) \to (Q, \widehat{q})$. Assuming, without loss of generality, that $T = Q$, gives us that $t \leq \widehat{q}$. On the other hand, the fact that $s_i \leq t \cdot \overline{f_i}$ gives us that $\widehat{q}([x]) = \widehat{q}(g_i(x)) = s_i(x) \leq t(\overline{f_i}(x)) = t([x])$, i.e. $t = \widehat{q}$ and $\overline{f_i}$ is strict. □

From the above description of pushouts, together with Lemma 4.7.1, Proposition 4.7.5 and the commutativity of (4.4), we obtain the following important facts about preservation of properties of morphisms under pushouts.

Theorem 4.7.10 *Assume that we have the following pushout square in* $\mathsf{Ar}(\Phi)$,

$$\begin{array}{ccc} (R, r) & \xrightarrow{f_2} & (S_2, s_2) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \overline{f_2}} \\ (S_1, s_1) & \xrightarrow{\overline{f_1}} & (T, t) \end{array}$$

We have the following

1. *If f_1, f_2 are monomorphisms then so are $\overline{f_1}, \overline{f_2}$.*
2. *If f_1, f_2 are pathwise-embedding then so are $\overline{f_1}, \overline{f_2}$*
3. *If f_1, f_2 are embeddings then so are $\overline{f_1}, \overline{f_2}$.*
4. *If f_1, f_2 are open pathwise-embedding then so are $\overline{f_1}, \overline{f_2}$.*

In particular, the same properties hold in any representable arboreal category.

Proof. (1) follows from the fact that pushouts of pairs of injective maps in $\mathcal{T}$ are injective. (2) follows from Lemmas 4.7.9 and 4.7.1 and (3) follows from (1), (2) and Lemma 4.7.3.

Finally, to check (4), we observe that due to (2), we know that $\overline{f_1}$ and $\overline{f_2}$ are pathwise embeddings and so, by item 3 of Proposition 4.7.5, it is enough to check that they are also p-morphisms between the corresponding trees.

The latter follows from the following sequence of standard facts:

i. Every tree T can be represented as a coalgebra of the powerset functor,[19] i.e. $\tau\colon T \to \mathfrak{P}(T)$, where $\tau(x)$ is the set of immediate successors $\{y \in T \mid x \prec y\}$.

19: Note that here we talk about functor coalgebras not Eilenberg–Moore coalgebras of a comonad.

ii. A tree morphism $f : T \to S$ is a p-morphism iff it is a coalgebra morphism $(T, \tau) \to (S, \sigma)$ i.e. $\sigma \cdot f = \mathfrak{P}(f) \cdot \tau$ [119, Example 149].
iii. The category of coalgebras has all colimits, and they are created by the forgetful functor to sets [111, Theorem 4.2].
iv. Every pushout of trees in $\mathcal{T}$ is also computed concretely, i.e. the pushouts of trees and the pushout of their corresponding powerset coalgebras agree. □

4.8 Four ways of expressing bisimulations

In what follows we consider b&f systems which are **strong**, that is, the b&f systems $\mathcal{S}$ such that if $X \xleftarrow{m} P \xrightarrow{n} Y$ is in $\mathcal{S}$ and $e : P' \rightarrowtail P$ then also $X \xleftarrow{m \cdot e} P' \xrightarrow{n \cdot e} Y$ is in $\mathcal{S}$.

Lemma 4.8.1 (Lemma 6.3 in [11]) *Given a b&f system $\mathcal{S}$ between X and Y in an arboreal category, then the class*

$$\{X \xleftarrow{m} P \xrightarrow{n} Y \mid \forall e : P' \rightarrowtail P,\ X \xleftarrow{m \cdot e} P' \xrightarrow{n \cdot e} Y \text{ is in } \mathcal{S}\}$$

is a strong b&f system between X and Y.

Recall that the one of the principal facts about arboreal categories is that their axioms imply that back-and-forth equivalence and bisimilarity are equivalent notions, i.e. that we also have the converse of Proposition 3.4.4.

Proposition 4.8.2 (Theorem 6.4 in [11]) *For objects X, Y in a concrete arboreal category $\mathfrak{X}$ such that the product $X \times Y$ exists*[20]*,*

$$X \rightleftharpoons Y \iff X \sim Y.$$

20: Note that we added 'concrete' to the list of assumptions when compared to Theorem 6.4. in [11]. This is because the proof uses that between any two paths there is at most one embedding, incorrectly interpreting the proof of Lemma 3.15(3) therein. Moreover, note that here and in Lemma 4.8.1 we use different axioms of arboreal categories from those in *op. cit.* but the result still applies, cf. Remark 4.3.4.

Proof sketch. For the remaining left-to-right implication, let $\mathcal{S}$ be a b&f system between X and Y. Since $\mathfrak{X}$ is representable, in particular, since it has a set of paths, we can assume $\mathcal{S}$ is given by a set $\{X \xleftarrow{m_i} P_i \xrightarrow{n_i} Y\}_i$ where each P_i is in the set of chosen paths $\widetilde{\mathcal{P}}_{\mathfrak{X}}$. By Lemma 4.8.1 we can further assume that $\mathcal{S}$ is strong.

Each pair m_i, n_i induces a morphism $\langle m_i, n_i \rangle : P_i \rightarrowtail X \times Y$ which is an embedding since, for the product projection $\pi_X : X \times Y \to X$, $m_i = \pi_X \cdot \langle m_i, n_i \rangle$ is an embedding.

We take the supremum $e : Z \rightarrowtail X \times Y$ of the set $\{\langle m_i, n_i \rangle\}_i$ in $\mathsf{Sub}(X \times Y)$. Using the fact that $\mathcal{S}$ is strong, it follows by a careful analysis that the composition with the projections

$$X \xleftarrow{\pi_X \cdot e} Z \xrightarrow{\pi_Y \cdot e} Y$$

gives a SOPE. □

Furthermore, in Proposition 3.4.10 we showed that $X \leftrightarroweq Y$ implies $X \sim Y$ under suitable conditions, e.g. in any arboreal category with pullbacks. The converse holds in any arboreal category with pushouts, due to Theorem 4.7.10.

Proposition 4.8.3 *Let $\mathcal{X}$ be a representable arboreal category with pushouts, then $X \sim Y$ implies $X \leftrightarroweq Y$.*

Proof. By Theorem 4.7.10 we know that OPEs are closed under pushouts. Therefore, a span of OPEs $X \leftarrow\!\!\bullet\!\!- R \bullet\!\!\rightarrow Y$ can be turned into a cospan of OPEs $X \bullet\!\!\rightarrow S \leftarrow\!\!\bullet\!\!- Y$ by taking the pushout. □

These propositions tell us that under quite mild assumptions on arboreal categories, the equivalences $\rightleftharpoons$, $\sim$ and $\leftrightarroweq$ are the same. Next we explore another notion arising from the fact that OPEs are stable under taking pushouts.

Simplifications

It is easy to observe that if $\mathcal{S}_i$ is a collection of strong back-and-forth systems between X and Y then so is their union $\bigcup_i \mathcal{S}_i$, cf. Lemma 3.3.4. Therefore, if there exists a b&f system witnessing $X \rightleftharpoons Y$ then there is some largest such strong b&f system.

In particular, since the identity relation is a strong b&f system witnessing $X \rightleftharpoons X$, there exists some *maximal* b&f system $\mathcal{S}_X$ on X. Then, if we work in a representable arboreal category, the left-to-right direction of Proposition 4.8.2 gives us a span

$$X \xleftarrow{p_1} Z_X \xrightarrow{p_2} X$$

obtained from $\mathcal{S}_X$ where p_1 and p_2 are open pathwise-embeddings. If, moreover, the pushout of this span exists, then Theorem 4.7.10 implies that the resulting square (depicted below) consists of open pathwise embeddings.

$$\begin{array}{ccc} & Z_X & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X & & X \\ {\scriptstyle q_1}\searrow & & \swarrow{\scriptstyle q_2} \\ & X/_\sim & \end{array} \tag{4.5}$$

We call $X/_\sim$ the **simplification**[21] of X. In fact, in Lemma 4.8.5 below we show that $q_1 = q_2$ and we denote this morphism by q^X or simply as q if X is implicit from the context.

In the following, since we work in a representable arboreal category $\mathcal{X}$, equivalence classes of path embeddings $\mathbb{P}X$ have uniquely chosen representatives $P \rightarrowtail X$ where $P \in \widetilde{\mathcal{P}}_{\mathcal{X}}$. Therefore, the largest b&f system $\mathcal{S}_X$ on X can be viewed as an equivalence relation[22] on $\mathbb{P}X$. Indeed, each span of path embeddings $X \xleftarrow{m_1} P \xrightarrow{m_2} X$ where $P \in \widetilde{\mathcal{P}}_{\mathcal{X}}$ gives a pair $(m_1, m_2) \in \mathbb{P}X \times \mathbb{P}X$. By abuse of notation, we denote by

$$\sim \subseteq \mathbb{P}X \times \mathbb{P}X$$

the equivalence relation on $\mathbb{P}X$ arising this way from the largest b&f system $\mathcal{S}_X$ on X.

21: This terminology comes from the Universal Coalgebra literature. Simple coalgebras are those for which bisimilar elements are equal [19, 111], i.e. elements witnessing the same behaviour. In some sources it is also called *observable* [63].

22: The fact that it is an equivalence follow from the fact that $\mathcal{S}_X$ contains the diagonal and that b&f systems are closed under taking inverse and composition, cf. Lemma 3.3.4.

Next, we show some basic facts about the diagrams that appear in the construction of $X/_\sim$.

Lemma 4.8.4 *Let $X/_\sim$ be defined as above. Then:*

1. *Z_X is the colimit of the tree diagram $\mathcal{S}_X \to \mathfrak{X}$, sending (e_1, e_2) to the common codomain of e_1 and e_2.*
2. *$\mathbb{P}Z_X \cong \mathcal{S}_X$.*
3. *$\mathbb{P}(p_1)$ and $\mathbb{P}(p_2)$ are the projections $(e_1, e_2) \mapsto e_i$.*
4. *The pushout $\mathbb{P}X +_{\mathcal{S}_X} \mathbb{P}X$ of $\mathbb{P}(p_1)$ and $\mathbb{P}(p_2)$ is isomorphic to the quotient $(\mathbb{P}X)/_\sim$ of $\mathbb{P}X$ by $\sim$.*
5. *$(\mathbb{P}X)/_\sim \cong \mathbb{P}(X/_\sim)$*

Proof. (1) follows from the proof of Proposition 4.8.2. Since $\mathcal{S}_X$ is strong, it is immediate that it a tree when ordered by: $(e_1, e_2) \leq (m_1, m_2)$ iff there is a (unique) n such that $e_i = m_i \cdot n$.

(2) Let $e\colon P \rightarrowtail Z_X$ be a path embedding (where P is from the chosen set of paths $\widetilde{\mathcal{P}}_{\mathfrak{X}}$). Since $Z_X \cong \operatorname{colim} D$ where D is the tree diagram from (1) and since P is tree-connected, there is a least pair (e_1, e_2) such that e factors $P \to D(e_1, e_2) \to Z_X$ where the latter component of the composition is the colimit inclusion. In fact, since $\mathcal{S}_X$ is strong, $D(e_1, e_2)$ has the same height as P and, moreover, the former component $P \to D(e_1, e_2)$ is an embedding since e is. Therefore, $P = D(e_1, e_2)$ since both are from $\widetilde{\mathcal{P}}_{\mathfrak{X}}$.

Conversely, any colimit inclusion $D(e_1, e_2) \to Z_X$ is a path embedding $P \rightarrowtail Z_X$ by Proposition 4.7.7.

(3) is the consequence of the fact that paths are tree connected and that the projections $p_i\colon Z_X \to X$ are obtained from the cocone

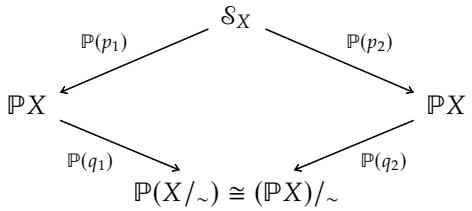
$$\{m_i\colon D(m_1, m_2) \to X\colon (m_1, m_2) \in \mathcal{S}_X\}.$$

(4) follows from (2) and the definition of pushouts in the category of trees.

(5) Observe that $\mathbb{P}$ preserves pushouts (cf. Lemma 4.7.9 and (4.4)). Hence, (5) follows from (4) and the fact that $X/_\sim$ is calculated by a pushout. □

Lemma 4.8.5 *$q_1 = q_2$ in the diagram (4.5) above.*

Proof. By Lemma 4.8.4 and Lemma 4.7.9 the following is the $\mathbb{P}$-image of the pushout diagram (4.5) that defines $X/_\sim$.



Since $\mathbb{P}$ is faithful (cf. Lemma 4.4.2[23]), it is enough to show that $\mathbb{P}(q_1) = \mathbb{P}(q_2)$. Now, let $e\colon P \rightarrowtail X$ be a path embedding. Since trivially (e, e), it belongs to the diagonal $\Delta \subseteq \mathcal{S}_X$, there is some $w \in \mathcal{S}_X$ such that

23: In fact, here it is enough that $\mathbb{P}$ is faithful on pathwise-embeddings which holds more generally, see the discussion following Remark 5.2 in [11].

$e = \mathbb{P}(p_i)(w)$ for $i = 1, 2$. Therefore, $\mathbb{P}(q_1)(e) = \mathbb{P}(q_1)(\mathbb{P}(p_1)(w)) = \mathbb{P}(q_2)(\mathbb{P}(p_2)(w)) = \mathbb{P}(q_2)(e)$. □

Concrete description of simplifications. Recall that in representable arboreal categories, every object is uniquely described by its canonical tree diagram $\mathsf{TDiag}_X \colon \mathbb{P}X \to \mathfrak{X}$. Since $\mathfrak{X}$ is representable, for each equivalence class of path embeddings in $\mathbb{P}X$, we have a canonical choice of a path embedding $P \rightarrowtail X$ where $P \in \widetilde{\mathcal{P}}_{\mathfrak{X}}$. Therefore, the diagram TDiag_X can be equivalently given as the mapping of the chosen $P \rightarrowtail X$ to P.

Then, since $\mathbb{P}(X/_\sim) \cong (\mathbb{P}X)/_\sim = \mathbb{P}X +_{\mathcal{S}_X} \mathbb{P}X$, we obtain that the morphism $q^X := q_1 = q_2$, that is,

$$q^X \colon X \to X/_\sim$$

is mapped by $\mathbb{P}(\text{-})$ to the quotient map

$$\mathbb{P}(q^X) \colon \mathbb{P}X \to (\mathbb{P}X)/_\sim$$

sending $e \colon P \rightarrowtail X$ to $[e] = \{e' \colon P \rightarrowtail X \mid (e, e') \in \mathcal{S}_X\}$. Furthermore, since q^X is a pathwise-embedding, we have that the canonical tree diagram $\mathsf{TDiag}_{X/_\sim}$ of $X/_\sim$ is equivalently given as the mapping $(\mathbb{P}X)/_\sim \to \mathfrak{X}$ mapping $[P \rightarrowtail X]$ to P.

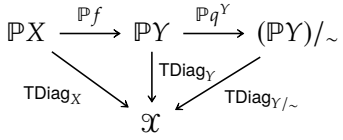
Using this concrete description of the 'quotient by bisimilarity', we directly obtain further useful properties

Lemma 4.8.6 *If there is an open morphism $f \colon X \to Y$ then $X/_\sim \cong Y/_\sim$.*

Proof. Observe that if $(e_1, e_2) \in \mathcal{S}_X$ then also $(f \cdot e_1, f \cdot e_2) \in \mathcal{S}_Y$. Indeed, this follows from the fact that strong b&f systems are closed under composition and taking inverse (Lemma 3.3.4 and Lemma 3.4.2) and also from the fact that $\mathcal{S}_Y$ is a maximal such, giving us that

$$(f \cdot e_1, f \cdot e_2) \in \mathcal{S}_f^{-1} ; \mathcal{S}_X ; \mathcal{S}_f \subseteq \mathcal{S}_Y.$$

Next, consider the following composition of morphisms of tree diagrams.

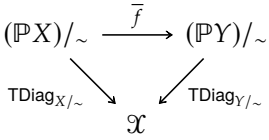


Note that the diagram commutes strictly because both f and q^Y are pathwise-embeddings.

By the above observation, if $(e_1, e_2) \in \mathcal{S}_X$ then $(f \cdot e_1, f \cdot e_2) \in \mathcal{S}_Y$ and so $\mathbb{P}q^Y(\mathbb{P}f(e_1)) = \mathbb{P}q^Y(\mathbb{P}f(e_2))$. Therefore, the mapping $\mathbb{P}q^Y \cdot \mathbb{P}$ lifts uniquely to $\overline{f} \colon (\mathbb{P}X)/_\sim \to (\mathbb{P}Y)/_\sim$ because $(\mathbb{P}X)/_\sim$ is defined as the quotient of $\mathbb{P}X$ by $\mathcal{S}_X$. In other words, we have the commutativity of the following square

$$\begin{array}{ccc} \mathbb{P}X & \xrightarrow{\mathbb{P}f} & \mathbb{P}Y \\ {\scriptstyle q^X}\downarrow & & \downarrow{\scriptstyle q^Y} \\ (\mathbb{P}X)/_\sim & \xrightarrow{\overline{f}} & (\mathbb{P}Y)/_\sim \end{array}$$

where $\overline{f}([e]) = [f \cdot e]$. Moreover, $\overline{f}$ is a strict morphism of tree diagrams



since q^X is surjective and since q^X, f and q^Y are pathwise-embeddings.

What is left to show is that $\overline{f}$ is a tree bijection since then we obtain that $\mathsf{TDiag}_{X/_\sim}$ and $\mathsf{TDiag}_{Y/_\sim}$ are equivalent diagrams, i.e. that $X/_\sim \cong Y/_\sim$.

For injectivity, let $\overline{f}([e]) = \overline{f}([e'])$ i.e. $(f \cdot e, f \cdot e') \in \mathcal{S}_Y$. Then, by Lemma 3.3.4 and Lemma 3.4.2, $(e, e') \in \mathcal{S}_f \,;\, \mathcal{S}_Y \,;\, \mathcal{S}_f^{-1}$ and, therefore, $(e, e') \in \mathcal{S}_X$ since $\mathcal{S}_X$ is the maximal strong b&f system on X, giving us that $[e] = [e']$ in $(\mathbb{P}X)/_\sim$.

Finally, for surjectivity, let $[n] \in (\mathbb{P}Y)/_\sim$. By Lemma 3.4.2, $\mathcal{S}_f$ is a b&f system between X and Y. Therefore, by (unit-back), there is some m such that $(m, n) \in \mathcal{S}_f$ and so $n = fm$ by definition of $\mathcal{S}_f$ or, in other words, $\overline{f}([m]) = [n]$. □

Remark 4.8.7 For $\mathfrak{X} = \mathsf{EM}(\mathbb{E}_k)$, the cofree coalgebra $F^{\mathbb{E}_k}(A)$ of some structure A consists of sequences $\overline{a}$ of length $\leq k$. However, $F^{\mathbb{E}_k}(A)/_\sim$ equates those sequences which have the same behaviour from the point of view of $\rightleftharpoons$ relation. This means that we can view $F^{\mathbb{E}_k}(A)/_\sim$ as the space of types with respect to logic $\mathtt{FO}_k$. We make use of this intuition later, in Chapter 9.

Implications for logic

The following theorem summarises the results from this section that all the notions of equivalence in terms of b&f system and equivalences expressed using open pathwise-embeddings align in arboreal categories.

Theorem 4.8.8 *Given objects X, Y in a representable arboreal category with pushouts, pullbacks and products, the following are equivalent.*

- $X \rightleftharpoons Y$ *i.e. X and Y are back-and-forth equivalent.*
- $X \leftrightarroweq Y$ *i.e. X and Y are behaviourally equivalent.*
- $X \sim Y$ *i.e. X and Y are bisimilar.*
- $X/_\sim \cong Y/_\sim$

Proof. The equivalence of back-and-forth equivalence $\rightleftharpoons$ and bisimilarity $\sim$ is shown in Proposition 4.8.2 and the equivalence of bisimilarity $\sim$ and behavioural equivalence $\leftrightarroweq$ follows from Proposition 3.4.10 and Proposition 4.8.3.

What is left to show is the equivalence of the last two items. The right-to-left implication is a consequence of the fact that we have a cospan

$$X \xrightarrow{q^X} X/_\sim \;\cong\; Y/_\sim \xleftarrow{q^Y} Y$$

where q^X and q^Y are open pathwise-embeddings by Theorem 4.7.10, since they are defined as pushouts of open pathwise-embeddings. Conversely, if we have any cospan

$$X \xrightarrow{p} W \xleftarrow{q} Y$$

where p and q are open pathwise-embeddings then, by Lemma 4.8.6, we obtain a sequence of isomorphisms $X/_\sim \cong W/_\sim \cong Y/_\sim$. □

Consequently, due to item 2 of Proposition 3.4.10, $\leftrightarrows$ is also transitive. Note that this can be also showed using pushout stability of open pathwise-embeddings (Theorem 4.7.10), by dualising the proof of item 2 of Proposition 3.4.10.

As a direct consequence of Theorem 4.8.8 we obtain the equivalence of different characterisations of the equivalence relation $\equiv_{\mathfrak{X}}$.

Corollary 4.8.9 *Given a functor $F\colon \mathcal{A} \to \mathfrak{X}$ where $\mathfrak{X}$ is a representable arboreal category, for A, B from $\mathcal{A}$, the following are equivalent:*

- $A \equiv_{\mathfrak{X}} B$, *that is,* $FA \rightleftharpoons FB$
- $FA \leftrightarrows FB$
- $FA \sim FB$
- $FA/_\sim \cong FB/_\sim$

Remark 4.8.10 Reggio–Riba [105] show that, for accessible arboreal adjunctions $\mathcal{A} \leftrightarrows \mathfrak{X}$ where $\mathcal{A}, \mathfrak{X}$ are locally finitely presentable and certain finiteness conditions are met, it follows that if A, B are $\mathcal{L}_\omega$-equivalent in $\mathcal{A}$ then $FA \rightleftharpoons FB$. In other words, their result gives an upper bound on the expressibility of logics captured by locally finitely presentable arboreal categories.

Comonadic meta-theorems

Homomorphism counting theorems

5

In this chapter we take a look at the first, probably most influential, application of game comonads to finite model theory. It is motivated by an observation Anuj Dawar, that recent theorems of Dvořák [46] and Grohe [55] when rephrased using game comonads, have the same categorical structure. We rephrase their results in the language of comonads, which leads to an elegant statement with a simple proof. Moreover, our result has shown to be very flexible to adjustments, with many follow-up results within and outside of the game comonad research programme.

5.1 Motivation 101
5.2 A Categorical Lovász Theorem 102
5.3 A Categorical Dvořák & Grohe theorem 106
The case of finite rank comonads 108
Related work 109
5.4 Equality elimination . . . 109

5.1 Motivation

Close to sixty years ago, Lovász [84] proved that two finite graphs (or, more generally, any two finite relational structures) A and B are isomorphic, if and only if, they admit the same number of homomorphisms from any other finite graph/structure, i.e.

$$A \cong B \iff \forall \text{ finite } C \quad |\hom(C, A)| = |\hom(C, B)|.$$

This seminal result has led to extensions and investigations in many different directions. One that concerns us is that, by restricting the collection of *all* finite graphs/structures to a natural sub-collection Δ, we can often obtain characterisations of natural coarsenings of the relation of isomorphism. For any subclass Δ of finite graphs/structures, we say that a pair of graphs/structures A and B are Δ-**homomorphism indistinguishable** if they cannot be distinguished by homomorphism counts from graphs/structures in Δ, that is,

$$A \simeq_\Delta B \iff \forall C \in \Delta \quad |\hom(C, A)| = |\hom(C, B)|.$$

For instance, Dvořák [46] shows that for any k, if we consider the collection $\mathcal{T}_k$ of all graphs of tree-width at most k, then finite graphs are $\mathcal{T}_k$-homomorphism indistinguishable if, and only if, they are $\equiv_{\mathsf{C}^{k+1}}$-equivalent. Another recent addition to this collection of results is that of Grohe [55]. He shows that, if $\mathcal{D}_n$ is the collection of graphs of *tree-depth* at most n, then two graphs are $\mathcal{D}_n$-homomorphism indistinguishable if, and only if, they are $\equiv_{\mathsf{C}_n}$-equivalent.

This variety of results relating homomorphism counts over restricted classes to approximations of isomorphism, often proved using very different methods, calls for the development of a more general theory. In this chapter we develop a general categorical result (using game comonads) which yields, as example applications, the results of Lovász, Dvořák and Grohe mentioned above. Moreover, we also obtain a new Lovász-style result for pointed Kripke structures, relating homomorphism counts from synchronisation trees of bounded height to equivalence in a modal logic with counting.

5.2 A Categorical Lovász Theorem

The categorical core of our abstract theorem is a general Lovász-type result which applies to any locally finite category with pushouts and a proper factorisation system. A category is **locally finite** if there are only finitely many morphisms between any two of its objects. A locally finite category $\mathcal{A}$ is said to be **combinatorial** if, for all $a, b \in \mathcal{A}$,

$$a \cong b \iff \forall k \in \mathcal{A} \quad |\hom_{\mathcal{A}}(k, a)| = |\hom_{\mathcal{A}}(k, b)|$$

Thus, Lovász' theorem [84] states that, for any finite relational signature σ, the category $\mathcal{S}tr_f(\sigma)$ of finite σ-structures with homomorphisms is combinatorial. The aim of this section is to prove the following generalisation of Lovász' result.

Theorem 5.2.1 *Let $\mathcal{A}$ be a locally finite category. If $\mathcal{A}$ has a proper factorisation system and pushouts of spans of quotients,[1] then it is combinatorial.*

1: I.e. it has pushouts of diagrams of the form $\bullet \twoheadleftarrow \bullet \twoheadrightarrow \bullet$.

Example 5.2.2 Observe that the categories of finite structures $\mathcal{S}tr_f(\sigma)$ and finite pointed structures $\mathcal{S}tr_{*,f}(\sigma)$ satisfy the assumptions of the theorem. In particular $\mathcal{S}tr_f(\sigma)$ and $\mathcal{S}tr_{*,f}(\sigma)$ have all finite colimits, including all pushouts. Therefore, we recover the original theorem of Lovász [84]. Moreover, the theorem also applies to the categories of finite sets, finite groups, finite monoids, etc.

We prove this theorem in the remainder of this section. To this end, we fix a category $\mathcal{A}$ admitting a proper factorisation system $(\mathcal{Q}, \mathcal{M})$.

To simplify the terminology, by a **$\mathcal{Q}$-pushout square** in $\mathcal{A}$ we refer to a commuting square which is a pushout consisting entirely of quotients, that is, a $\mathcal{Q}$-pushout square is a pushout of quotients.[2] We are interested in functors $\mathcal{A}^{op} \to \mathbf{Set}_f$, where $\mathbf{Set}_f$ is the category of finite sets and functions between them, that turn $\mathcal{Q}$-pushout squares in $\mathcal{A}$ into pullbacks in $\mathbf{Set}_f$. Recall that pullbacks in $\mathbf{Set}_f$ admit the following explicit description: a commutative square in $\mathbf{Set}_f$ is a pullback if, and only if,

2: Indeed, by the dual of Lemma 2.9.3, pushout of a quotient along any morphism is again a quotient.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ C & \xrightarrow{h} & D \end{array} \qquad \begin{array}{l} \forall b \in B, \forall c \in C, \text{ if } i(b) = h(c) \text{ then} \\ \exists! a \in A.\ (f(a) = b \text{ and } g(a) = c). \end{array}$$

If h and i are injective then, by identifying B and C with subsets of D, the pullback A can be identified with $B \cap C$.

The following lemma shows that the hom-functors involved in the homomorphism counting theorem send pushout squares to pullbacks.[3] Recall that, for any object n of a locally finite category $\mathcal{B}$, the functor $\hom_{\mathcal{B}}(-, n) \colon \mathcal{B}^{op} \to \mathbf{Set}_f$ sends a morphism $f \colon a \to b$ in $\mathcal{B}$ to the function $- \circ f \colon \hom_{\mathcal{B}}(b, n) \to \hom_{\mathcal{B}}(a, n)$.

3: This is a consequence of the more general fact that representable functors preserve all limits that exist in their domain, cf. [86, pp. 116-117], but we provide here also a direct elementary proof.

Lemma 5.2.3 *For any locally finite category $\mathcal{B}$ and $n \in \mathcal{B}$, the functor*

$$\hom_{\mathcal{B}}(-, n) \colon \mathcal{B}^{op} \to \mathbf{Set}_f$$

sends all pushout squares that exist in $\mathcal{B}$ to pullbacks in $\mathbf{Set}_f$.

Proof. Just observe that, given a pushout square in $\mathcal{B}$ and the corresponding diagram in $\mathbf{Set}_f$, as displayed below,

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ c & \xrightarrow{h} & d \end{array} \qquad \begin{array}{ccc} \hom_{\mathcal{B}}(d,n) & \xrightarrow{-\circ i} & \hom_{\mathcal{B}}(b,n) \\ {\scriptstyle -\circ h}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \hom_{\mathcal{B}}(c,n) & \xrightarrow{-\circ g} & \hom_{\mathcal{B}}(a,n) \end{array}$$

if $\alpha \in \hom_{\mathcal{B}}(b,n)$ and $\beta \in \hom_{\mathcal{B}}(c,n)$ are such that $\alpha \circ f = \beta \circ g$ then, by the universal property of the pushout, there is a unique $\gamma \in \hom_{\mathcal{B}}(d,n)$ satisfying $\gamma \circ i = \alpha$ and $\gamma \circ h = \beta$. □

Lemma 5.2.4 *If a functor $F\colon \mathcal{A}^{op} \to \mathbf{Set}_f$ sends $\mathbb{Q}$-pushout squares in $\mathcal{A}$ to pullbacks, then it sends $\mathbb{Q}$-morphisms to injections.*

Proof. Let $e\colon n \twoheadrightarrow m$ be an $\mathbb{Q}$-morphism in the category $\mathcal{A}$. Because e is an epimorphism, it follows directly that the square on the left-hand side below is a pushout square in $\mathcal{A}$.

$$\begin{array}{ccc} n & \xrightarrow{e} & m \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ m & \xrightarrow{\mathrm{id}} & m \end{array} \qquad \begin{array}{ccc} F(m) & \xrightarrow{\mathrm{id}} & F(m) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle F(e)} \\ F(m) & \xrightarrow{F(e)} & F(n) \end{array}$$

Since identities are $\mathbb{Q}$-morphisms, the square on the right-hand side above is a pullback in $\mathbf{Set}_f$. In turn, this is equivalent to $F(e)$ being an injection. □

Generic and degenerate elements. The key step in the proof of the classical Lovász' theorem consists in showing that, if $|\hom_{\mathcal{S}tr_f(\sigma)}(C,A)| = |\hom_{\mathcal{S}tr_f(\sigma)}(C,B)|$ for all C in $\mathcal{S}tr_f(\sigma)$, then also $|\mathrm{inj}_{\mathcal{S}tr_f(\sigma)}(C,A)| = |\mathrm{inj}_{\mathcal{S}tr_f(\sigma)}(C,B)|$ for all C in $\mathcal{S}tr_f(\sigma)$, where $\mathrm{inj}_{\mathcal{S}tr_f(\sigma)}(C,A)$ is the set of all injective homomorphisms $C \hookrightarrow A$, and similarly for $\mathrm{inj}_{\mathcal{S}tr_f(\sigma)}(C,B)$. We aim to show the analogous fact in our setting, where injective homomorphisms are replaced by $\mathcal{M}$-morphisms.

The intuition behind the following definition is that a σ-homomorphism $f\colon C \to A$ is non-injective precisely when there exist an onto non-injective σ-homomorphism $e\colon C \twoheadrightarrow C'$ and a σ-homomorphism $g\colon C' \to A$ such that $f = g \circ e$. In terms of the hom-functor $E \coloneqq \hom_{\mathcal{S}tr_f(\sigma)}(-,A)$, $f \in E(C)$ is non-injective if, and only if, $f = E(e)(g)$ for some onto non-injective σ-homomorphism $e\colon C \twoheadrightarrow C'$ and $g \in E(C')$.

Definition 5.2.5 *A **strict quotient** in $\mathcal{A}$ is an $\mathbb{Q}$-morphism which is not an isomorphism. (Equivalently, a strict quotient is an arrow in $\mathbb{Q} \setminus \mathcal{M}$.*[4]*)*

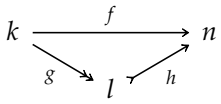
*Further, consider a functor $E\colon \mathcal{A}^{op} \to \mathbf{Set}_f$. For every $k \in \mathcal{A}$, we say that $s \in E(k)$ is **degenerate** if there exist a strict quotient $f\colon k \twoheadrightarrow l$ and $t \in E(l)$ such that $E(f)(t) = s$. Otherwise, s is called **generic**. The subset of $E(k)$ consisting of the generic elements is denoted by $E[\![k]\!]$.*

4: Cf. Lemma 2.9.3.

The next lemma shows that this definition matches our intuition for hom-functors on $\mathscr{A}$.

Lemma 5.2.6 *Let $E = \hom_{\mathscr{A}}(-, n)$ for some $n \in \mathscr{A}$. For any $k \in \mathscr{A}$, $E[\![k]\!]$ is the set of all $\mathscr{M}$-morphisms $k \to n$.*

Proof. Let f be an arbitrary element of $E(k) = \hom_{\mathscr{A}}(k, n)$. Assume f is generic, and take its $(\mathbb{Q}, \mathscr{M})$ factorisation:



Then $E(g)(h) = f$, and so g must be an $\mathscr{M}$-morphism. By Lemma 2.9.3, $f = h \circ g$ is also an $\mathscr{M}$-morphism.

Conversely, suppose f is an $\mathscr{M}$-morphism and pick an $\mathbb{Q}$-morphism $g\colon k \twoheadrightarrow l$ and $h \in E(l)$ such that $E(g)(h) = f$, i.e. $h \circ g = f$. By Lemma 2.9.3, g is an $\mathscr{M}$-morphism, so it is not a strict quotient. That is, f is generic. □

The following is the main technical lemma of this section, and it ultimately relies on an application of the inclusion-exclusion principle.

Lemma 5.2.7 *Assume $\mathscr{A}$ has pushouts, and let E and F be functors $\mathscr{A}^{op} \to \mathbf{Set}_f$ sending $\mathbb{Q}$-pushout squares in $\mathscr{A}$ to pullbacks. If $E(k) \cong F(k)$ for all $k \in \mathscr{A}$, then $E[\![k]\!] \cong F[\![k]\!]$ for all $k \in \mathscr{A}$.*

Proof. Let E, F be as in the statement, and suppose that $E(k) \cong F(k)$ for all $k \in \mathscr{A}$. We must show that, for every $k \in \mathscr{A}$, $E[\![k]\!] \cong F[\![k]\!]$. Equivalently, we show that the set $E\langle k\rangle$ of degenerate elements of $E(k)$ is in bijection with the set $F\langle k\rangle$ of degenerate elements of $F(k)$.

Observe that $E\langle k\rangle$ coincides with

$$\bigcup \{\mathrm{Im}(E(f)) \mid f\colon k \twoheadrightarrow l \text{ is a strict quotient in } \mathscr{A}\} \subseteq E(k),$$

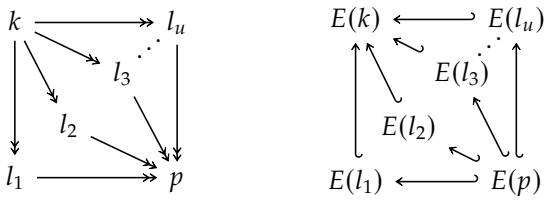
and similarly for $F\langle k\rangle$. Since $E(k)$ and $F(k)$ are finite, there are finite sets $\mathcal{S}_1, \mathcal{S}_2$ of strict quotients of k such that

$$E\langle k\rangle = \bigcup \{\mathrm{Im}(E(f)) \mid f \in \mathcal{S}_1\},$$
$$F\langle k\rangle = \bigcup \{\mathrm{Im}(F(f)) \mid f \in \mathcal{S}_2\}.$$

Let $\mathcal{S} \coloneqq \mathcal{S}_1 \cup \mathcal{S}_2$. By the inclusion-exclusion principle,

$$|E\langle k\rangle| = \sum_{J \subseteq \mathcal{S}, J \neq \emptyset} (-1)^{|J|+1} \left|\bigcap_{f \in J} \mathrm{Im}(E(f))\right|$$

and similarly for $|F\langle k\rangle|$. Therefore, to prove $E\langle k\rangle \cong F\langle k\rangle$ it suffices to show that $\left|\bigcap_{f\in J} \mathrm{Im}(E(f))\right| = \left|\bigcap_{f\in J} \mathrm{Im}(F(f))\right|$ for any non-empty $J \subseteq \mathcal{S}$. To this end, fix such a J and consider the *wide pushout* in $\mathscr{A}$ of the strict quotients in J, i.e. the diagram obtained by taking consecutive pushouts of the elements of J, as shown in the diagram on the left-hand side below.



By Lemma 2.9.3, all arrows in this diagram are $\mathbb{Q}$-morphisms. Hence, by Lemma 5.2.4, the diagram obtained by applying the functor E, depicted on the right-hand side above, consists of injections. Since E sends $\mathbb{Q}$-pushouts to pullbacks, the diagram on the right is a wide pullback of injections in $\mathbf{Set}_f$, and so $E(p) \cong \bigcap_{f\in J} \mathrm{Im}(E(f))$. Similarly, $F(p) \cong \bigcap_{f\in J} \mathrm{Im}(F(f))$. As $E(p) \cong F(p)$ by assumption, we get $\left|\bigcap_{f\in J} \mathrm{Im}(E(f))\right| = \left|\bigcap_{f\in J} \mathrm{Im}(F(f))\right|$, thus concluding the proof. □

We are now in a position to prove the main result of this section.

Proof of Theorem 5.2.1. Fix arbitrary objects $m, n \in \mathcal{A}$. For the non-trivial direction, assume $|\hom_{\mathcal{A}}(k, m)| = |\hom_{\mathcal{A}}(k, n)|$ for all $k \in \mathcal{A}$. Let E and F denote, respectively, the functors $\hom_{\mathcal{A}}(-, m)\colon \mathcal{A}^{op} \to \mathbf{Set}_f$ and $\hom_{\mathcal{A}}(-, n)\colon \mathcal{A}^{op} \to \mathbf{Set}_f$. These functors send $\mathbb{Q}$-pushout squares in $\mathcal{A}$ to pullbacks by Lemma 5.2.3. Thus, by Lemma 5.2.7, $E[\![k]\!] \cong F[\![k]\!]$ for all $k \in \mathcal{A}$. According to Lemma 5.2.6, there is a bijection

$$\{\mathcal{M}\text{-morphisms } k \rightarrowtail m\} \;\cong\; \{\mathcal{M}\text{-morphisms } k \rightarrowtail n\}.$$

Setting $k = m$, the existence of an $\mathcal{M}$-morphism $m \rightarrowtail m$ (namely the identity) entails the existence of an $\mathcal{M}$-morphism $i\colon m \rightarrowtail n$. Similarly, there exists an $\mathcal{M}$-morphism $j\colon n \rightarrowtail m$. A standard argument then shows that $m \cong n$ (see e.g. [101]). We briefly sketch a proof. The set L of all $\mathcal{M}$-morphisms $m \rightarrowtail m$ is a monoid with respect to composition and contains $j \circ i$. Since all $\mathcal{M}$-morphisms are monos, L satisfies the left cancellation law $ab = ac \Rightarrow b = c$. But every finite monoid satisfying the left cancellation law is a group, hence $j \circ i$ has an inverse. It follows from Lemma 2.9.3, that j is an isomorphism. □

Related work. Other categorical generalisations of Lovász' theorem have been obtained by Pultr [101] and Isbell [61]. Cf. also Lovász' paper [83]. In modern terms, Pultr requires a finitely well-powered, locally finite category with (extremal epi, mono) factorisations and Isbell requires the category to be locally finite with a special type of factorisation system (which he calls "bicategory"). Unlike the results in op. cit., our Theorem 5.2.1 does not hinge on a combinatorial counting argument, but rather on an application of the inclusion-exclusion principle, and appears to be better suited for applications to game comonads. More recently Reggio [104] proved a new Lovász-type theorem requiring only a locally finite category with a proper factorisation system which is $\mathbb{Q}$-well-founded, i.e. that each object has a well-founded poset of quotients. Reggio's statement is also proved in categorical terms and, instead of the inclusion-exclusion principle that we applied in our proof, Reggio uses the Möbius inversion formula.

5.3 A Categorical Dvořák & Grohe theorem

Recall that, for a comonad $\mathbb{C}$ on a category $\mathcal{A}$, we defined the relation $A \equiv_{\#\mathbb{C}} B$ as $F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B)$, cf. Section 3.1. Furthermore, in case when $\mathbb{C} = \mathbb{E}_k$, $\equiv_{\#\mathbb{C}}$ is the same as $\equiv_{\mathsf{C}_k}$ and if $\mathbb{C} = \mathbb{P}_k$, then $\equiv_{\#\mathbb{C}}$ is precisely $\equiv_{\mathsf{C}^k}$. Similarly, recall that $\mathbb{C}$-coalgebras of these comonads characterise structures of tree-depth and tree-width bounded from above by k.

Therefore, the following theorem is precisely in the shape of the theorems of Dvořák and Grohe.

Lemma 5.3.1 *For a comonad $\mathbb{C}$ on a category $\mathcal{A}$, if $\mathsf{EM}(\mathbb{C})$ is combinatorial then, for $A, B \in \mathcal{A}$,*

$$A \equiv_{\#\mathbb{C}} B \iff \forall C \in \Delta_{\mathbb{C}} \quad |\hom_{\mathcal{A}}(C, A)| = |\hom_{\mathcal{A}}(C, B)|$$

where $\Delta_{\mathbb{C}}$ consists of all C in $\mathcal{A}$ which admit a $\mathbb{C}$-coalgebra.

Proof. The result follows by unwrapping definitions and using the fact that $U^{\mathbb{C}}$ is the left adjoint of $F^{\mathbb{C}}$:

$$\begin{aligned} F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B) &\iff \hom_{\mathsf{EM}(\mathbb{C})}(X, F^{\mathbb{C}}(A)) \cong \hom_{\mathsf{EM}(\mathbb{C})}(X, F^{\mathbb{C}}(B)) \\ &\quad \forall X \in \mathsf{EM}(\mathbb{C}) \\ &\iff \hom_{\mathcal{A}}(U^G(X), A) \cong \hom_{\mathcal{A}}(U^G(X), B) \\ &\quad \forall X \in \mathsf{EM}(\mathbb{C}) \\ &\iff \hom_{\mathcal{A}}(C, A) \cong \hom_{\mathcal{A}}(C, B) \quad \forall C \in \Delta_{\mathbb{C}} \end{aligned}$$

□

It is, therefore, crucial to establish conditions under which the category of coalgebras is combinatorial. Below we give sufficient conditions.

Proposition 5.3.2 *Let $\mathcal{A}$ be a category with a proper factorisation system and a comonad $\mathbb{C}$ such that:*

1. *$\mathbb{C}$ is a regular comonad, i.e. it preserves embeddings;*
2. *$\mathcal{A}$ has pushouts of spans of quotients; and*
3. *$\mathcal{A}$ is locally finite.*

Then, $\mathsf{EM}(\mathbb{C})$ is combinatorial.

Proof. We show that the category $\mathsf{EM}(\mathbb{C})$ satisfies the hypotheses of Theorem 5.2.1. The forgetful functor $U^{\mathbb{C}} \colon \mathsf{EM}(\mathbb{C}) \to \mathcal{A}$ is faithful and, thus, for all $(A, \alpha), (B, \beta) \in \mathsf{EM}(\mathbb{C})$, there is an injection $\hom_{\mathsf{EM}(\mathbb{C})}((A, \alpha), (B, \beta)) \hookrightarrow \hom_{\mathcal{A}}(A, B)$. Because the latter set is finite, so is the former. Therefore, $\mathsf{EM}(\mathbb{C})$ is locally finite.

Since $\mathbb{C}$ is regular, by Lemma 2.9.6, it lifts the proper factorisation system from $\mathcal{A}$ to $\mathsf{EM}(\mathbb{C})$. Furthermore, since $U^{\mathbb{C}}$ creates colimits [18, Proposition 20.12], $\mathsf{EM}(\mathbb{C})$ has pushouts of spans of quotients. □

Observe that the categories of finite structures $\mathcal{S}tr_f(\sigma)$ and finite pointed structures $\mathcal{S}tr_{*,f}(\sigma)$ automatically satisfy the conditions of the proposition, i.e. the factorisation system from $\mathcal{S}tr(\sigma)$ restricts to finite structures, $\mathcal{S}tr(\sigma)$

has all pushouts and is locally finite, and similarly for $\mathcal{S}tr_{*,f}(\sigma)$. Therefore, we obtain the following.

Corollary 5.3.3 *For any regular comonad $\mathbb{C}$ on $\mathcal{S}tr_f(\sigma)$ or $\mathcal{S}tr_{*,f}(\sigma)$, the category of coalgebras $\mathsf{EM}(\mathbb{C})$ is combinatorial.*

Remark 5.3.4 In [42, Corollary 14] we show that the subcategory of $\mathsf{EM}(\mathbb{C})$ consisting of finite $\mathbb{C}$-coalgebras is combinatorial, for any comonad $\mathbb{C}$ on either $\mathcal{S}tr(\sigma)$ or $\mathcal{S}tr_*(\sigma)$. However, the proof uses the fact that the base categories are cocomplete and well-powered in order to create a factorisation system on $\mathsf{EM}(\mathbb{C})$ from this data. Our proof here is more direct and still applies to our examples, since our comonads are regular.

Putting Theorem 5.2.1, Lemma 5.3.1 and Proposition 5.3.2 together yields a categorical version of Dvořák and Grohe theorems.

Theorem 5.3.5 *Let $\mathcal{A}$ be a locally finite category with a proper factorisation system, pushouts of spans of quotients and a regular comonad $\mathbb{C}$, then, for $A, B \in \mathcal{A}$,*

$$A \equiv_{\#\mathbb{C}} B \iff \forall C \in \Delta_{\mathbb{C}} \quad |\hom_{\mathcal{A}}(C, A)| = |\hom_{\mathcal{A}}(C, B)|$$

where $\Delta_{\mathbb{C}}$ consists of all C in $\mathcal{A}$ which admit a $\mathbb{C}$-coalgebra.

Example 5.3.6 Using the fact that $\mathbb{E}_k$ is regular and restricts to $\mathcal{S}tr_f(\sigma)$ which, in turn, satisfies the assumptions of Theorem 5.3.5, we immediately recover Grohe's theorem [55].

Following exactly the same lines for $\mathbb{M}_k$, we obtain the following new theorem. For finite pointed Kripke structures A and B,[5]

$$\begin{aligned} A \equiv_{\#\mathtt{ML}_k} B \iff & \forall \text{synchronisation tree } C \text{ of depth} \leq k \\ & |\hom_{\mathcal{S}tr_{*,f}(\sigma)}(C, A)| = |\hom_{\mathcal{S}tr_{*,f}(\sigma)}(C, B)|. \end{aligned}$$

5: Instead of (A, a_0) and (B, b_0) we write just A, B, for simplicity.

For the next example, recall from Section 2.5 that every $\mathsf{EM}(\mathbb{P}_k)$ coalgebra can be equivalently described as a forest-ordered structure $(A, \leq, p)$ with a pebble function $p : A \to \{1, \dots, k\}$. We say that A has a **k-pebble forest-decomposition** if there exist $\leq$ and p such that $(A, \leq, p) \in \mathsf{EM}(\mathbb{P}_k)$ or, equivalently, if there exists some X such that $A \cong U^{\mathbb{P}_k}(X)$. Furthermore, we say that A has **k-pebble forest-decomposition of depth $\leq n$** if there exist $\leq$ and p such that $(A, \leq, p) \in \mathsf{EM}(\mathbb{P}_k)$ and $(A, \leq)$ has depth $\leq n$.

Example 5.3.7 To recover Dvořák's theorem, we have to do a little more work because $\mathbb{P}_k$ does not restrict to a comonad $\mathcal{S}tr_f(\sigma) \to \mathcal{S}tr_f(\sigma)$.

To this end, define $\mathbb{P}_{k,n}$ as the restriction of $\mathbb{P}_k$ to sequences of length $\leq n$. It is immediate that all the usual theorems for $\mathbb{P}_k$ adapt directly to $\mathbb{P}_{k,n}$ as well. In particular, the equivalence $\equiv_{\#\mathbb{P}_{k,n}}$ characterises the logic $\mathsf{C}_n^k := \mathsf{C}_n \cap \mathsf{C}^k$ and $\mathbb{P}_{k,n}$-coalgebras characterise k-pebble forest-decomposition of depth $\leq n$.

Also, it is immediate that $\mathbb{P}_{k,n}$ restrict to $\mathcal{S}tr_f(\sigma) \to \mathcal{S}tr_f(\sigma)$, therefore,

we obtain that, for finite structures A, B,

$$A \equiv_{\mathsf{C}^k_n} B \iff \forall C \text{ with a } k\text{-pebble forest-decomposition of depth} \leq n \\ |\hom_{\mathcal{S}tr_f(\sigma)}(C, A)| = |\hom_{\mathcal{S}tr_f(\sigma)}(C, B)|.$$

Then, since $\mathsf{C}^k = \bigcup_n \mathsf{C}^k_n$ and since, for every *finite* structure C of tree-width $< k$ there is some n such that C admits a k-pebble forest-decomposition of depth $\leq n$, we recover Dvořák's theorem [46] by Theorem 5.3.5 as follows:

$$\begin{aligned} & A \equiv_{\mathsf{C}^k} B \\ \iff & \forall n\, (A \equiv_{\mathsf{C}^k_n} B) \\ \iff & \forall n\, \forall C \text{ with a } k\text{-pebble forest-decomposition of depth} \leq n \\ & |\hom_{\mathcal{S}tr_f(\sigma)}(C, A)| = |\hom_{\mathcal{S}tr_f(\sigma)}(C, B)| \\ \iff & \forall C \text{ of tree-width} < k \\ & |\hom_{\mathcal{S}tr_f(\sigma)}(C, A)| = |\hom_{\mathcal{S}tr_f(\sigma)}(C, B)|. \end{aligned}$$

Example 5.3.8 Lovász' original theorem [84] is also recovered using Theorem 5.3.5, by setting $\mathbb{C} = \mathrm{Id}$ to be the identity comonad on $\mathcal{S}tr_f(\sigma)$.

Remark 5.3.9 The flexibility of this theorem is demonstrated by the fact that other examples of comonads automatically satisfy the required properties and, hence, homomorphism counting theorems are derived using this for free. See, for example, [4, 93, 117].

The case of finite rank comonads

To provide a more systematic approach to recovering Dvořák's theorem by directly working with $\mathbb{P}_k$, instead of $\mathbb{P}_{k,n}$, as in Example 5.3.7, Reggio [104] devised a version of Theorem 5.3.5 applicable to comonads of *finite rank*,[6] that is, in particular to $\mathbb{P}_k$ as well as $\mathbb{P}_{k,n}$, $\mathbb{E}_k$, $\mathbb{M}_k$, etc.

6: Cf. Subsection"Locally finitely presentable categories" on page 40.

Theorem 5.3.10 (Corollary 5.15 in [104]) *Let $\mathbb{C}$ be a finite rank comonad on a locally finitely presentable category $\mathcal{A}$, then for $A, B \in \mathcal{A}$ of* finite fp-type,[7]

$$A \equiv_{\#\mathbb{C}} B \iff \forall C \in \Delta_{\mathbb{C}} \quad |\hom_{\mathcal{A}}(C, A)| = |\hom_{\mathcal{A}}(C, B)|$$

where $\Delta_{\mathbb{C}}$ consists of all finitely generated[8] *$C \in \mathcal{A}$ admitting a $\mathbb{C}$-coalgebra.*

7: An object A is of 'finite fp-type' if, for every finitely presentable C, $\hom(C, A)$ is finite.

8: Being 'finitely generated' is a notion similar to 'finitely presentable.' It asks, for an object C, that the functor $\hom(C, \text{-})$ preserves directed colimits *of monomorphisms*.

This statement uses extra terminology specific to the theory of locally finitely presentable categories. However, in the case of (pointed) structures, since finitely generated and finitely presentable objects are simply the finite structures, Reggio's theorem simplifies as follows.

Corollary 5.3.11 *Let $\mathbb{C}$ be a comonad of finite rank on $\mathcal{S}tr(\sigma)$ or $\mathcal{S}tr_*(\sigma)$. Then, for (pointed) finite structures A, B,*

$$A \equiv_{\#\mathbb{C}} B \iff \forall C \in \Delta_{\mathbb{C}} \quad |\hom_{\mathcal{A}}(C, A)| = |\hom_{\mathcal{A}}(C, B)|$$

where $\Delta_{\mathbb{C}}$ consists of all finite *C which admit a $\mathbb{C}$-coalgebra.*

Remark 5.3.12 As shown in [24], $\equiv_{\mathtt{FO}_k}$ is not a homomorphism indistinguishability relation. Consequently, by the above, there cannot be a finite rank comonad $\mathbb{C}$ such that $\equiv_{\#\mathbb{C}}$ characterises $\equiv_{\mathtt{FO}_k}$.

In a similar vein, [81] derives a similar result for the invertible-map equivalence $G \equiv^{\mathrm{IM}}_{k,Q} H$ which expresses the fact that G and H cannot be distinguished by the set of k-tuples in both graphs with respect to any linear-algebraic operator acting on vector spaces of characteristic q, for any $q \in Q$. *Op. cit.* shows that, again, $\equiv^{\mathrm{IM}}_{k,Q}$ is not a homomorphism indistinguishability relation and, therefore, by Corollary 5.3.11, there is no finite rank comonad $\mathbb{C}$ such that $\equiv_{\#\mathbb{C}}$ characterises $\equiv^{\mathrm{IM}}_{k,Q}$.

Related work

Many equivalence classes $\equiv_{\#\mathbb{C}}$ given by some comonad $\mathbb{C}$ can be expressed using the language of matrices [56], e.g.

- $G \equiv_{\#\mathbb{P}_{k,n}} H$ iff there exists a pseudo-stochastic matrix X such that $X\widehat{\mathbb{A}}_G = \widehat{\mathbb{A}}_H X$ for every matrix map $A \in \mathcal{A}^n_k$

where $\mathcal{A}^n_k$ is a certain class of matrices and $\widehat{\mathbb{A}}_G$ is the associated adjacency matrix. In [103, Appendix C] the authors show how this equivalence can be derived by counting homomorphisms from finite bilabelled $\mathbb{P}_{k,n}$-coalgebras.

5.4 Equality elimination

When comparing our proofs with the original theorems of Dvořák [46] and Grohe [55], our results are way more general and way less technical to prove. However, in order to achieve this, we used a crucial property of counting logic fragments, that is, that they have elimination of equality.

Recall from Chapter 2 that for the Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$, for finite structures A, B,

$$A \equiv_{\mathbb{E}_k} B \text{ i.e. } F^{\mathbb{E}_k}(A) \rightleftharpoons F^{\mathbb{E}_k}(B)$$

$$\Longleftrightarrow$$

$A \equiv_{\mathtt{FO}_k} B$ i.e. A, B satisfy the same first-order sentences of quantifier depth at most k, *without equality*.

When we wish to express the logical equivalence with respect to the fragment, where equality is allowed (e.g. $\mathtt{FO}^=_k$), we take the expansions A^I, B^I of structures A, B, where (as in Section 2.8)

- $(\text{-})^I : \mathcal{S}tr(\sigma) \to \mathcal{S}tr(\sigma^I)$ sends a structure to its expansion with an interpretation of an additional binary relation I, expressing equality, and
- $\mathbb{C}^I$ denotes (for emphasis) the version of $\mathbb{C}$ on $\mathcal{S}tr(\sigma^I)$.

Then, $A \equiv_{\mathrm{FO}^=_k} B$ iff $F^{\mathbb{E}^I_k}(A^I) \rightleftharpoons F^{\mathbb{E}^I_k}(B^I)$. On the other hand, when it comes to the counting fragments, we have that, for example,

$$A \equiv_{\#\mathbb{E}_k} B \text{ i.e. } F^{\mathbb{E}_k}(A) \cong F^{\mathbb{E}_k}(B) \iff A \equiv_{\mathsf{C}^=_k} B \iff A \equiv_{\mathsf{C}_k} B \quad (5.1)$$

where $\mathsf{C}^=_k$ and C_k denote the fragments of infinitary counting logics consisting of sentences of quantifier depth at most k, *with* and *without equality*, respectively.

The same discrepancy also holds for $\mathbb{P}_k$ and the k-variable fragments of first-order and infinitary counting logics.

Consequently, for these comonads $\mathbb{C}$, the equivalence relation expressed using the *relaxed* bijective back-and-forth games (i.e. $F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B)$) is the same as the equivalence expressed using the *non-relaxed* bijective back-and-forth games (i.e. $F^{\mathbb{C}^I}(A^I) \cong F^{\mathbb{C}^I}(B^I)$).

Elimination of equality of these fragments (i.e. equivalences similar to the second equivalence in (5.1)) can be proved using syntactic methods [42] or, as we show below, using categorical methods and using the homomorphism counting theorem Theorem 5.3.5. The first crucial observation is that the functor $(\text{-})^I$ has a left adjoint.

Lemma 5.4.1 *Define a functor*

$$(\text{-})/_I \colon \mathcal{S}tr(\sigma^I) \to \mathcal{S}tr(\sigma)$$

which sends (A, I) to the quotient $A/_I$ of the σ-reduct A of (A, I) by the least equivalence relation containing I. Then, $(\text{-})/_I \dashv (\text{-})^I$.

We observe that this adjunction restricts to the subcategories of structures identified by the $\mathbb{P}_{k,n}$ comonad from Example 5.3.7, since structures which admit a $\mathbb{P}_{k,n}$-coalgebra are precisely the structures which admit a k-pebble forest-decomposition of depth $\leq n$.

Note that, for $n = k$, a σ-structure A (or a σ^I-structure A) admits an $\mathbb{E}_k$-coalgebra if and only if it admits a $\mathbb{P}_{k,n}$-coalgebra since admitting a k-pebble forest-decomposition of depth $\leq k$ is the same as admitting a forest-decomposition of depth $\leq k$ (without pebble function). Therefore, the following also applies to the case of structures admitting a $\mathbb{E}_k$-coalgebra.

Lemma 5.4.2 (Combinatorial Lemma) *The adjunction*

$$\mathcal{S}tr(\sigma) \underset{(\text{-})^I}{\overset{(\text{-})/_I}{\rightleftarrows}} \mathcal{S}tr(\sigma^I)$$

restricts to the full subcategories

- $\mathcal{T}^k_n \subseteq \mathcal{S}tr(\sigma)$ *of σ-structures and*
- σ^I*-structures* $(\mathcal{T}^I)^k_n \subseteq \mathcal{S}tr(\sigma^I)$

comprising of σ-structures or σ^I-structures, respectively, which admit a k-pebble forest-decomposition of depth $\leq n$.

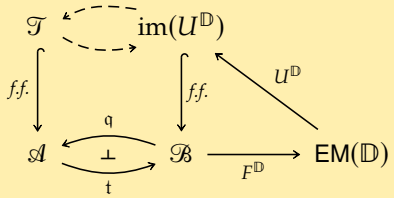
Proof idea. The fact that $(\text{-})^I$ restricts to $\mathcal{T}_n^k \to (\mathcal{T}^I)_n^k$ is trivial since, for $A \in \mathcal{S}tr(\sigma)$, any tree decomposition $\leq$ of A is also a tree decomposition for A^I.

The converse direction is shown in [42, Proposition 23]. The idea of the proof is to view a σ^I-structure A with a k-pebble forest-decomposition of depth $\leq n$ as a k-variable primitive positive formula of quantifier rank n, via the Chandra–Merlin correspondence [35], and observe that elimination of quantifiers on primitive positive formulas can be done without changing the number of variables or quantifier rank. See also [21] for a direct proof of this fact. □

We show that the Combinatorial Lemma is sufficient for transporting logical equivalence given by a comonad on σ^I-structures whilst still counting homomorphisms with respect to a comonad on σ-structures.

To this end, given a comonad $\mathbb{D}$, we denote by $\mathrm{im}(U^{\mathbb{D}})$ the full subcategory consisting of all objects which admit a $\mathbb{D}$-coalgebra, i.e. $\mathrm{im}(U^{\mathbb{D}})$ is the class $\{B \in \mathcal{B} \mid \exists X \in \mathsf{EM}(\mathbb{D}),\ B \cong U^{\mathbb{D}}(X)\}$.

Proposition 5.4.3 *Let $\mathsf{t}\colon \mathcal{A} \to \mathcal{B}$ be a fully faithful functor with a left adjoint q, and let $\mathcal{T}$ be a full subcategory of $\mathcal{A}$ and $\mathbb{D}$ a comonad on $\mathcal{B}$.*



Then, if $\mathsf{EM}(\mathbb{D})$ is combinatorial and if $\mathsf{q} \dashv \mathsf{t}$ restricts to an adjunction between $\mathcal{T}$ and $\mathrm{im}(U^{\mathbb{D}})$ then, for A, B from $\mathcal{A}$,

$$\mathsf{t}(A) \equiv_{\#\mathbb{D}} \mathsf{t}(B) \iff \forall C \in \mathcal{T} \quad |\hom(C, A)| = |\hom(C, B)|.$$

Proof. By Lemma 5.3.1 and using the fact that $\mathsf{EM}(\mathbb{D})$ is combinatorial we have that

$$\begin{aligned} \mathsf{t}(A) \equiv_{\#\mathbb{D}} \mathsf{t}(B) &\iff \forall C \in \mathrm{im}(U^{\mathbb{D}})\ \hom(C, \mathsf{t}(A)) \cong \hom(C, \mathsf{t}(B)) \\ &\iff \forall T \in \mathcal{T}\ \hom(T, A) \cong \hom(T, B) \end{aligned}$$

where the last equivalence holds by the following justification.

For the right-to-left direction, for any $C \in \mathrm{im}(U^{\mathbb{D}})$, we have

$$\begin{aligned} \hom_{\mathcal{B}}(C', \mathsf{t}(A)) &\cong \hom_{\mathcal{A}}(\mathsf{q}(C), A) && (\mathsf{q} \dashv \mathsf{t}) \\ &\cong \hom_{\mathcal{A}}(\mathsf{q}(C), B) && (\mathsf{q}(C) \in \mathcal{T}) \\ &\cong \hom_{\mathcal{B}}(C, \mathsf{t}(B)). && (\mathsf{q} \dashv \mathsf{t}) \end{aligned}$$

Conversely, for a $T \in \mathcal{T}$,

$$\begin{aligned} \hom_{\mathcal{A}}(C, A) &\cong \hom_{\mathcal{B}}(\mathsf{t}(C), \mathsf{t}(A)) && (\mathsf{t} \text{ fully faithful}) \\ &\cong \hom_{\mathcal{B}}(\mathsf{t}(C), \mathsf{t}(B)) && (\mathsf{t}(C) \in \mathrm{im}(U^{\mathbb{D}})) \\ &\cong \hom_{\mathcal{A}}(C, B). && (\mathsf{t} \text{ fully faithful}) \end{aligned}$$

□

We can now state a general categorical theorem, establishing elimination of equalities for counting fragments which admit a homomorphism counting result, in the style of Dvořák and Grohe.

Theorem 5.4.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, with comonads $\mathbb{C}$ and $\mathbb{D}$, respectively. Let $\mathsf{q} \dashv \mathsf{t}$ be an adjunction between $\mathcal{A}$ and $\mathcal{B}$ which restricts to $\mathrm{im}(U^{\mathbb{C}})$ and $\mathrm{im}(U^{\mathbb{D}})$. If $\mathsf{EM}(\mathbb{C})$ and $\mathsf{EM}(\mathbb{D})$ are combinatorial then, for A, B in $\mathcal{A}$,*

$$A \equiv_{\#\mathbb{C}} B \iff \mathsf{t}(A) \equiv_{\#\mathbb{D}} \mathsf{t}(B).$$

Proof. The result follows by invoking Lemma 5.3.1 for comonad $\mathbb{C}$ and Proposition 5.4.3 for $\mathcal{T} = \mathrm{im}(U^{\mathbb{C}})$, since

$$\begin{aligned} A \equiv_{\#\mathbb{C}} B &\iff \forall T \in \mathrm{im}(U^{\mathbb{C}})\ \hom(T, A) \cong \hom(T, B) \\ &\iff \mathsf{t}(A) \equiv_{\#\mathbb{D}} \mathsf{t}(B). \end{aligned}$$

□

Therefore, assuming that $\equiv_{\#\mathbb{E}_k}$ for $\mathbb{E}_k$ on $\mathcal{S}tr(\sigma)$ characterises k-variable counting logic without equality and $\equiv_{\#\mathbb{E}_k^I}$ for $\mathbb{E}_k^I$ on $\mathcal{S}tr(\sigma^I)$ characterises the same fragment but with equality, Theorem 5.4.4 entails elimination of equality for C_k. The same argument but for $\mathbb{P}_k$ applies to show elimination of equality for C^k, although, as in Example 5.3.7, this has to be done iteratively, using $\mathbb{P}_{k,n}$ and the fact that $\mathsf{C}^k = \bigcup_n \mathsf{C}^k_n$.

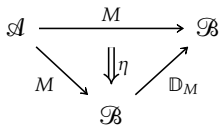
Density comonads 6

In this chapter we focus on the density comonad construction, which allows us to describe a comonad using a different (usually simpler) functor. We show that a special case of this construction, when the domain of the description functor is discrete, allows us to classify many classes of interest, as those classes whose objects admit a coalgebra for a given comonad, similarly to Section 2.5. Moreover, we show that the comonad obtained this way satisfies the assumptions of the abstract homomorphism counting theorem for $\equiv_{\#\mathbb{C}}$, similar to that of the categorical Dvořák & Grohe theorem, from Section 5.3. Finally, we also use the density construction to give a proof-theoretic description of $\mathbb{E}_k$.

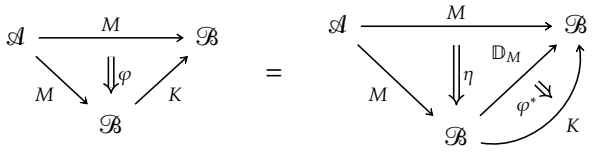
6.1 The density construction 113
The comonad structure . . 114
6.2 Discrete density comonads 115
6.3 The abstract classification theorem 115
6.4 Concrete examples 118
Examples 120
6.5 Comonad morphisms, a brief overview 121
Comonad morphisms from composites of functors . . 122
6.6 Graph Parameters 122
Grading graph parameters 123
Comparison with game comonads 124
Nowhere dense comonads 126
6.7 Homomorphism-Counting Theorems 126
6.8 Game comonads from proof systems 129
6.9 Dense graph parameters and more 133

6.1 The density construction

We start by reviewing the basics of the theory of density comonads, introduced independently by Appelgate and Tierney [22] and Kock [77] (who studied the dual notion of codensity monads). The **density comonad** of a functor $M\colon \mathcal{A} \to \mathcal{B}$ is a functor $\mathbb{D}_M\colon \mathcal{B} \to \mathcal{B}$ with a natural transformation $\eta\colon M \Rightarrow \mathbb{D}_M M$.



Moreover, η is required to be the initial natural transformation with this property. In other words, for any functor $K\colon \mathcal{B} \to \mathcal{B}$ and a natural transformation $\varphi\colon M \Rightarrow KM$ there is a *unique* $\varphi^*\colon \mathbb{D}_M \Rightarrow K$ such that $\varphi = \varphi^* M \cdot \eta$, i.e. diagramatically

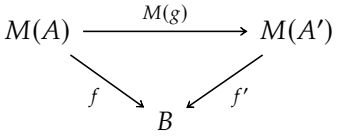


Density comonads are special types of left Kan extensions. They do not exist for all functors. However, when $\mathcal{A}$ is a small category and $\mathcal{B}$ is cocomplete then $\mathbb{D}_M$ exists for every functor $M\colon \mathcal{A} \to \mathcal{B}$. In such case, $\mathbb{D}_M(B)$ is computed as the colimit of the diagram:

$$M \downarrow B \xrightarrow{V} \mathcal{A} \xrightarrow{M} \mathcal{B}$$

where V is the forgetful functor from the comma category $M \downarrow B$, which consists of pairs (A, f) where $f\colon M(A) \to B$ is a morphism in $\mathcal{B}$, and morphisms $(A, f) \to (A', f')$ between such pairs are morphisms

$g\colon A \to A'$ in $\mathscr{A}$ making the following triangle commute.



We may express the same fact by the formula:

$$\mathbb{D}_M(B) \;=\; \operatorname*{colim}_{A\in\mathscr{A},\ M(A)\to B} M(A). \tag{6.1}$$

Note that $\mathscr{A}$ does not have to be small nor $\mathscr{B}$ cocomplete in general. It is enough that the colimit above exists. In such case we speak of **pointwise** density comonads. Denote by

$$\iota_f\colon M(A) \to \mathbb{D}_M(B)$$

the inclusion morphism of the copy of $M(A)$ corresponding to the morphism $f\colon M(A) \to B$ into the colimit. Then, the component

$$\eta_A\colon M(A) \to \mathbb{D}_M(M(A))$$

of the natural transformation $\eta\colon M \Rightarrow \mathbb{D}_M M$ is given as ι_f for f equal to the identity morphism $\mathrm{id}\colon M(A) \to M(A)$.

The comonad structure

The initiality of $\eta\colon M \Rightarrow \mathbb{D}_M M$ ensures that we can equip $\mathbb{D}_M$ with a comonad structure. In particular, the identity natural transformation $M \Rightarrow \mathrm{Id} \circ M$ uniquely factors as the composition of η with the counit $\varepsilon\colon \mathbb{D}_M \Rightarrow \mathrm{Id}$ and, similarly, $\mathbb{D}_M(\eta)\cdot\eta\colon M \Rightarrow \mathbb{D}_M \circ \mathbb{D}_M \circ M$ factors through the comultiplication $\delta\colon \mathbb{D}_M \Rightarrow \mathbb{D}_M^2$. In other words, the counit and the comultiplication are uniquely determined by the equations

$$\varepsilon M \cdot \eta = \mathrm{id} \quad \text{and} \quad \delta M \cdot \eta = \mathbb{D}_M(\eta)\cdot\eta. \tag{6.2}$$

Moreover, these two equations guarantee that the functor

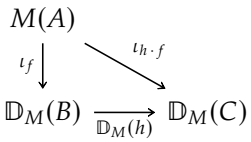
$$M^{\dagger}\colon \mathscr{A} \to \mathsf{EM}(\mathbb{D}_M) \tag{6.3}$$

which sends $A \in \mathscr{A}$ to the coalgebra $\eta_A\colon M(A) \to \mathbb{D}_M(M(A))$ is well-defined.

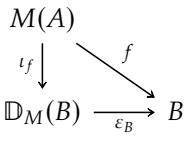
Lastly, we recall three equations of density comonads, following from (6.2), which we use extensively throughout the chapter. For morphisms $f\colon M(A) \to B$ and $h\colon B \to C$, the following equations hold.

(DC1) $\mathbb{D}_M(h)\cdot\iota_f = \iota_{h\cdot f}$
(DC2) $\varepsilon_B \cdot \iota_f = f$
(DC3) $\delta_B \cdot \iota_f = \iota_{\iota_f}$

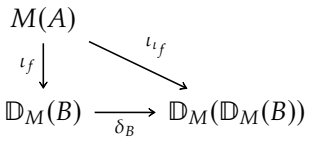
Equivalently, we represent the same with the following three commutative triangles.



(DC1)



(DC2)



(DC3)

6.2 Discrete density comonads

Recall that a category is **discrete** whenever it only has identity morphisms. By **discrete density comonads** we mean density comonads for functors whose domain is discrete. An important feature of discrete density comonads is that the formula in (6.1) simplifies dramatically. Indeed, assume that

$$M \colon \mathscr{A} \to \mathscr{B}$$

is a fixed functor from a small and discrete category $\mathscr{A}$. Then, since $\mathscr{A}$ is discrete, there are no morphisms between objects $M(A)$, given by morphisms $M(A) \to B$, in the colimit formula (6.1). Therefore, the density comonad $\mathbb{D}_M \colon \mathscr{B} \to \mathscr{B}$ is computed as a coproduct, that is, the colimit of a discrete diagram. Concretely, for an object B in $\mathscr{B}$,

$$\mathbb{D}_M(B) \;=\; \coprod_{A \in \mathscr{A}} \; \coprod_{f \colon M(A) \to B} M(A). \tag{6.4}$$

Note that $\mathbb{D}_M$ exists whenever the above coproduct exists in $\mathscr{B}$, for every object $B \in \mathscr{B}$. In particular, $\mathbb{D}_M$ exists whenever $\mathscr{B}$ has all coproducts.

As with general density comonads, we have inclusion morphisms

$$\iota_f \colon M(A) \to \mathbb{D}_M(B)$$

for every $f \colon M(A) \to B$, which satisfy axioms (DC1)–(DC3) from Subsection"The comonad structure" on the facing page.

6.3 The abstract classification theorem

In this section we investigate the question of what classes of objects $\Delta \subseteq \mathscr{B}$ of a category $\mathscr{B}$ admit a comonad $\mathbb{C}$ which **classifies** Δ. This means that, for every object $B \in \mathscr{B}$,

$$B \in \Delta \iff B \text{ admits a } \mathbb{C}\text{-coalgebra.}$$

We know that Δ needs to be closed under isomorphism for this to make any sense. Furthermore, if $\mathscr{B}$ has coproducts, then $\mathbb{C}$ has to be closed under coproducts as well since comonad coalgebras are closed under coproducts. It turns out that an additional requirement that the class is "closed under connected substructures" (to be defined precisely later) suffices for some $\mathbb{C}$ to exist. Furthermore, the construction makes use of the discrete density comonad construction outlined above.

The abstract categorical result, Theorem 6.4.1 below, gives us that the classes of relational structures classifiable by comonads include many

well-known and well-studied examples of classes, such as classes of bounded degree, clique-width, twin-width, etc.

Instead of fixing a class Δ of objects of $\mathscr{B}$, we work in a slightly more general setting and we fix a functor

$$M\colon \mathscr{A} \to \mathscr{B}$$

from a discrete category $\mathscr{A}$ into $\mathscr{B}$. We further assume that the pointwise density comonad $\mathbb{D}_M$ exists on $\mathscr{B}$ (i.e. it is given by the formula (6.4)).

We start with a useful observation. Recall that an object C is **connected** iff, for every morphism $f\colon C \to \coprod_i A_i$ into a coproduct, the morphism f factors uniquely through one of the inclusion morphisms $\iota_i\colon A_i \to \coprod_i A_i$.[1] Whenever a connected C is such that $A \cong C + X$ for some A, X, we say that C is a **component** of A.

1: Equivalently, C is connected iff $\hom(C, -)$ preserves coproducts.

Lemma 6.3.1 *Let $\xi\colon X \to \mathbb{D}_M(X)$ be a $\mathbb{D}_M$-coalgebra and let $\iota_C\colon C \to X$ be a component inclusion. Furthermore, let $f\colon M(A) \to X$ be the morphism for which $\xi \cdot \iota_C$ decomposes through ι_f, as shown below.*

$$\begin{array}{ccc} C & \xrightarrow{\iota_C} & X \\ {\scriptstyle z}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle \xi} \\ M(A) & \xrightarrow[\iota_f]{} & \mathbb{D}_M(X) \end{array} \tag{6.5}$$

Then, also the two triangles in the diagram above commute, that is, $\iota_C = f \cdot z$ and $\iota_f = \xi \cdot f$.

Proof. The first equality is obtained immediately from the triangle law of coalgebras (cf. (2.6) from Chapter 2) together with (DC2) from Subsection"The comonad structure" on page 114 as

$$\iota_C = \varepsilon_C \cdot \xi \cdot \iota_C = \varepsilon_C \cdot \iota_f \cdot z = f \cdot z.$$

To show that also $\iota_f = \xi \cdot f$ we apply the square law of coalgebras (cf. (2.6)). Observe that

- $\delta_X \cdot \xi \cdot \iota_C = \delta_X \cdot \iota_f \cdot z = \iota_{\iota_f} \cdot z$ by (DC3), and
- $\mathbb{D}_M(\xi) \cdot \xi \cdot \iota_C = \mathbb{D}_M(\xi) \cdot \iota_f \cdot z = \iota_{\xi \cdot f} \cdot z$ by (DC1).

Because C is connected, the factorisation $C \to M(A) \to \mathbb{D}_M(\mathbb{D}_M(C))$ into the coproduct must be unique, hence $\iota_f = \xi \cdot f$. □

In the following we need to assume that $\mathscr{B}$ is a **componental category**, i.e. that

- every object in $\mathscr{B}$ is (isomorphic to) a coproduct of connected objects, and
- inclusion morphisms into coproducts $\iota_i\colon a_i \to \coprod_i a_i$ are monomorphisms.

We say that an object of $\mathscr{B}$ is **essentially in** $\mathscr{A}$ if it is isomorphic to $M(A)$, for some A in $\mathscr{A}$.

Lemma 6.3.1 directly implies a version of Theorem 6.4.1 for connected objects.

Lemma 6.3.2 *If $\mathscr{B}$ is a componental category, then a connected object of $\mathscr{B}$ is essentially in $\mathscr{A}$ iff it admits a $\mathbb{D}_M$-coalgebra.*

Proof. Let $C \in \mathscr{B}$ be connected. The left-to-right implication follows immediately from the fact that, $M^+(A)$ is a coalgebra on $M(A)$, for every $A \in \mathscr{A}$ (cf. (6.3)). For the right-to-left implication, let $\beta\colon C \to \mathbb{D}_M(C)$ be a coalgebra. Since C is connected,

$$\beta\colon C \to \coprod_{A\in\mathscr{A}} \coprod_{f\colon M(A)\to C} M(A)$$

uniquely factors as $\beta = \iota_f \cdot \beta_0$, for some $\beta_0\colon C \to M(A)$ and $f\colon M(A) \to C$. By Lemma 6.3.1, the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{id}} & C \\ {\scriptstyle\beta_0}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle\beta} \\ M(A) & \xrightarrow{\iota_f} & \mathbb{D}_M(C) \end{array}$$

One triangle immediately gives $f \cdot \beta_0 = \mathrm{id}$ and the other triangle together with $\beta = \iota_f \cdot \beta_0$ and the fact that inclusions are monomorphisms (in componental categories) entails $\beta_0 \cdot f = \mathrm{id}$. □

This proof serves us as a stepping stone and inspiration for the proof of Lemma 6.3.4 below. We start with a technical observation.

Lemma 6.3.3 *In a componental category, a component inclusion $C \to Y$ which factors through a monomorphism $X \to Y$ is a component of X as well.*

Proof. Assume Y is equal to the coproduct $\coprod_i C_i$ and X is equal to $\coprod_j D_j$ for some collections of connected objects $\{C_i\}_i$ and $\{D_j\}_j$. By assumption, the component inclusion $\iota\colon C \to Y$ factors as $m \cdot h$ for some monomorphism $m\colon X \to Y$ and a morphism $h\colon C \to X$.

Since C is connected, h factors through a component inclusion $\iota_j\colon D_j \to X$ as shown in the left diagram below:

$$\begin{array}{ccc} D_j & \xleftarrow{h_0} & C \\ {\scriptstyle\iota_j}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle\iota} \\ X & \xrightarrow{m} & Y \end{array} \qquad\qquad \begin{array}{ccc} D_j & \xrightarrow{u} & C_k \\ & {\scriptstyle m\cdot\iota_j}\searrow & \downarrow{\scriptstyle\iota_k} \\ & & Y \end{array}$$

Since D_j is connected, $m \cdot \iota_j$ factors through some inclusion $\iota_k\colon C_k \to Y$, as shown in the right diagram above. But then $i = k$ and $u \cdot h_0 = \mathrm{id}$ since $\iota_i = m \cdot h = m \cdot \iota_j \cdot h_0 = \iota_k \cdot u \cdot h_0$ and component inclusions are unique. Furthermore, since both m and ι_j are monomorphisms by our assumptions, so must be u because $\iota_\ell \cdot u = m \cdot \iota_j$. Consequently, u is an isomorphism because it is both a monomorphism and a split epimorphism. □

To make progress, we need to assume that $\mathscr{A}$ is **component-based**. This means that, whenever $B \in \mathscr{B}$ is essentially in $\mathscr{A}$ then so is every component of B. Note that this condition mirrors the third item in the

definition of component-based classes. With this we show the main technical lemma of this section.

Lemma 6.3.4 *If $\mathcal{A}$ is component-based and $\mathcal{B}$ is a componental category, then any component C of an object X of $\mathcal{B}$ which admits a $\mathbb{D}_M$-coalgebra $\xi\colon X \to \mathbb{D}_M(X)$ is essentially in $\mathcal{A}$.*

Proof. Let $\iota_C\colon C \to X$ be the inclusion morphism of C into X. Furthermore, let $f\colon M(A) \to X$ be the morphism such that $\xi \cdot \iota_C$ decomposes through $\iota_f\colon M(A) \to \mathbb{D}_M(X)$ and recall that, by Lemma 6.3.1, the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\iota_C} & X \\ {\scriptstyle z}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle \xi} \\ M(A) & \xrightarrow{\iota_f} & \mathbb{D}_M(X) \end{array} \tag{6.6}$$

Observe that f is a monomorphism since ι_f is. Therefore, by Lemma 6.3.3, C is a component of $M(A)$. Consequently, C is essentially in $\mathcal{A}$ because $\mathcal{A}$ is component-based. □

The main classification theorem, which we state in full, is obtained as a consequence of the previous lemma.

Theorem 6.3.5 *Let $M\colon \mathcal{A} \to \mathcal{B}$ be a functor from a discrete component-based category $\mathcal{A}$ into a componental category $\mathcal{B}$ such that the pointwise density comonad $\mathbb{D}_M$ exists.*
Then an object $b \in \mathcal{B}$ is isomorphic to a coproduct of objects essentially in $\mathcal{A}$ if and only if b admits a $\mathbb{D}_M$-coalgebra.

Proof. The left-to-right implication follows the fact that $M^{\dagger}(A)$ is a coalgebra on $M(A)$, for every $A \in \mathcal{A}$ (cf. (6.3)), and that coalgebras are closed under coproducts that exist in $\mathcal{B}$[2]. Conversely, if $\xi\colon X \to \mathbb{D}_M(X)$ is a coalgebra then, by our assumptions, X is isomorphic to a coproduct $\coprod_i C_i$ of connected objects by Lemma 6.3.4 and all those components are essentially in $\mathcal{A}$. □

2: Follows from the fact that the forgetful functor $\mathsf{EM}(\mathbb{D}_M) \to \mathcal{B}$ creates colimits, cf. [18, Proposition 20.12].

6.4 Concrete examples

The abstract Theorem 6.3.5 has very concrete applications, when $\mathcal{B}$ is instantiated as the category of relational structures $\mathcal{S}tr(\sigma)$ or the category **Graph** of graphs and their homomorphisms.

To this end, for a class Δ of finite relational structures or graphs, we say that it is **component-based** if it is closed under

- isomorphisms,
- finite coproducts, and
- summands (i.e. if $A + B$ is in Δ then so are A and B).

We can directly use the above machinery to prove the following concrete results.

Theorem 6.4.1 *Any component-based class Δ is classified by a comonad.*

Proof. Observe that the category of relational structures $\mathcal{S}tr(\sigma)$ is a componental category.[3] Next, given a component-based class Δ of relational structures or graphs, let Δ_C be the subclass of Δ consisting of connected structures only. We then set $\mathcal{A}$ to be a discrete subcategory of $\mathcal{S}tr(\sigma)$ or **Graph** consisting of one representative from every isomorphism class in Δ_C. Since we picked only one representative from every equivalence class, the category $\mathcal{A}$ is small. Therefore, the density comonad $\mathbb{D}_M$, for the inclusion functor $M\colon \mathcal{A} \to \mathcal{S}tr(\sigma)$, exists because both $\mathcal{S}tr(\sigma)$ and **Graph** have all (small) coproducts. Observe that the comonad $\mathbb{D}_M$ classifies Δ. Indeed, by Theorem 6.3.5, a finite relational structure B has a $\mathbb{D}_M$-coalgebra if and only if there exist $C_1, \ldots, C_n$ in Δ_C such that $B \cong C_1 + \cdots + C_n$. In turn, this is equivalent to B being in Δ, which follows from being component-based as then $C_1 + \cdots + C_n$ is in Δ iff all the individual structures $C_1, \ldots, C_n$ are. □

3: Note that for $\mathcal{S}tr(\sigma)$, the connected objects are those structures whose Gaifman graphs are connected.

Remark 6.4.2 In the proof of Theorem 6.4.1 we made sure that all objects in the image of M are connected. This is stronger than assuming that $\mathcal{A}$ is component-based. By carefully inspecting the proof of Theorem 6.3.5 and the preceding lemmas one can check that this extra assumption allows us to drop the requirement that $\mathcal{B}$ is componental. See Lemma 6.7.2 below for details.

A concrete description of discrete density comonads. Let $\mathbb{D}_\Delta$ be obtained for a component-based class Δ of σ-structures as in the proof of Theorem 6.4.1 above. For a σ-structure B, $\mathbb{D}_M(B)$ can be concretely described as follows. Its universe consists of tuples

$$(A, f, x)$$

where $f\colon M(A) \to B$ is a homomorphism of relational structures and x is an element of $M(A)$. Further, an n-ary relation R in σ is interpreted as the set of all tuples

$$(A, f, x_1), \quad \ldots, \quad (A, f, x_n)$$

such that $R(x_1, \ldots, x_n)$ in $M(A)$.

The axiom (DC1) determines the action of $\mathbb{D}_M$ on morphisms. Given a homomorphism $g\colon B \to B'$, we have a homomorphism $\mathbb{D}_M(g)\colon \mathbb{D}_M(B) \to \mathbb{D}_M(B')$ defined by

$$(M(A), h, x) \mapsto (M(A), g \cdot h, x).$$

Moreover, by (DC2), the counit $\varepsilon_B\colon \mathbb{D}_M(B) \to B$ is the following map

$$(M(A), h, x) \mapsto h(x)$$

and the comultiplication $\delta_B\colon \mathbb{D}_M(B) \to \mathbb{D}^2_M(B)$ is given by

$$(M(A), h, x) \mapsto (M(A), \iota_h, x),$$

as dictated by (DC3).

Example 6.4.3 Let Δ be the class consisting of only the triangle graph. Then, given an arbitrary graph G, the graph computed as $\mathbb{D}_\Delta(G)$ is the disjoint union of $k \times l$ triangles, where k is the number of triangles in G and l is the number of automorphisms of the triangle graph, i.e. $l = 6$.

Note that the extra copies arising from the automorphisms can be eliminated by adding all automorphisms of the triangle graph into $\mathcal{A}$ of the generating functor $M : \mathcal{A} \to \mathbf{Graph}$. However, then $\mathcal{A}$ is a groupoid and not just a discrete category. In this case our proofs would still probably go through, albeit with some additional bookkeeping for the extra automorphisms.

Remark 6.4.4 Theorem 6.4.1 says that, for a class Δ of structures closed under isomorphisms and finite coproducts, if Δ is also closed under summands then it can be classified by a comonad. However, the converse does not hold. Let C and D be two graphs with no homomorphism $C \to D$ nor any homomorphisms $D \to C$. This happens, for example, if C is the triangle and D is the cycle on five vertices. Take $\mathcal{A}$ to be the discrete subcategory of **Graph** consisting of $C + D$ only and let $M : \mathcal{A} \to \mathbf{Graph}$ be the subcategory inclusion. Then, despite $C + D$ admitting a $\mathbb{D}_M$-coalgebra, no connected graph admits a $\mathbb{D}_M$-coalgebra (by Lemma 6.3.2). It is easy to see that the class of finite structures classified by $\mathbb{D}_M$ is the class consisting of graphs isomorphic to

$$C + \cdots + C + D + \cdots + D$$

where both C and D appear in at least one copy in the coproduct. Consequently, the class of structures classified by $\mathbb{D}_M$ is not closed under summands.

Examples

The category $\mathcal{S}tr(\sigma)$ of σ-structures is a componental category. Therefore, in our applications we only need to check that a class Δ of σ-structures is closed under finite coproducts and summands. These are fairly weak conditions, satisfied by many well-known examples of classes from the literature. In particular, this includes classes of finite structures closed under finite coproducts which are

1. **monotone**, i.e. class closed under taking substructures,
2. hereditary, i.e. class closed under taking induced substructures, or
3. closed under taking graph minors.

Further examples include

4. Fraïssé classes closed under free amalgamations, or
5. classes of coproducts of connected cores.

Recall that a core is a structure with the property that all of its endomorphisms are automorphisms. An example of a class from (5) is the class of coproducts of cycles. Note that the discrete density comonad for this class captures co-spectrality, see Section 6.7.

As an example of a non-example, take the class of graphs that can be drawn on a surface of genus n, for $n > 1$. This class is characterised by a finite set of forbidden minors. However, it is not closed under taking

coproducts and hence is not a component-based class. On the other hand, any minor-closed class can be completed under finite coproducts. The resulting class is then still minor closed [32, Lemma 5] and hence is classified by a comonad.[4]

4: I am grateful to Anuj Dawar for pointing out these facts.

Remark 6.4.5 The proof of Theorem 6.4.1 is carried out abstractly, in the language of category theory, and thus can be dualised. In the dual statement we have monads instead of comonads and instead of component-based classes we have classes closed under isomorphisms, products, and *factors*, i.e. with the property that if $A \times B$ is in the class then so are A and B. For such classes there is a monad which classifies the class, i.e. a finite structure is in the class iff it admits an algebra for the monad. An example of a class of graphs which can be classified in this way is the class of connected non-bipartite graphs, cf. Chapter 8 in [58].

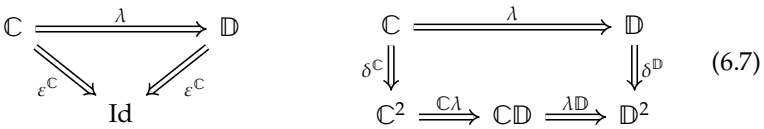
6.5 Comonad morphisms, a brief overview

Monad morphisms (from the dual theory of monads) can be thought of as transformations of algebraic theories, e.g. we have a monad morphism from the monad specifying monoids to that of specifying groups, which corresponds to adding the extra axioms which turn a monoid into a group.

Similarly, comonads have their morphisms too, obtained simply by dualising those of monads. However, since there is no inherent specification logic of comonads, similarly to how algebraic theories specify monads, we cannot explain comonad morphisms by a similar analogy.

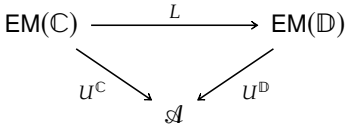
We take a closer look at comonads morphisms and their effect on the theory of game comonads in Chapter 7. For now, we just review some facts and definitions, following Appelgate and Tierney [22], which are required in the rest of this chapter.

Given two comonads $(\mathbb{C}, \varepsilon^{\mathbb{C}}, \delta^{\mathbb{C}})$ and $(\mathbb{D}, \varepsilon^{\mathbb{D}}, \delta^{\mathbb{D}})$ on $\mathcal{A}$, a natural transformation $\lambda \colon \mathbb{C} \Rightarrow \mathbb{D}$ is a **comonad morphism** if the following two diagrams of natural transformations commute.



(6.7)

Note that comonad morphisms can be equivalently presented as functors $L \colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$ such that the following diagram of functors commutes.



The functor L is constructed from the comonad morphism given as a natural transformation λ by sending $A \xrightarrow{\alpha} \mathbb{C}(A)$ to $A \xrightarrow{\alpha} \mathbb{C}(A) \xrightarrow{\lambda_A} \mathbb{D}(A)$. We study this situation in detail in Chapter 7. In particular, see Theorem 7.1.1.

Comonad morphisms from composites of functors

Let M be the composite of functors

$$\mathcal{A}_0 \xrightarrow{M_0} \mathcal{A} \xrightarrow{M_1} \mathcal{B}$$

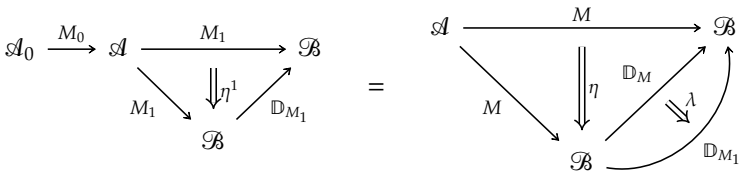
such that the density comonads $\mathbb{D}_M$ and $\mathbb{D}_{M_1}$ exist. Let

$$\eta \colon M \Rightarrow \mathbb{D}_M M \qquad \eta^1 \colon M_1 \Rightarrow \mathbb{D}_{M_1} M_1$$

be the corresponding initial natural transformations. By initiality of η, there is a unique natural transformation

$$\lambda \colon \mathbb{D}_M \Rightarrow \mathbb{D}_{M_1}$$

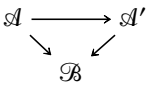
such that:



Moreover, the following holds too.

Lemma 6.5.1 *$\lambda \colon \mathbb{D}_M \Rightarrow \mathbb{D}_{M_1}$ is a comonad morphism.*

In fact, $\mathbb{D}_{(-)}$ is a functor from the category of functors $X \to \mathcal{B}$ which admit density comonads with morphisms the commutative triangles of the form



into the category of comonads and comonad morphisms (cf. page 73 in [45], see also [79]).

6.6 Graph Parameters

A **graph parameter** is a mapping $\mu \colon \mathbf{Graph}_{\mathsf{fin}} \to \overline{\mathbb{R}}$, from the class of finite graphs $\mathbf{Graph}_{\mathsf{fin}}$ to the class of extended real numbers $\overline{\mathbb{R}} = [-\infty, +\infty]$, which gives the same value to any two isomorphic graphs. Moreover, we say that it is **standard**[5] if $\mu(G_1 + G_2) = \max\{\mu(G_1), \mu(G_2)\}$.

5: Also known as *maxing*, e.g. in [85].

Standard graph parameters cover many well-known examples of graph parameters from the literature, such as

- clique number, chromatic number, max-degree,
- tree-depth, tree-width, path-width, clique-width, etc.

In this section we show that every standard graph parameter μ gives rise to a **graded comonad** $(\mathbb{C}_k)_k$, that is, a sequence of comonads $(\mathbb{C}_k)_k$ indexed by extended real numbers and comonad morphisms $g_{k,l} \colon \mathbb{C}_k \Rightarrow \mathbb{C}_l$, for every $k \leq l$ in $\overline{\mathbb{R}}$, such that $g_{k,l} = g_{j,l} \cdot g_{k,j}$ for any $k \leq j \leq l$ in $\overline{\mathbb{R}}$.[6] Given a graded comonad $(\mathbb{C}_k)_k$ we define the **coalgebra number** $\kappa^{\mathbb{C}}(G)$, of a graph

6: In fact, this is a special type of graded comonad, with the grading being over the fixed monoid $(\overline{\mathbb{R}}, \min, +\infty)$. For details see [14].

G, to be the infimum of $k \in \overline{\mathbb{R}}$ such that G admits a $\mathbb{C}_k$-coalgebra [3, 14]. We show that the coalgebra number for the constructed graded comonad agrees with the standard graph parameter μ we started with. In other words, we have that $\mu(G) \leq k$ iff G admits a $\mathbb{C}_k$-coalgebra.

Note that every graded comonad $(\mathbb{C}_k)_k$ trivially determines a graph parameter, by setting $\mu(G) := \kappa^{\mathbb{C}}(G)$. We have already mentioned graded comonads characterising graph parameters this way. For example, the (graded) Ehrenfeucht–Fraïssé comonad $(\mathbb{E}_k)_k$ characterises tree-depth [14], the pebbling comonad $(\mathbb{P}_k)_k$ characterises tree-width [3] and the pebble-relation comonad $(\mathbb{PR}_k)_k$ classifies path-width [92].

To start with, observe that there is a one-to-one correspondence between **graph properties**, i.e. graph parameters valued in $\{0, 1\}$, and classes of finite graphs which are closed under isomorphisms. Furthermore, it is easy to see that the correspondence restricts to that of standard graph properties and component-based classes of graphs:

Lemma 6.6.1 *Given a standard graph property $\mu \colon \mathbf{Graph}_{\mathsf{fin}} \to \{0, 1\}$, the class of graphs G such that $\mu(G) = 0$ is closed under isomorphisms, finite coproducts, and summands. In fact, every such class is obtained from a standard graph property this way.* □

Therefore, by Theorem 6.4.1, there is a comonad $\mathbb{C}^\mu$ which classifies the class Δ of finite graphs G such that $\mu(G) = 0$, for every standard graph property μ. We construct $\mathbb{C}^\mu$ explicitly, as the pointwise density comonad for the inclusion functor

$$\mathscr{A} \to \mathbf{Graph}, \tag{6.8}$$

where $\mathscr{A}$ is a discrete subcategory of finite connected graphs consisting precisely of one graph from every isomorphism class in Δ. Then, by Theorem 6.3.5, $\mathbb{C}^\mu$ classifies Δ.

Grading graph parameters

We use this to construct a sequence of comonads for a given standard graph parameter μ. For every extended real number k, we turn μ into a graph property

$$\mu_{\leq k} \colon \mathbf{Graph} \to \{0, 1\}$$

by setting $\mu_{\leq k}(G) = 0$ iff $\mu(G) \leq k$. Then, the density comonad $\mathbb{C}^\mu_k$, defined as $\mathbb{C}^\mu$ for $\mu := \mu_{\leq k}$, classifies finite graphs G such that $\mu(G) \leq k$.

Moreover, we can make sure that there is a linearly ordered chain of embeddings of discrete categories

$$\mathscr{A}_{-\infty} \hookrightarrow \cdots \hookrightarrow \mathscr{A}_k \hookrightarrow \mathscr{A}_l \hookrightarrow \cdots \hookrightarrow \mathscr{A}_{+\infty} \qquad (\text{with } k \leq l)$$

where each $\mathscr{A}_k$ is a category as in (6.8), for the class of graphs G such that $\mu(G) \leq k$. Then, by Lemma 6.5.1, the composite

$$\mathscr{A}_k \hookrightarrow \mathscr{A}_l \to \mathbf{Graph},$$

for $k \leq l$, gives rise to a comonad morphism $g_{k,l} \colon \mathbb{C}^\mu_k \Rightarrow \mathbb{C}^\mu_l$. In fact, we have $g_{k,l} = g_{j,l} \cdot g_{k,j}$ for every $k \leq j \leq l$, by functoriality of $\mathbb{D}_{(-)}$. As a

result, $(\mathbb{C}^{\mu}_k)_k$ is a graded comonad with the property that $\kappa^{\mathbb{C}^{\mu}}(G) = \mu(G)$ for every finite graph G. We obtain a sequence of comonad morphisms

$$\mathbb{C}^{\mu}_0 \Rightarrow \mathbb{C}^{\mu}_1 \Rightarrow \mathbb{C}^{\mu}_2 \Rightarrow \mathbb{C}^{\mu}_3 \Rightarrow \dots$$

and, consequently, also a sequence of functors

$$\mathsf{EM}(\mathbb{C}^{\mu}_0) \to \mathsf{EM}(\mathbb{C}^{\mu}_1) \to \mathsf{EM}(\mathbb{C}^{\mu}_2) \to \mathsf{EM}(\mathbb{C}^{\mu}_3) \to \dots$$

Remark 6.6.2 The procedure to produce sequences of comonads for standard graph parameters can be defined dually for graph parameters μ with the property that $-\mu$ is standard, i.e. graph parameters such that $\mu(G_1 + G_2) = \min\{\mu(G_1), \mu(G_2)\}$. This is done by constructing a sequence of standard graph properties

$$\mu_{\geq k} \colon \mathbf{Graph} \to \{0, 1\}$$

and inducing comonads classifying the classes of graphs such that $\mu(G) \geq k$ in a similar spirit as before. This then covers examples of graph parameters such as min-degree and girth.

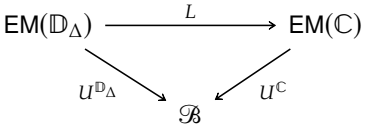
Comparison with game comonads

For some graph parameters and classes of structures we already knew how to construct comonads that classify them. In particular, this holds for the classes of structures classified by the comonads $\mathbb{P}_k$, $\mathbb{E}_k$, and $\mathbb{PR}_k$. In this section, we explain the relationship between those comonads and discrete density comonads constructed directly for given classes.

In fact, we show that discrete density comonads are weakly initial in the category of comonads that classify the same class. To this end, denote by $\mathbb{D}_\Delta$ the discrete density comonad constructed as in Section 6.4, for a component-based class Δ.

Theorem 6.6.3 *Let Δ be a component-based class of relational structures or graphs and let $\mathbb{C}$ be a comonad that classifies a class Γ. Then, $\Delta \subseteq \Gamma$ if, and only if, there exists a comonad morphism $\mathbb{D}_\Delta \Rightarrow \mathbb{C}$.*

Observe that the right-to-left direction is immediate as a comonad morphism $\mathbb{D}_\Delta \Rightarrow \mathbb{C}$ lifts to a functor $L \colon \mathsf{EM}(\mathbb{D}_\Delta) \to \mathsf{EM}(\mathbb{C})$ making the following diagram commute (cf. Section 6.5).



(Here $\mathscr{B}$ is either the category of relational structures or graphs.) For a structure B in Δ, let $\beta \colon B \to \mathbb{D}_\Delta(B)$ be a $\mathbb{D}_\Delta$-coalgebra, which exists because $\mathbb{D}_\Delta$ classifies Δ. Then, by the commutativity of the above triangle $L(\beta)$ is a $\mathbb{C}$-coalgebra $B \to \mathbb{C}(B)$ making $B \in \Gamma$ because $\mathbb{C}$ classifies Γ.

We carry out the left-to-right direction of the proof abstractly, for arbitrary categories rather than just relational structures or graphs. Let $\mathbb{D} := \mathbb{D}_M$ be the density comonad of a functor $M \colon \mathscr{A} \to \mathscr{B}$ from a discrete category $\mathscr{A}$.

Further, assume that $\mathbb{C}$ is a comonad on $\mathscr{B}$ such that for every $A \in \mathscr{A}$, there exists a coalgebra

$$\varphi_A \colon M(A) \to \mathbb{C}(M(A)).$$

Observe that, since $\mathscr{A}$ is a discrete category, the collection of morphisms $\{\varphi_A \mid A \in \mathscr{A}\}$ trivially forms a natural transformation $\varphi \colon M \Rightarrow \mathbb{C}M$. Since $\mathbb{D}$ is a density comonad of M, there is a natural transformation $\varphi^* \colon \mathbb{D} \Rightarrow \mathbb{C}$ such that

$$\varphi = \varphi^* M \cdot \eta \tag{6.9}$$

where $\eta \colon M \Rightarrow \mathbb{D}M$ is the initial natural transformation determining $\mathbb{D}$ (cf. Section 6.1). Then, Theorem 6.6.3 follows from the following lemma, which is a direct consequence of Theorem II.1.1 in [45]. We include its proof for completeness.

Lemma 6.6.4 *$\varphi^* \colon \mathbb{D} \Rightarrow \mathbb{C}$ is a morphism of comonads.*

Proof. First, let us check commutativity of the triangle diagram in (6.7). By (6.2), $\varepsilon^{\mathbb{D}} M \cdot \eta = \mathrm{id}$. Similarly, by (6.9), $\varepsilon^{\mathbb{C}} \cdot \varphi^* \cdot \eta = \varepsilon^{\mathbb{C}} \cdot \varphi = \mathrm{id}$ where the last equality follows from commutativity of the triangle law in (2.6) for every individual coalgebra $\varphi_A \colon M(A) \to \mathbb{C}(M(A))$. We see that $\varepsilon^{\mathbb{D}} M \cdot \eta = (\varepsilon^{\mathbb{C}} \cdot \varphi^*) M \cdot \eta$ from which it follows that $\varepsilon^{\mathbb{D}} = \varepsilon^{\mathbb{C}} \cdot \varphi^*$ by the universal property of η.

Next, we check the oblong law in (6.7). The composition $\delta^{\mathbb{C}} M \cdot \varphi^* M \cdot \eta$ is equal to $\delta^{\mathbb{C}} M \cdot \varphi$, by (6.9), which is equal to $\mathbb{C}\varphi \cdot \varphi$ by the square law of coalgebras (2.6). Similarly,

$$\begin{aligned} \varphi^* \mathbb{C} M \cdot \mathbb{D}\varphi^* M \cdot \delta^{\mathbb{D}} M \cdot \eta &= \varphi^* \mathbb{C} M \cdot \mathbb{D}\varphi^* M \cdot \mathbb{D}\eta \cdot \eta \\ &= \varphi^* \mathbb{C} M \cdot \mathbb{D}\varphi \cdot \eta \\ &= \mathbb{C}\varphi \cdot \varphi^* M \cdot \eta \\ &= \mathbb{C}\varphi \cdot \varphi \end{aligned}$$

The first equality follows from (6.2), the second and fourth from (6.9) and the third is naturality of φ^*. We have shown that $(\varphi^* \mathbb{C} \cdot \mathbb{D}\varphi^* \cdot \delta^{\mathbb{D}}) M \cdot \eta$ is equal to $(\delta^{\mathbb{C}} \cdot \varphi^*) M \cdot \eta$ which, by the universal property of η, implies that $\varphi^* \mathbb{C} \cdot \mathbb{D}\varphi^* \cdot \delta^{\mathbb{D}} = \delta^{\mathbb{C}} \cdot \varphi^*$. □

Example 6.6.5 Theorem 6.6.3 gives us that that for our running examples of comonads $\mathbb{E}_k$, $\mathbb{P}_k$, $\mathbb{PR}_k$, there exist comonad morphisms

$$\mathbb{D}_{\mathcal{TD}_k} \Rightarrow \mathbb{E}_k, \quad \mathbb{D}_{\mathcal{TW}_k} \Rightarrow \mathbb{P}_k, \quad \text{and} \quad \mathbb{D}_{\mathcal{PW}_k} \Rightarrow \mathbb{PR}_k,$$

where $\mathcal{TD}_k$, $\mathcal{TW}_k$, and $\mathcal{PW}_k$ are the classes of finite structures of tree-depth, tree-width, and path-width $\leq k$, respectively.

Note that unlike the game comonads $\mathbb{E}_k$, $\mathbb{P}_k$, and $\mathbb{PR}_k$, the discrete density comonads $\mathbb{D}_{\mathcal{TD}_k}$, $\mathbb{D}_{\mathcal{TW}_k}$, and $\mathbb{D}_{\mathcal{PW}_k}$ do not classify infinite structures with the corresponding properties.

Nowhere dense comonads

A direct consequence of Theorem 6.6.3 is that we can characterise comonads that classify monotone nowhere dense classes of graphs in terms of non-existence of certain comonad morphisms. Recall that a class Δ is **somewhere dense** if there exists a natural number p such that, for every n, the p-th subdivision K_n^p of all edges in the clique graph K_n on n vertices is a subgraph of some graph in Δ.[7] Then, a class is **nowhere dense** [97] if it is not somewhere dense.

7: A subdivision of a set of edges in a graph replaces each edge in the set by a path of length 2 through a new vertex.

It is immediate that a monotone class of graphs Δ (i.e. a class closed under substructures) is somewhere dense if and only if $\mathbf{Cli}_p \subseteq \Delta$, for some p, where

$$\mathbf{Cli}_p = \{K_n^p \mid n \in \mathbb{N}\}$$

is the class of p-th subdivisions of all cliques. We can now state the characterisation.

Proposition 6.6.6 *Assume $\mathbb{C}$ classifies a monotone class of graphs Δ. Then, Δ is nowhere dense if, and only if, there is no comonad morphism $\mathbb{D}_{\mathbf{Cli}_p} \Rightarrow \mathbb{C}$ for any $p \in \mathbb{N}$.*

Proof. Define $\overline{\mathbf{Cli}_p}$ to be the closure of $\mathbf{Cli}_p$ under finite coproducts. Observe that $\overline{\mathbf{Cli}_p}$ is component-based and, since the connected objects in $\overline{\mathbf{Cli}_p}$ are precisely the objects in $\mathbf{Cli}_p$, the comonad $\mathbb{D}_{\mathbf{Cli}_p}$ classifies $\overline{\mathbf{Cli}_p}$. Moreover, since any class classified by a comonad needs to be closed under finite coproducts, $\mathbf{Cli}_p \subseteq \Delta$ iff $\overline{\mathbf{Cli}_p} \subseteq \Delta$. The result follows by monotonicity of Δ and by Theorem 6.6.3. □

6.7 Homomorphism-Counting Theorems

In Chapter 5 we establish a number of homomorphism counting results for counting fragments of the logics captured as $\equiv_{\#\mathbb{C}}$ by our game comonads $\mathbb{C}$. This fits a general scheme of results where, for a selected equivalence relation $\asymp$ on finite structures we ask if it is a homomorphism indistinguishability relation, that is, if there is a class of finite structures Δ such that, for finite structures A, B,

$$A \asymp B \iff \forall C \in \Delta \quad |\hom(C, A)| = |\hom(C, B)|.$$

A number of well-known equivalence relations on finite structures have been characterised in this way, see Table 6.1 for an overview.

The equivalences $A \asymp B$ characterised using comonadic methods, in Chapter 5, are of the form $F^{\mathbb{C}}(A) \cong F^{\mathbb{C}}(B)$, for some comonad $\mathbb{C}$. Moreover, by being formalised using game comonads, they express logical

* The *bipartite double cover* of a graph G is the product graph $G \times K_2$ where K_2 is the clique on two vertices.

The fact that isomorphic bipartite double covers correspond to counting homomorphisms from bipartite graphs was worked out by Böker in his master's thesis [29]. He later observed (in private communication) that the same result already follows from Section 5.4.2 in [85].

Table 6.1: Examples of Lovász-type theorems

Δ	$\asymp$	reference
cycles	co-spectrality	(folklore)
trees	fractional isomorphism	[102]
bipartite graphs	isomorphic bipartite double covers	[29, 85] *
planar graphs	quantum isomorphism	[88]
tree-depth $\leq k$	Duplicator wins the bijective k-round Ehrenfeucht–Fraïssé game	[55]
tree-width $< k$	Duplicator wins the bijective k-pebble game	[46]
path-width $< k$	Duplicator wins the bijective k-pebble relation game	[92]
admitting a k-pebble tree cover of height $\leq n$	Duplicator wins the bijective n-round k-pebble E.F. game	[42]
synchronization trees of height $\leq k$	equivalence in graded modal logic of modal depth $\leq k$	[42]
an inner-product compatible class	existence of a certain unitary map between homomorphism tensor spaces	[56]

equivalence in a counting fragment. However, not all homomorphism indistinguishability relations need to be of this form.

Anuj Dawar has asked (in private communication) whether there are comonads covering the other listed cases as well. In our terminology, this means finding comonads that classify both the class Δ as well as the corresponding $\asymp$ relation, in the same row of the table. We answer this question in the positive for any component-based class Δ. In Theorem 6.7.5 below, we show that the discrete density comonad $\mathbb{C}$ that classifies such class also classifies the relation $\asymp$ where $A \asymp B$ precisely whenever, for every $C \in \Delta$, $\hom(C, A) \cong \hom(C, B)$.

The main ingredient of our proof is the theorem of Reggio (Corollary 5.3.11), establishing that $\equiv_{\#\mathbb{C}}$, for a comonad on (pointed) relational structures, is homomorphism indistinguishable whenever $\mathbb{C}$ has finite rank.[8]

8: Cf. Subsection"Locally finitely presentable categories" on page 40.

In the following we fix a functor

$$M : \mathcal{A} \to \mathcal{B}$$

from a discrete category $\mathcal{A}$ and assume that the pointwise density comonad $\mathbb{D}_M$ for M exists. We show that, under sufficient conditions, $\mathbb{D}_M$ has finite rank.

Proposition 6.7.1 *If all objects in the image of M are finitely presentable, then $\mathbb{D}_M$ is finitary.*

Proof. It is enough to check the proof for $M : \mathbf{1} \to \mathcal{B}$ where $\mathbf{1} = \{\star\}$ is the discrete category with one object. Indeed, the density comonad for any $\mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ discrete is computed as a coproduct of density comonads for individual restrictions $\mathbf{1} \to \mathcal{B}$ and coproducts commute

with colimits. Then, for an $M\colon \mathbf{1} \to \mathscr{B}$, the density comonad is

$$\mathbb{D}_M(X) = \hom_{\mathscr{B}}(M(\star), X) \otimes M(\star)$$

where $S \otimes b$ denotes copower, i.e. $b + \ldots + b$ $|S|$-times. Consequently, since $M(\star)$ is finitary in $\mathscr{B}$, copowers commute with colimits, for a (small) directed diagram $D\colon I \to \mathscr{B}$ with a colimit $\operatorname{colim}_i D(i)$,

$$\begin{aligned}
\mathbb{D}_M(\operatorname*{colim}_i D(i)) &= \hom_{\mathscr{B}}(M(\star), \operatorname*{colim}_i D(i)) \otimes M(\star) \\
&\cong (\operatorname*{colim}_i \hom_{\mathscr{B}}(M(\star), D(i))) \otimes M(\star) \\
&\cong \operatorname*{colim}_i (\hom_{\mathscr{B}}(M(\star), D(i)) \otimes M(\star)) \\
&\cong \operatorname*{colim}_i \mathbb{D}_M(D(i)). \qquad \square
\end{aligned}$$

Next, we prove a technical lemma, which is (in some sense) a strengthening of Lemma 6.3.4, and which is needed in the proof of Proposition 6.7.3 below.

Lemma 6.7.2 *Assume that all objects in the image of M are connected. Let C be a component of X and let $\xi\colon X \to \mathbb{D}_M(X)$ be a $\mathbb{D}_M$-coalgebra. Then, C can be equipped with a $\mathbb{D}_M$-coalgebra $\gamma\colon C \to \mathbb{D}_M(C)$ such that the inclusion $\iota_C\colon C \to X$ is a coalgebra homomorphism $(C, \gamma) \to (X, \xi)$.*

Proof. As in the proof of Lemma 6.3.4, we arrive at the diagram (6.6). This time $M(A)$ is connected. Therefore, since f is a monomorphism (because so is ι_f), by Lemma 6.3.3, z is an isomorphism. Next we show that f is a coalgebra morphism from $\eta_A\colon M(A) \to \mathbb{D}_M(M(A))$ to $\xi\colon X \to \mathbb{D}_M(X)$. To this end, recall that $\eta_A = \iota_g$ for $g = \mathrm{id}\colon M(A) \to M(A)$. Therefore, we obtain the desired $\mathbb{D}_M(f) \cdot \eta_A = \iota_{f \cdot \mathrm{id}} = \iota_f = \xi \cdot f$ by (DC1). Consequently, ι_C is also a coalgebra morphism, for C equipped with the coalgebra structure of η_A transported along the isomorphism z. $\square$

Lastly, we show that also conditions (2) and (3) of finite rank comonads are satisfied for discrete density comonads.

Proposition 6.7.3 *Assume that all objects in the image of M are connected and finitely presentable, and that $\mathscr{B}$ is a componental category with finite coproducts. Then, every morphism $f\colon A \to U(\xi)$ in $\mathscr{B}$, for finitely presentable A in $\mathscr{B}$, admits a unique factorisation (up to isomorphism)*

$$f \;=\; A \xrightarrow{g} U(\xi_0) \xrightarrow{U(\gamma)} U(\xi)$$

where $U(\xi_0)$ is finitely presentable.

Proof. By Theorem 6.3.5, we may assume that the underlying object X of $\xi\colon X \to \mathbb{D}_M(X)$ is a coproduct $\coprod_{i\in I} C_i$ of a collection of connected, finitely presentable objects C_i essentially in $\mathscr{A}$.

Recall that $\coprod_{i\in I} C_i$ is isomorphic to the directed colimit of the following directed diagram

$$\{\coprod_{i\in F} C_i \mid F \text{ is a finite subset of } I\}$$

with the obvious morphisms between these finite coproducts. Since A is finitely presentable, $f : A \to X$ decomposes as

$$f = A \xrightarrow{g} \coprod_{i \in F} C_i \xrightarrow{\iota_F} X$$

for some finite $F \subseteq I$. By Lemma 6.7.2, for each $i \in F$, the inclusion morphism $\iota_i : C_i \to X$ is a coalgebra morphism $(C_i, \xi_i) \to (X, \xi)$, for some comonad coalgebra $\xi_i : C_i \to \mathbb{D}_M(C_i)$.

Lastly, because the forgetful functor $U^{\mathbb{C}} : \mathsf{EM}(\mathbb{D}_M) \to \mathscr{B}$ creates colimits [17, Proposition 20.12], $\coprod_{i \in F} C_i$ can be equipped with the coalgebra structure of the coproduct of the coalgebras $\xi_i : C_i \to \mathbb{D}_M(C_j)$. Moreover, the morphism $\iota_F : \coprod_{i \in F} C_i \to X$ is a coalgebra morphism because each of its components is. Also, $\coprod_{i \in F} C_j$ is finitely presentable because it is a finite coproduct of finitely presentable objects (see e.g. Proposition 1.3 in [20]). Finally, g is unique (and, therefore, essentially unique) because $\mathscr{B}$ is a componental category and ι_F is the inclusion morphism of $\coprod_F C_i$ into the coproduct $\coprod_I C_i$ and hence a monomorphism. □

As a corollary of Proposition 6.7.1 and Proposition 6.7.3 we obtain the main theorem of this section.

Theorem 6.7.4 *Let $M : \mathscr{A} \to \mathscr{B}$ be a functor from a discrete category $\mathscr{A}$ to a componental category $\mathscr{B}$ with finite coproducts and assume that the pointwise density comonad $\mathbb{D}_M$ for M exists. If all objects in the image of M are connected and finitely presentable then $\mathbb{D}_M$ has finite rank.*

Observe that the constructed functor $M : \mathscr{A} \to \mathscr{B}$ in the proof of Theorem 6.4.1 (cf. the paragraph following the theorem) automatically satisfies the assumptions of Theorem 6.7.4. Indeed, M is an inclusion of a class of finite connected structures and finite structures are precisely the finitely presentable objects in the category of relational structures or the category of graphs. Therefore, Theorem 6.7.4 together with Corollary 5.3.11 yields the following.

Theorem 6.7.5 *For a component-based class Δ of finite (pointed) structures or graphs and its corresponding discrete density comonad $\mathbb{D}_\Delta$,*

$$A \equiv_{\#\mathbb{D}_\Delta} B \iff \forall C \in \Delta \quad |\hom(C, A)| = |\hom(C, B)|.$$

Example 6.7.6 Using this theorem we recover the remaining examples from Table 6.1 which are not covered by game comonads, such as cospectrality (from Δ consisting of cycles), quantum isomorphism (from Δ consisting of finite planar graphs) and isomorphism of bipartite double covers (from Δ consisting of finite bipartite graphs).

6.8 Game comonads from proof systems

Next we turn to a different application of density comonads. By the example of $\mathbb{E}_k$ we show that some game comonads can be recovered as density comonads of an interpretation of a category of primitive formulas and proofs between them (as morphisms) in the category of structures.

That is, we show that game comonads arise from a certain proof theory by a syntax-to-semantic interpretation. Unlike before, the density comonads arising this way are given by a functor whose domain category is not discrete.

More general facts about density constructions. It is well-known that the density comonad construction gives the same result when we restrict the domain of the generating functor to its dense subcategory. More precisely, we further have to require that the generating functor preserves (suitable) colimits, as is spelled out formally below.[9] To this end, recall that a functor $F\colon \mathscr{C} \to \mathscr{D}$ is **dense** iff the pointwise density comonad $\mathbb{D}_F$ exists and $\mathbb{D}_F \cong \mathrm{Id}$.

9: This fact is probably folklore, however, the author is grateful to Gabriel Goren who reminded him of this more general version.

Lemma 6.8.1 *Let $M\colon \mathscr{A} \to \mathscr{B}$ be a functor which preserves colimits and let $I\colon \mathscr{A}' \to \mathscr{A}$ be a dense functor then, if the pointwise density comonad $\mathbb{D}_M$ exists then so does $\mathbb{D}_{M\circ I}$ and, moreover, $\mathbb{D}_M \cong \mathbb{D}_{M\circ I}$.*

Proof. Since I is dense, we have that, for every $A \in \mathscr{A}$, we have the isomorphism

$$A \cong \operatorname*{colim}_{A'\in\mathscr{A}',\, I(A')\to A} I(A').$$

Therefore, we obtain that

$$\begin{aligned}
\mathbb{D}_M(B) &= \operatorname*{colim}_{A\in\mathscr{A},\, M(A)\to B} M(A) \\
&\cong \operatorname*{colim}_{A\in\mathscr{A},\, M(A)\to B} M\Big(\operatorname*{colim}_{A'\in\mathscr{A}',\, A'\to A} I(A')\Big) \\
&\cong \operatorname*{colim}_{A\in\mathscr{A},\, M(A)\to B}\ \operatorname*{colim}_{A'\in\mathscr{A}',\, I(A')\to A} M(I(A')) \\
&\cong \operatorname*{colim}_{A'\in\mathscr{A}',\, M(A')\to B} M(I(A')) \\
&\cong \mathbb{D}_{M\circ I}(B).
\end{aligned}$$

□

Typical applications of this fact are comonadic functors M, i.e. when M is the forgetful functor $U^{\mathbb{C}}$ for some comonad $\mathbb{C}$ on $\mathscr{B}$. In such case, we can use the following important fact about density constructions.

Lemma 6.8.2 ([79, Proposition 6.1]) *Let $L \dashv R$ be an adjunction, then $\mathbb{D}_L$ exists and is isomorphic to the comonad LR.*

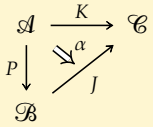
Recall that in case $M = U^{\mathbb{C}}$, it is the left adjoint[10] to the cofree functor $F^{\mathbb{C}}$ and, moreover, we have that $U^{\mathbb{C}}F^{\mathbb{C}} = \mathbb{C}$. Also, since $U^{\mathbb{C}}$ is a left adjoint, it preserves colimits. Therefore, by Lemma 6.8.1 and Lemma 6.8.2, we have the following.

10: Cf. Subsection"Eilenberg–Moore category" on page 22.

Proposition 6.8.3 *Given a comonad $\mathbb{C}$ on a category $\mathscr{B}$ and a dense functor $I\colon \mathscr{A} \to \mathsf{EM}(\mathbb{C})$ then $\mathbb{D}_{U^{\mathbb{C}}\circ I} \cong \mathbb{C}$.*

Finally, we recall a useful observation that a category which is appropriately 'sandwiched' between a dense and another category is dense in the latter. This is useful when establishing that the assumptions of Proposition 6.8.3 hold.

Lemma 6.8.4 ([73, Proposition 5.7]) *Let α be a natural transformation as shown in the diagram below*



which is initial among the natural transformations of the type $K \Rightarrow JP$,[11] then J is dense if and only if K is dense.

11: Or, in other words, α exhibits J as the left Kan extension of K along P.

A proof-theoretic description of $\mathbb{E}_k$. In the following we show, using Proposition 6.8.3, that $\mathbb{E}_k \cong \mathbb{D}_M$ for a suitable contravariant functor

$$M \colon \mathbf{PP}_k^{op} \to \mathcal{S}tr(\sigma).$$

which interprets **primitive positive sentences** as structures. In this section, we define primitive positive sentences as those first-order sentences in signature σ built only using $\exists$, true, $\wedge$, and relational symbols $R \in \sigma$ as atoms, i.e. without equality. Note that this is a discrepancy as equality is usually allowed in the finite model theory literature [80].

The category $\mathbf{PP}_k$ consists of primitive positive sentences of quantifier rank $\leq k$ as objects. Since we want to reconstruct $\mathbb{E}_k$ which, from the point of view of logic, does not care about variable names, we further assume that formulas in $\mathbf{PP}_k$ quantify variables in the sequence $x_1, x_2, \ldots$, based on their nesting depth.[12] For example, the primitive positive sentence $(\exists x_1\, E(x_1, x_1)) \wedge (\exists x_2\, S(x_2))$ is not in $\mathbf{PP}_k$ while, on the other hand, the following sentence is an object of $\mathbf{PP}_k$.

12: This way we avoid having to introduce additional invertible proof rules, for variable renaming, later on.

$$\varphi = \exists x_1\, \exists x_2\, (E(x_1, x_2) \wedge (\exists x_3\, E(x_1, x_3)) \wedge (\exists x_3\, T(x_3, x_2, x_1))) \quad (6.10)$$

The morphisms in $\mathbf{PP}_k$ are the proofs generated from the following **basic proof rules**, applicable to any subformula of a sentence,

$$\frac{\varphi \wedge \mathsf{true}}{\varphi} \qquad \frac{\mathsf{true} \wedge \varphi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi \wedge \varphi} \qquad \frac{\varphi \wedge (\psi \wedge \rho)}{(\varphi \wedge \psi) \wedge \rho}$$
$$\frac{R(y_1, \ldots, y_n)}{\mathsf{true}} \qquad \frac{\exists x\ \mathsf{true}}{\mathsf{true}} \quad (6.11)$$

where the first four rules are *invertible*, i.e. they are isomorphisms in $\mathbf{PP}_k$. Furthermore, we also add the following invertible rule.[13]

13: This allows us to ignore the exact position of atoms. For example, we can assume that they appear right after their last relevant quantifier.

$$\frac{\exists x\, (R(y_1, \ldots, y_n) \wedge \varphi)}{R(y_1, \ldots, y_n) \wedge \exists x\, \varphi} \quad (\text{if } x \notin \{y_1, \ldots, y_n\}) \quad (6.12)$$

Defining M on objects. We define the action of the functor $M \colon \mathbf{PP}_k^{op} \to \mathcal{S}tr(\sigma)$ on objects based on the Chandra–Merlin correspondence [35] between finite structures and primitive positive sentences. Given a formula φ, define $M(\varphi)$ as the structure on the set V where V is the enumeration of all existential quantifications of variables appearing in φ.

For example, $M(\varphi)$ for the formula φ from 6.10 has the universe consisting of $\{x_1, x_2, x_3^1, x_3^2\}$ where x_3^1 and x_3^2 correspond to the two different occurrences of x_3 in φ. Similarly, $M(\mathsf{true}) = \emptyset$.

Next, the relational structure of $M(\varphi)$ is defined as follows. We add the tuple $(v_1, \ldots, v_n)$ into the relation $R^{M(\varphi)}$ whenever $R(v_1, \ldots, v_n)$ appears as an atom in φ, for the copies of variables $v_1, \ldots, v_n$ in scope of the atom.

For example, the interpretation of relations on $M(\varphi)$ for φ from 6.10 is as follows: $E^{M(\varphi)} = \{(x_1, x_2), (x_1, x_3^1)\}$ and $T^{M(\varphi)} = \{(x_3^2, x_2, x_1)\}$.

Defining M on morphisms. In order to explain how M acts on morphisms, we define its action on basic proof rules. If $\pi\colon \varphi \to \varphi'$ is given by one of the invertible rules, we set $M(\pi)$ to be the identity function, which is valid since $M(\varphi)$ and $M(\varphi')$ are identical structures.

The proof $\pi\colon \varphi \to \varphi'$ introduced by the second to last rule in (6.11) introduces the inclusion homomorphism $M(\varphi') \to M(\varphi)$, which corresponds to the inclusion of the structure $M(\varphi')$ as a substructure of $M(\varphi)$ which differs only by the relation dropped by the rule.

Similarly, the proof $\pi\colon \varphi \to \varphi'$ introduced by the last rule in (6.11) corresponds to the inclusion homomorphism $M(\varphi') \to M(\varphi)$ where $M(\varphi)$ has an additional vertex in its universe, when compared to $M(\varphi')$.

Example 6.8.5 Let $\pi\colon \varphi \to \varphi'$ be the proof given by the following derivation.

$$\dfrac{\dfrac{\exists x_1\, (\exists x_2\, E(x_1, x_2)) \wedge (\exists x_2\, E(x_2, x_1))}{\exists x_1\, (\exists x_2\, E(x_1, x_2)) \wedge (\exists x_2\, \mathsf{true})}}{\exists x_1\, (\exists x_2\, E(x_1, x_2)) \wedge \mathsf{true}}$$

Then, the morphism $M(\pi)\colon M(\varphi') \to M(\varphi)$ is the composition of the following two inclusions.

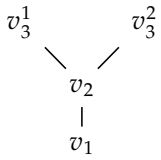
$$(\{x_1, x_2^1\}, E = \{(x_1, x_2^1)\}) \hookrightarrow (\{x_1, x_2^1, x_2^2\}, E = \{(x_1, x_2^1)\}) \hookrightarrow (\{x_1, x_2^1, x_2^2\}, E = \{(x_1, x_2^1$$

Factorisation via $\mathsf{EM}(\mathbb{E}_k)$. Crucially, we observe that M factors via $U^{\mathbb{E}_k}\colon \mathsf{EM}(\mathbb{E}_k) \to \mathcal{S}tr(\sigma)$, by the functor

$$I\colon \mathbf{PP}_k \to \mathsf{EM}(\mathbb{E}_k)$$

which maps φ to $M(\varphi)$ equipped with the forest order: $v_1 \leq v_2$ iff the quantification of v_2 appears in the scope of the quantification of v_1.

For example, $M(\varphi)$ for the formula φ from 6.10 has the following tree structure.



Observe that our basic proof rules maintain variable nesting and, therefore, the induced morphisms are forest morphisms. Furthermore, from

the Chandra–Merlin correspondence [35] it follows that every finite $\mathbb{E}_k$-coalgebra can be obtained this way.

Lemma 6.8.6 *The image* $\mathrm{im}(I)$ *of* $I\colon \mathbf{PP}_k \to \mathsf{EM}(\mathbb{E}_k)$ *is equivalent to the subcategory of* $\mathsf{EM}(\mathbb{E}_k)$ *consisting of finite coalgebras and injective morphisms.*

Finally, by Lemma 6.8.4 we observe that the finite coalgebras with injective morphisms between them are dense in $\mathsf{EM}(\mathbb{E}_k)$. Consider the following inclusion of categories

$$\mathcal{P} \hookrightarrow \mathrm{im}(I) \hookrightarrow \mathsf{EM}(\mathbb{E}_k)$$

where $\mathcal{P}$ denotes the (implicitly given) paths of $\mathsf{EM}(\mathbb{E}_k)$. Since paths are dense in arboreal categories (Lemma 4.3.5) and since, clearly, the inclusion $\mathrm{im}(I) \hookrightarrow \mathsf{EM}(\mathbb{E}_k)$ is the initial extension of the inclusion $\mathcal{P} \hookrightarrow \mathsf{EM}(\mathbb{E}_k)$ along $\mathcal{P} \hookrightarrow \mathrm{im}(I)$, we see that $\mathrm{im}(I) \hookrightarrow \mathsf{EM}(\mathbb{E}_k)$ is dense. Consequently, using Proposition 6.8.3 and the fact that $M = U^{\mathbb{E}_k} \circ I$ gives us the following theorem.

Theorem 6.8.7 $\mathbb{E}_k$ *is isomorphic to the density comonad of the functor* $M\colon \mathbf{PP}_k \to \mathcal{S}tr(\sigma)$ *defined above.*

Note that this procedure can be easily modified to obtain the other game comonads, such as $\mathbb{P}_k$ and $\mathbb{M}_k$, in a similar way.

6.9 Dense graph parameters and more

Observe that Theorem 6.4.1 enables us to extend comonadic classification to most classes of interest (cf. Section 6.4 and Section 6.6). However, the disadvantage of the discrete density comonad approach is that the comonad has a trivial structure. For a structure A, having a $\mathbb{D}_\Delta$-coalgebra, for a fixed component-based class Δ, does not tell us how to describe the decomposition structure of A, justifying why it belongs to the class Δ. For example, for Δ consisting of finite structures of tree-depth $\leq k$, having a $\mathbb{D}_\Delta$-coalgebra does not describe the compatible forest order on A, unlike with $\mathbb{E}_k$-coalgebras which encode the forest order, cf. Section 2.5.

Consequently, the quest of finding comonads to classify classes of interest is not over. However, instead of just requiring to find *any* comonad classifying a given class, we should require to describe a comonad which *encodes structural decompositions*.

In an ongoing joint work with Dušan Knop, the author describes such a comonad for *clique-width* and *modular-width*, important dense graph parameters. The comonad is obtained as a density comonad $\mathbb{D}_M$ for a suitable $M\colon \mathcal{A} \to \mathcal{S}tr(\sigma)$ where $\mathcal{A}$ is a non-discrete category encoding decompositions and morphisms between them. It is still an open problem if there is a comonad encoding structural decomposition of twin-width or flip-width.

Composition methods and types

7 Lifting theorems

7.1 Eilenberg–Moore laws . . 137
7.2 Kleisli laws 139
7.3 Extended Kleisli liftings 143
Historical remarks 145
7.4 Beyond unary operations 146
7.5 Heteromorphisms 147
7.6 Lifting adjunctions 152
On compositions of conjugations 154
Right adjoints versus extended Kleisli liftings . 156
Comonad morphisms . . . 156

Essential for the study of game comonads is the question of when a functor $\mathcal{A} \to \mathcal{B}$ between categories with game comonads $\mathbb{C}$ and $\mathbb{D}$, respectively, *lifts* to either a functor between the Kleisli categories $\mathsf{Kl}(\mathbb{C}) \to \mathsf{Kl}(\mathbb{D})$ or the Eilenberg–Moore categories $\mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$. This is because these categories are used for expressing logical equivalence in the logic fragments captured by our comonads or, similarly, for expressing given combinatorial properties, cf. Chapter 2.

In this chapter we establish the necessary categorical foundations for the existence of such liftings. Recall, from Section 2.6, that expressing logical equivalence with respect to standard fragments is done using bisimulations in the categories of coalgebras. In Chapter 8 we use the categorical foundations established here to further study the question of preservation of bisimulations, and hence also preservation of equivalence with respect to the standard fragments.

7.1 Eilenberg–Moore laws

The following folklore theorem establishes that **Eilenberg–Moore liftings**, i.e. functors between EM categories which commute with the forgetful functors, correspond to the so-called **Eilenberg–Moore laws** which are a generalisation of comonad morphisms, cf. Section 6.5.

Theorem 7.1.1 *Given a functor $H \colon \mathcal{A} \to \mathcal{B}$ and comonads $\mathbb{C}, \mathbb{D}$ on categories $\mathcal{A}, \mathcal{B}$, respectively, there is a bijective correspondence between:*

1. ***EM lifting*** *$H^{\mathsf{EM}} \colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$, which are the functors making the following diagram commute*

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{H^{\mathsf{EM}}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle U^{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle U^{\mathbb{D}}} \\ \mathcal{A} & \xrightarrow{H} & \mathcal{B} \end{array}$$

2. ***EM laws*** *$\lambda \colon H\mathbb{C} \Rightarrow \mathbb{D}H$, which are natural transformations such that*

$$\begin{array}{ccc} H\mathbb{C} & \xrightarrow{\lambda} & \mathbb{D}H \\ & {\scriptstyle H\varepsilon}\searrow \quad \swarrow{\scriptstyle \varepsilon_H} & \\ & H & \end{array} \qquad \begin{array}{ccc} H\mathbb{C}A & \xrightarrow{\lambda} & \mathbb{D}HA \\ {\scriptstyle H\delta}\downarrow & & \\ H\mathbb{C}^2A & & \downarrow{\scriptstyle \delta} \\ {\scriptstyle \lambda}\downarrow & & \\ \mathbb{D}H\mathbb{C}A & \xrightarrow{\mathbb{D}\lambda} & \mathbb{D}^2HA \end{array}$$

Proof sketch. The functor H^{EM} is obtained from an EM law by sending a

coalgebra $\alpha\colon A \to \mathbb{C}A$ to the coalgebra

$$H^{\mathsf{EM}}(\alpha)\colon HA \xrightarrow{H\alpha} H\mathbb{C}A \xrightarrow{\lambda} \mathbb{D}HA$$

The EM laws validate that $H^{\mathsf{EM}}(\alpha)$ is an EM coalgebra for $\mathbb{D}$.

Conversely, λ is obtained in two steps. First, we define a natural transformation $\lambda^{\mathsf{EM}}\colon H^{\mathsf{EM}}F^{\mathbb{C}} \Rightarrow F^{\mathbb{D}}H$, componentwise as $(H\varepsilon)^{\#}$ since $H\varepsilon$ can be written as $U^{\mathbb{D}}H^{\mathsf{EM}}F^{\mathbb{C}} = HU^{\mathbb{C}}F^{\mathbb{C}} = H\mathbb{C} \Rightarrow H$. Then, we define λ to be the composition

$$H\mathbb{C} = HU^{\mathbb{C}}F^{\mathbb{C}} = U^{\mathbb{D}}H^{\mathsf{EM}}F^{\mathbb{C}} \xrightarrow{U^{\mathbb{D}}(\lambda^{\mathsf{EM}})} U^{\mathbb{D}}F^{\mathbb{D}}H = \mathbb{D}H$$

It is a routine verification that this natural transformation is an EM law. □

Observe, from the proof, that λ is a coalgebra morphism $H^{\mathsf{EM}}FA \to FHA$. The original statement is usually attributed to Appelgate's PhD thesis [23] and we sketched the proof from [69].

Next we state a consequence of this theorem, which allows us to compare combinatorial properties using comonads.

Corollary 7.1.2 *If a functor $H\colon \mathcal{A} \to \mathcal{B}$ admits an EM law $H\mathbb{C} \Rightarrow \mathbb{D}H$ then, for any $A \in \mathcal{A}$, if A admits a $\mathbb{C}$-coalgebra then HA admits a $\mathbb{D}$-coalgebra.*

Proof. Since H admits an EM law, by Theorem 7.1.1, there exists H^{EM} such that:

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{H^{\mathsf{EM}}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle U^{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle U^{\mathbb{D}}} \\ \mathcal{A} & \xrightarrow{H} & \mathcal{B} \end{array}$$

Next, for $A \in \mathcal{A}$, if it admits $\mathbb{C}$-coalgebra, then there is some $X \in \mathsf{EM}(\mathbb{C})$ such that $A \cong U^{\mathbb{C}}(X)$. Then, by chasing the diagram we obtain that $HA \cong U^{\mathbb{D}}(H^{\mathsf{EM}}(X))$, i.e. HA admits a $\mathbb{D}$-coalgebra. □

Example 7.1.3 Observe that the inclusion $\mathbb{E}_kA \hookrightarrow \mathbb{E}_{k+1}A$ defines a comonad morphism, i.e. natural transformation $\mathrm{Id}\mathbb{E}_k \Rightarrow \mathbb{E}_{k+1}\mathrm{Id}$ which satisfies the conditions for the EM law. Then, by Corollary 7.1.2 and Proposition 2.5.4, we obtain the trivial fact that if a structure has tree-depth $\leq k$ then it has tree-depth $\leq k+1$. Similarly, we have EM laws $\mathrm{Id}\mathbb{P}_k \Rightarrow \mathbb{P}_{k+1}\mathrm{Id}$ and $\mathrm{Id}\mathbb{M}_k \Rightarrow \mathbb{M}_{k+1}\mathrm{Id}$.

Example 7.1.4 Let $\mathbb{E}_k$ and $\mathbb{P}_k$ be the Ehrenfeucht–Fraïssé and pebble comonads on $\mathcal{S}tr(\sigma)$, for some fixed relational signature σ. Define a homomorphism $\lambda_A\colon \mathbb{E}_k(A) \to \mathbb{P}_k(A)$ as the mapping which sends the word $[a_1, \ldots, a_n]$ to the word $[(1, a_1), (2, a_2), \ldots, (n, a_n)]$. One can readily verify that this mapping is natural in A and, moreover, is a comonad morphism. In particular, when typed as $\lambda\colon \mathrm{Id}\mathbb{E}_k \Rightarrow \mathbb{P}_k\mathrm{Id}$, it satisfies the axioms of an EM law. Then, by Corollary 7.1.2 and Proposition 2.5.4, we obtain the well-known fact that if a structure has tree-depth $\leq k$ then it has tree-width $< k$.

In fact, the functor $\mathrm{Id}^{\mathsf{EM}}$ corresponding to this EM law has a simple description using the concrete representation of categories $\mathsf{EM}(\mathbb{E}_k)$ and $\mathsf{EM}(\mathbb{P}_k)$ from Section 2.5. Using the definition from the proof of Theorem 7.1.1 gives us that $\mathrm{Id}^{\mathsf{EM}}(A, \alpha) = (A, \lambda \cdot \alpha)$ which translates as the mapping which sends a forest-ordered structure $(A, \leq)$ to the forest-ordered structure with a pebble function $(A, \leq, p)$ where $p(x) = \mathrm{ht}(x) + 1$.

Example 7.1.5 For a modal signature σ, let $V : \mathcal{S}tr_*(\sigma) \to \mathcal{S}tr(\sigma)$ be the functor which sends a Kripke structure (A, a_0) to its σ-structure reduct A, i.e. forgetting the designated point. Let $\mathbb{M}_k$ and $\mathbb{P}_2$ be the modal and pebble comonads on $\mathcal{S}tr_*(\sigma)$ and $\mathcal{S}tr(\sigma)$, respectively. Then, we can define an EM law $\lambda : V\mathbb{M}_k \Rightarrow \mathbb{P}_k V$, by sending a sequence $a_0 \xrightarrow{R_1} a_1 \xrightarrow{R_2} a_2 \xrightarrow{R_3} \ldots \xrightarrow{R_n} a_n$ from $V(\mathbb{M}_k(A, a_0))$ to the sequence $[(0, a_0), (1, a_1), (0, a_2), \ldots, (n \mod 2, a_n)]$ in $\mathbb{P}_2(A)$. The fact that this is a valid EM law for every k gives us, by Corollary 7.1.2 and Proposition 2.5.4, that every synchronisation tree has tree-width 1, as expected.

Also, the functor V^{EM} obtained from λ can be described using the concrete representation of EM categories from Section 2.5 as the mapping sending a synchronisation tree (A, a_0) to the forest-ordered structure with a pebble function $(A, \leq, p)$ where $x \leq y$ if there exists a path from x to y in A and $p(x) = (\mathrm{ht}(x) \mod 2) + 1$.

7.2 Kleisli laws

The other important folklore theorem says that **Kleisli liftings**, i.e. functors lifted to the Kleisli categories which commute with the cofree functors, correspond to the so-called **Kleisli laws**. Kleisli laws are, again, a generalisation of comonad morphisms, cf. Section 6.5, however, defined dually to EM laws.

Theorem 7.2.1 ([95]) *Given a functor $H : \mathcal{A} \to \mathcal{B}$ and comonads $\mathbb{C}, \mathbb{D}$ on categories $\mathcal{A}, \mathcal{B}$, respectively, there is a bijective correspondence between:*

1. ***Kleisli lifting*** $H^{\mathsf{Kl}} : \mathsf{Kl}(\mathbb{C}) \to \mathsf{Kl}(\mathbb{D})$, *which are the functors making the following diagram commute*

$$\begin{array}{ccc} \mathsf{Kl}(\mathbb{C}) & \xrightarrow{H^{\mathsf{Kl}}} & \mathsf{Kl}(\mathbb{D}) \\ {\scriptstyle F_{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F_{\mathbb{D}}} \\ \mathcal{A} & \xrightarrow{H} & \mathcal{B} \end{array}$$

2. ***Kleisli laws*** $\kappa : \mathbb{D}H \Rightarrow H\mathbb{C}$, *which are natural transformations such that*

$$\begin{array}{ccc} \mathbb{D}H & \xrightarrow{\kappa} & H\mathbb{C} \\ & {\scriptstyle \varepsilon_H}\searrow \quad \swarrow{\scriptstyle H\varepsilon} & \\ & H & \end{array} \qquad \begin{array}{ccc} \mathbb{D}HA & \xrightarrow{\kappa} & H\mathbb{C}A \\ {\scriptstyle \delta_H}\downarrow & & \downarrow{\scriptstyle H\delta} \\ \mathbb{D}^2HA & & \\ {\scriptstyle \mathbb{D}\kappa}\downarrow & & \\ \mathbb{D}H\mathbb{C}A & \xrightarrow{\kappa_{\mathbb{C}}} & H\mathbb{C}^2A \end{array}$$

Proof sketch. Given a Kleisli law κ, the Kleisli lifting is obtained as the mapping which sends an object $A \in \mathsf{Kl}(\mathbb{C})$ to $HA \in \mathsf{Kl}(\mathbb{D})$ and a morphism $\mathbb{C}A \xrightarrow{f} B$ to $\mathbb{D}HA \xrightarrow{\kappa} H\mathbb{C}A \xrightarrow{Hf} HB$.

Conversely, from a Kleisli lifting, we need to construct a natural transformation $U_{\mathbb{D}}H^{\mathsf{Kl}}F_{\mathbb{C}} = U_{\mathbb{D}}F_{\mathbb{D}}H \Rightarrow HU_{\mathbb{C}}F_{\mathbb{C}}$ which, by conjugation,[1] amounts to finding $H^{\mathsf{Kl}}F_{\mathbb{C}} = U_{\mathbb{D}}F_{\mathbb{D}}H \Rightarrow F_{\mathbb{D}}HU_{\mathbb{C}}F_{\mathbb{C}} = H^{\mathsf{Kl}}F_{\mathbb{C}}U_{\mathbb{C}}F_{\mathbb{C}}$. Such a morphism can be constructed as $H^{\mathsf{Kl}}(\eta_{F_{\mathbb{C}}})$ where $\eta_{F_{\mathbb{C}}} : F_{\mathbb{C}} \Rightarrow F_{\mathbb{C}}U_{\mathbb{C}}F_{\mathbb{C}}$ is the unit of the monad $F_{\mathbb{C}}U_{\mathbb{C}}$ on $\mathsf{Kl}(\mathbb{C})$. □

1: Cf. Subsection "Adjunctions" on page 20.

For what follows it is useful to observe that Kleisli laws can be characterised in more elementary terms, using the coextension operation, as follows.

Lemma 7.2.2 *Let $\{\kappa_A : \mathbb{D}HA \to H\mathbb{C}A\}_{A \in \mathrm{Ob}(\mathcal{A})}$ be a collection of morphisms which satisfies*

$$H(\varepsilon_A) \cdot \kappa_A = \varepsilon_{H(A)} \tag{7.1}$$

for every object A of $\mathcal{A}$. Then it further satisfies

$$H(f^*) \cdot \kappa_A = \kappa_B \cdot (H(f) \cdot \kappa_A)^* \tag{7.2}$$

for every $f : \mathbb{C}(A) \to B$ iff κ is a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$.

Proof. First, we show that (7.2) implies that κ is natural, that is, we check[2] the following

$$H(\mathbb{C}(g)) \cdot \kappa_A = \kappa_B \cdot \mathbb{D}(H(g))$$

for any $g : A \to B$. For

$$f = \mathbb{C}(A) \xrightarrow{\varepsilon} A \xrightarrow{g} B,$$

we have that

$$H(\mathbb{C}(g)) = H(f^*)$$

and similarly

$$\begin{aligned}
\mathbb{D}(H(g)) &= \left(\mathbb{D}(H(A)) \xrightarrow{\varepsilon} H(A) \xrightarrow{H(g)} H(B)\right)^* \\
&= \left(\mathbb{D}(H(A) \xrightarrow{\kappa} H(\mathbb{C}(A)) \xrightarrow{H(\varepsilon_A)} H(A) \xrightarrow{H(g)} H(B)\right)^* \\
&= \left(\mathbb{D}(H(A) \xrightarrow{\kappa} H(\mathbb{C}(A)) \xrightarrow{H(f)} H(B)\right)^*.
\end{aligned}$$

Consequently, (7.2) directly entails naturality of κ. Similarly, to check the oblong law of Kleisli laws, we set

$$f = \mathbb{C}(A) \xrightarrow{\mathrm{id}} \mathbb{C}(A).$$

2: Recall the following basic facts about comonads from Section 2.2, which we use without mentioning:

$$\mathbb{C}(f) = (\mathbb{C}(A) \xrightarrow{\varepsilon_A} A \xrightarrow{f} B)^*$$
$$\delta_A = (\mathbb{C}(A) \xrightarrow{\mathrm{id}_{\mathbb{C}(A)}} \mathbb{C}(A))^*$$
$$f^* = \mathbb{C}(A) \xrightarrow{\delta_A} \mathbb{C}^2(A) \xrightarrow{\mathbb{C}(f)} \mathbb{C}(B)$$

Then, we have the following:

$$\begin{aligned}\big(H(f)\cdot\kappa_A\big)^* &= \Big(\mathbb{D}(H(A)) \xrightarrow{\kappa} H(\mathbb{C}(A))\Big)^* \\ &= \mathbb{D}(H(A)) \xrightarrow{\delta} \mathbb{D}^2(H(A)) \xrightarrow{\mathbb{D}(\kappa)} \mathbb{D}(H(\mathbb{C}(A)))\end{aligned}$$

Consequently, the oblong law follows from (7.2) since

$$\kappa\cdot\mathbb{D}(\kappa)\cdot\delta = \kappa\cdot\big(H(f)\cdot\kappa\big)^* = H\big(f^*\big)\cdot\kappa = H\big(\delta_A\big)\cdot\kappa.$$

Conversely, we show that from naturality of κ and the oblong law we obtain that (7.2) holds for any morphism $f\colon \mathbb{C}(A) \to B$, by a direct computation:

$$\begin{aligned}H\big(f^*\big)\cdot\kappa &= H\big(\mathbb{C}(f)\big)\cdot H(\delta)\cdot\kappa \\ &= H\big(\mathbb{C}(f)\big)\cdot\kappa\cdot\mathbb{D}(\kappa)\cdot\delta && (\star)\\ &= \kappa\cdot\mathbb{D}\big(H(f)\big)\cdot\mathbb{D}(\kappa)\cdot\delta && (\dagger)\\ &= \kappa\cdot\big(H(f)\cdot\kappa\big)^*\end{aligned}$$

where the oblong law was used in ($\star$) and naturality of κ in ($\dagger$). □

Remark 7.2.3 Using the representation of Kleisli laws in terms the equivalent characterisation, from Lemma 7.2.2, it is easier to see why H^{Kl} defined using the same formula as in Theorem 7.2.1 is functorial. Given $f\colon \mathbb{C}A \to B$ and $g\colon \mathbb{C}B \to C$, we see that:[3]

$$\begin{aligned}H^{\mathsf{Kl}}(g)\bullet H^{\mathsf{Kl}}(f) &= (H(g)\cdot\kappa_B)\bullet(H(f)\cdot\kappa_A)\\ &= H(g)\cdot\kappa_B\cdot(H(f)\cdot\kappa_A)^*\\ &\overset{(7.2)}{=} H(g)\cdot H(f^*)\cdot\kappa_A\\ &= H(g\cdot f^*)\cdot\kappa_A\\ &= H(g\bullet f)\cdot\kappa_A = H^{\mathsf{Kl}}(g\bullet f)\end{aligned}$$

3: Cf. Subsection"Kleisli category" on page 21 for the definition of composition in $\mathsf{Kl}(\mathbb{C})$.

In Section 7.1 we observed that EM laws can be used to compare the combinatorial properties characterised by comonads. Kleisli laws, on the other hand, can be used to compare strengths of logics characterised by comonads.[4] Below we look at two examples of this phenomenon, related to the examples we saw earlier, in Section 7.1. The description of the general method is postponed until Section 8.1 in Chapter 8.

4: Observe that the comparison of logical strengths goes in the opposite direction, when comparing to the comparison of combinatorial properties given by EM laws!

Example 7.2.4 Observe that the comonad morphism from Example 7.1.4 is also a Kleisli law, i.e. the same collection of functions λ is a natural transformation $\mathbb{E}_k\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{P}_k$ satisfying the Kleisli law properties.

Consequently, by Lemma 7.2.2, there is a Kleisli lifting $\mathrm{Id}^{\mathsf{Kl}}$ displayed below.

$$\begin{array}{ccc}\mathsf{Kl}(\mathbb{P}_k) & \xrightarrow{\mathrm{Id}^{\mathsf{Kl}}} & \mathsf{Kl}(\mathbb{E}_k)\\ {\scriptstyle F_{\mathbb{P}_k}}\uparrow & & \uparrow{\scriptstyle F_{\mathbb{E}_k}}\\ \mathcal{S}tr(\sigma) & \xrightarrow{\mathrm{Id}} & \mathcal{S}tr(\sigma)\end{array}$$

For $A \in \mathcal{S}tr(\sigma)$, by commutativity of the diagram above, $\mathrm{Id}^{\mathsf{Kl}}(F_{\mathbb{P}_k}(A)) = F_{\mathbb{E}_k}(\mathrm{Id}(A)) = F_{\mathbb{E}_k}(A)$. Therefore, if $A \equiv_{\mathsf{C}^k} B$, i.e. if $F_{\mathbb{P}_k}(A) \cong F_{\mathbb{P}_k}(B)$ then,

by functoriality of $\mathrm{Id}^{\mathsf{Kl}}$,

$$F_{\mathbb{E}_k}(A) = \mathrm{Id}^{\mathsf{Kl}}(F_{\mathbb{P}_k}(A)) \cong \mathrm{Id}^{\mathsf{Kl}}(F_{\mathbb{P}_k}(B)) = F_{\mathbb{E}_k}(B),$$

i.e. $A \equiv_{\mathsf{C}_k} B$.

Similarly, using the Kleisli laws $\mathbb{C}_k\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{C}_{k+1}$ derived from the EM laws $\mathrm{Id}\mathbb{C}_k \Rightarrow \mathbb{C}_{k+1}\mathrm{Id}$ for our game comonads, from Example 7.1.3, we obtain the trivial facts that $A \equiv_{\mathsf{C}_{k+1}} B$ implies $A \equiv_{\mathsf{C}_k} B$, $A \equiv_{\mathsf{C}^{k+1}} B$ implies $A \equiv_{\mathsf{C}^k} B$ and that $(A, a_0) \equiv_{\#\mathtt{ML}_{k+1}} (B, b_0)$ implies $(A, a_0) \equiv_{\#\mathtt{ML}_k} (B, b_0)$.

Example 7.2.5 Unlike in Example 7.2.4 we cannot use the EM law from Example 7.1.5, to define a Kleisli law since the translation functor V is not the identity functor or, in other words, the EM law from Example 7.2.4 is not a comonad morphism.

However, we can define a version of the pebble comonad $\mathbb{P}_k$ over Kripke structures $\mathcal{S}tr_*(\sigma)$ and use this comonad to build a Kleisli law from $\mathbb{M}_k$. To this end, we define $\mathbb{P}_k(A, a_0)$ for a Kripke structure (A, a_0) as follows. When we disregard the designated point a_0, we can apply the usual pebble comonad $\mathbb{P}_k$ defined over the non-pointed σ-structures $\mathcal{S}tr(\sigma)$. Then, the σ-structure part of $\mathbb{P}_k(A, a_0)$ is a restriction of $\mathbb{P}_k(A)$ to the sequences $[(0, a_0), (p_1, a_1), \ldots, (p_n, a_n)]$, i.e. the sequences where the first element is the pair $(0, a_0)$. The designated point of $\mathbb{P}_k(A, a_0)$ is the sequence $[(0, a_0)]$ and the relations are defined as in $\mathbb{P}_k(A)$. It is immediate to see that $\mathbb{P}_k$ is a comonad on $\mathcal{S}tr_*(\sigma)$ and that the characterisation of $\mathbb{P}_k$-coalgebras carries naturally to this setting too.

Then, as before, the Kleisli law (comonad morphism) $\kappa\colon \mathbb{M}_k\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{P}_2$ sends a sequence $a_0 \xrightarrow{R_1} a_1 \xrightarrow{R_2} a_2 \xrightarrow{R_3} \ldots \xrightarrow{R_n} a_n$ from $\mathbb{M}_k(A, a_0)$ to the sequence $[(0, a_0), (1, a_1), (0, a_2), \ldots, (n \mod 2, a_n)]$ in $\mathbb{P}_2(A, a_0)$. Similarly to Example 7.2.4, this witnesses the fact that $(A, a_0) \equiv_{\mathsf{C}^2} (B, b_0)$ implies that $(A, a_0) \equiv_{\#\mathtt{ML}_k} (B, b_0)$.

Example 7.2.6 Fix a structure $B \in \mathcal{S}tr(\sigma)$ and define a functor $H(A) = A \times B$ of the type $\mathcal{S}tr(\sigma) \to \mathcal{S}tr(\sigma)$. Observe that we have a Kleisli law $\kappa\colon \mathbb{E}_kH \Rightarrow H\mathbb{E}_k$ defined, componentwise $\kappa_A\colon \mathbb{E}_k(A \times B) \to (\mathbb{E}_kA) \times B$ as the map $[(a_1, b_1), \ldots, (a_n, b_n)] \mapsto ([a_1, \ldots, a_n], b_n)$.

To quickly verify that this is a Kleisli law, we use Lemma 7.2.2, so that we do not have to check naturality by hand. The fact that (7.1) holds is immediate. To check (7.2), fix a $w = [(a_1, b_1), \ldots, (a_n, b_n)]$ and $f\colon \mathbb{E}_kA \to B$. Recall that $f^*([a_1, \ldots, a_n]) = [c_1, \ldots, c_n]$ where $c_i = f([a_1, \ldots, a_i])$. Then, the left-hand side of (7.1) is

$$H(f^*)(\kappa(w)) = H(f^*)([a_1, \ldots, a_n], b_n) = (f^*([a_1, \ldots, a_n]), b_n).$$

Conversely, for the right-hand side of (7.1), observe that $(H(f) \cdot \kappa_A)^*(w) = [d_1, \ldots, d_n]$ where

$$\begin{aligned} d_i &= H(f)(\kappa_A([(a_1, b_1), \ldots, (a_i, b_i)])) \\ &= H(f)([a_1, \ldots, a_i], b_i) \\ &= (f([a_1, \ldots, a_i]), b_i). \end{aligned}$$

Therefore, $\kappa_B((H(f) \cdot \kappa_A)^*(w)) = (f^*([a_1, \ldots, a_n]), b_n)$, i.e. the equality (7.1) holds. Then, by reasoning similar to Example 7.2.4, we obtain

that $A \equiv_{\mathsf{C}_k} A'$ implies $A \times B \equiv_{\mathsf{C}_k} A' \times B$.

7.3 Extended Kleisli liftings

In the examples above we described a general method of obtaining preservation of the $\equiv_{\#\mathbb{C}}$ relation, expressing equivalence with respect to the counting fragments. In order to obtain similar results for the $\equiv_{\mathbb{C}}$ relation, expressing equivalence with respect to the standard fragments, we need a lifting theorem similar to Theorem 7.2.1, however, lifting all the way to the EM category. Furthermore, such lifted functor needs to preserve the $\rightleftharpoons$ relation. In this section we focus solely on the existence of the lifting to the categories of coalgebras and we leave the analysis of when these functors preserve the $\rightleftharpoons$ relation to Chapter 8.

In order to extend Corollary 8.1.1 to logical equivalences of the form $\equiv_{\mathbb{C}}$, assume that we have a Kleisli law $\kappa \colon \mathbb{D}H \Rightarrow H\mathbb{C}$ and we wish to "extend" the Kleisli lifting $H^{\mathsf{Kl}} \colon \mathsf{Kl}(\mathbb{C}) \to \mathsf{Kl}(\mathbb{D})$ further on, to the categories of coalgebras. Recall from Subsection"Comparison functors" on page 23 that $\mathsf{Kl}(\mathbb{C})$ can be identified with the subcategory of $\mathsf{EM}(\mathbb{C})$ consisting of cofree coalgebras, with the functor $K_{\mathbb{C}}$. Therefore, by the **extended Kleisli lifting** we mean a functor $\widehat{H} \colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$ extending H^{Kl}, that is, a functor $\widehat{H}$ such that the following diagram of functors commutes (on the nose).

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{\widehat{H}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle K_{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle K_{\mathbb{D}}} \\ \mathsf{Kl}(\mathbb{C}) & \xrightarrow{H^{\mathsf{Kl}}} & \mathsf{Kl}(\mathbb{D}) \\ {\scriptstyle F_{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F_{\mathbb{D}}} \\ \mathscr{A} & \xrightarrow{H} & \mathscr{B} \end{array} \qquad F^{\mathbb{C}} \qquad F^{\mathbb{D}} \tag{7.3}$$

To this end, we adapt a standard category-theoretic technique of lifting functors to the Eilenberg–Moore categories of coalgebras. In fact, our approach generalises that of lifting the monoidal operation to the categories of Eilenberg–Moore categories of coalgebras for monoidal monads [62, 114], and lifting of adjoints to the categories of algebras [69].

This technique needs an extra requirement that the category $\mathsf{EM}(\mathbb{D})$ has **equalisers of coreflexive pairs**.[5] These are equalisers

$$A \xrightarrow{e} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

in some category $\mathfrak{X}$ where f, g is a **coreflexive pair**, that is, there is a morphism $r \colon C \to B$ such that $rf = rg = \mathrm{id}$.

In order to define $\widehat{H}$ from (7.3), let (A, α) be an object of $\mathsf{EM}(\mathbb{C})$. We form the following parallel pair of morphisms in $\mathsf{EM}(\mathbb{D})$.

$$\widehat{H}(A,\alpha) \overset{\iota_\alpha}{\dashrightarrow} F^{\mathbb{D}}(HA) \underset{F^{\mathbb{D}}(H\alpha)}{\overset{F^{\mathbb{D}}(\kappa)\cdot\eta^{\mathbb{D}}}{\rightrightarrows}} F^{\mathbb{D}}(H\mathbb{C}A) \tag{7.4}$$

5: Recall from Chapter 2 that the category $\mathsf{EM}(\mathbb{C})$ for a game comonad $\mathbb{C}$ is usually complete, due to Linton's theorem (Proposition 2.9.9). In particular, $\mathsf{EM}(\mathbb{C})$ typically has all equalisers.

In fact, by examining the proof of Linton's theorem, one can observe that all limits are computed as equalisers of certain coreflexive pairs in $\mathsf{EM}(\mathbb{C})$. Hence, $\mathsf{EM}(\mathbb{C})$ is complete if and only if it has equalisers of coreflexive pairs.

where $\eta^{\mathbb{D}}$ is the unit of the monad $F^{\mathbb{D}}U^{\mathbb{D}}$ on $\mathsf{EM}(\mathbb{D})$ whose underlying morphism in $\mathscr{B}$ is simply the comultiplication $\delta_{HA}\colon \mathbb{D}(HA) \to \mathbb{D}^2(HA)$ of the comonad.[6]

6: In general, for a coalgebra (B,β) in $\mathsf{EM}(\mathbb{D})$, the underlying morphisms in $\mathscr{B}$ of the coalgebra morphism $\eta^{\mathbb{D}}_{(B,\beta)}\colon (B,\beta) \to F^{\mathbb{D}}B$ is just $\beta\colon B \to \mathbb{D}B$.

Observation 7.3.1 *If κ is a Kleisli law, then the two solid arrows in (7.4) form a coreflexive pair.*

Proof. Set the common retract r to be $F^{\mathbb{D}}(H\varepsilon^{\mathbb{C}}_A)\colon F^{\mathbb{D}}(H\mathbb{C}A) \to F^{\mathbb{D}}(HA)$. Then, r composed with the bottom arrow in (7.4) is the identity, since (A,α) is an EM coalgebra. The fact that r is also a retract follows by

$$F^{\mathbb{D}}(H\varepsilon^{\mathbb{C}}_A)\cdot F^{\mathbb{D}}(\kappa)\cdot\eta^{\mathbb{D}} = F^{\mathbb{D}}(\varepsilon^{\mathbb{C}}_{HA})\cdot\eta^{\mathbb{D}} = \mathrm{id}$$

where the first equality is the triangle diagram of Kleisli laws and the second equality is a consequence of the definition of η, i.e. $\eta_\beta = \beta$, as then $F^{\mathbb{D}}(\varepsilon_H)\cdot\eta_\delta = \mathbb{D}(\varepsilon_H)\cdot\delta = \mathrm{id}$. □

If equaliser like in (7.4) exists for coalgebras $(A,\alpha),(B,\beta)$ in $\mathsf{EM}(\mathbb{C})$, then for a morphism $f\colon (A,\alpha) \to (B,\beta)$ we define $\widehat{H}(f)$ by the universal property of equalisers. Observe that in the following diagram the two squares on the right-hand side commute by naturality of κ and η and the fact that f is a coalgebra morphism.

$$\begin{array}{ccccc}
\widehat{H}(A,\alpha) & \xrightarrow{\iota_\alpha} & F^{\mathbb{D}}(HA) & \underset{F^{\mathbb{D}}(H\alpha)}{\overset{F^{\mathbb{D}}(\kappa)\cdot\eta^{\mathbb{D}}}{\rightrightarrows}} & F^{\mathbb{D}}(H\mathbb{C}A) \\
\downarrow{\scriptstyle \widehat{H}(f)} & & \downarrow{\scriptstyle F^{\mathbb{D}}(Hf)} & & \downarrow{\scriptstyle H\mathbb{C}f} \\
\widehat{H}(B,\beta) & \xrightarrow{\iota_\beta} & F^{\mathbb{D}}(HB) & \underset{F^{\mathbb{D}}(H\beta)}{\overset{F^{\mathbb{D}}(\kappa)\cdot\eta^{\mathbb{D}}}{\rightrightarrows}} & F^{\mathbb{D}}(H\mathbb{C}B)
\end{array} \tag{7.5}$$

Therefore, $F^{\mathbb{D}}(Hf)\cdot\iota_\alpha$ equalises the bottom pair of morphisms and so we have a unique morphism $\widehat{H}(f)$ making the left-hand side diagram commute. We summarise these observations in the following lemma.

Lemma 7.3.2 *If $\mathsf{EM}(\mathbb{D})$ has equalisers of coreflexive pairs, then the above construction defines a functor $\widehat{H}\colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$.*

Next we look at the action of $\widehat{H}$ on cofree coalgebras. It turns out that in this case the equaliser (7.4) has a very explicit description.

Lemma 7.3.3 *For any object A of $\mathscr{A}$, the following diagram is an equaliser in the category $\mathsf{EM}(\mathbb{D})$*

$$F^{\mathbb{D}}(HA) \xrightarrow{\eta^{\mathbb{D}}} F^{\mathbb{D}}(\mathbb{D}HA) \xrightarrow{F^{\mathbb{D}}(\kappa)} F^{\mathbb{D}}(H\mathbb{C}A) \underset{F^{\mathbb{D}}(H\delta^{\mathbb{C}})}{\overset{F^{\mathbb{D}}(\kappa)\cdot\eta^{\mathbb{D}}}{\rightrightarrows}} F^{\mathbb{D}}(H\mathbb{C}^2A)$$

where, for any other coalgebra morphism $g\colon (B,\beta) \to F^{\mathbb{D}}(H\mathbb{C}B)$ commuting with the parallel morphisms on the right, the unique map $\overline{g}\colon (B,\beta) \to F^{\mathbb{D}}(HB)$ is obtained as $F^{\mathbb{D}}(H\varepsilon)\cdot g$.

Proof. First, we see that $F^{\mathbb{D}}(\kappa)\cdot\delta$ equalizes $F^{\mathbb{D}}(\kappa)\cdot\eta$ and $F^{\mathbb{D}}(H(\delta))$ as the following computation in $\mathcal{B}$ shows:

$$\begin{aligned}\mathbb{D}(\kappa)\cdot\delta\cdot\mathbb{D}(\kappa)\cdot\delta &= \mathbb{D}(\kappa)\cdot\mathbb{D}^2(\kappa)\cdot\delta\cdot\delta\\ &= \mathbb{D}(\kappa)\cdot\mathbb{D}^2(\kappa)\cdot\mathbb{D}(\delta)\cdot\delta\\ &= \mathbb{D}(H\delta)\cdot\mathbb{D}(\kappa)\cdot\delta\end{aligned}$$

Next, assuming a coalgebra morphism $g\colon (B,\beta)\to F^{\mathbb{D}}(H\mathbb{C}B)$ equalizes $F^{\mathbb{D}}\kappa\cdot\eta^{\mathbb{D}}$ and $F^{\mathbb{D}}(H\delta)$, then g is equal to the composition of $F^{\mathbb{D}}(H\varepsilon)\cdot g$ and $F^{\mathbb{D}}(\kappa)\cdot\delta$:

$$\begin{aligned}\mathbb{D}\kappa\cdot\delta\cdot\mathbb{D}(H\varepsilon)\cdot g &= \mathbb{D}\kappa\cdot\mathbb{D}^2(H\varepsilon)\cdot\delta\cdot g && \text{(naturality of } \delta)\\ &= \mathbb{D}(\kappa\cdot\mathbb{D}H\varepsilon)\cdot\delta\cdot g && \text{(functoriality of } \mathbb{D})\\ &= \mathbb{D}(H\mathbb{C}\varepsilon\cdot\kappa)\cdot\delta\cdot g && \text{(naturality of } \kappa)\\ &= \mathbb{D}(H\mathbb{C}\varepsilon)\cdot\mathbb{D}(H\delta)\cdot g = g && (g \text{ equalizes})\end{aligned}$$

However, for $F^{\mathbb{D}}(\kappa)\cdot\delta$ to be an equaliser, we still need to show unicity of the above factorisation. This follows from the fact that $F^{\mathbb{D}}(\kappa)\cdot\delta$ is a monomorphism. In fact, it is a split monomorphism as:

$$\mathbb{D}(H\varepsilon)\cdot\mathbb{D}\kappa\cdot\delta = \mathbb{D}\varepsilon\cdot\delta = \mathrm{id}$$ □

Observe that Lemma 7.3.3 shows that $F^{\mathbb{D}}(HA)$ is the equaliser of the diagram (7.4) for $\widehat{H}(F^{\mathbb{C}}A)$ or, in other words, $\widehat{H}$ commutes with the cofree coalgebra functors. As a corollary we obtain that the functor $\widehat{H}$ defines the desired extended Kleisli lifting.

Theorem 7.3.4 *If there is a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ and $\mathsf{EM}(\mathbb{D})$ has equalisers of coreflexive pairs, then there is an* extended Kleisli lifting *$\widehat{H}\colon \mathsf{EM}(\mathbb{C})\to\mathsf{EM}(\mathbb{D})$ of H; in particular the following commutes.*

$$\begin{array}{ccc}\mathsf{EM}(\mathbb{C}) & \xrightarrow{\widehat{H}} & \mathsf{EM}(\mathbb{D})\\ {\scriptstyle F^{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F^{\mathbb{D}}}\\ \mathcal{A} & \xrightarrow{H} & \mathcal{B}\end{array}$$

Historical remarks

The usual reference for extended Kleisli liftings (also sometimes referred to as *extensions*) is the preprint Wolff [122], see e.g. Meyer [90], however, Wolff's paper seems to be lost and with it its proofs. This is why we include the full proof of Theorem 7.3.4 here.

A similar construction appears in Meyer [90, Lemma 3.3]. In fact, *op. cit.* also shows that every extended Kleisli lifting also determines a Kleisli law. Observe that this follows directly from Theorem 7.2.1 since any functor $\widehat{H}$ which makes the diagram in Theorem 7.3.4 commute restricts to a Kleisli lifting, in sense of Theorem 7.2.1 and hence induces a Kleisli law. For further results along these lines see also [96] or [26, Appendix A].

7.4 Beyond unary operations

So far, our theorems only apply to unary operations, expressed as functors $H\colon \mathscr{A} \to \mathscr{B}$ between the base categories. Consider, for example, the operation of taking the disjoint union of two structures $+\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$. Following the same lines as above, if there is a natural transformation $\kappa_{A,B}\colon \mathbb{C}(A+B) \to \mathbb{C}A + \mathbb{C}B$, satisfying axioms similar to those of Kleisli laws in Theorem 7.2.1 and if $\mathsf{EM}(\mathbb{C})$ has equalisers of coreflexive pairs then one obtains a lifting:

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) \times \mathsf{EM}(\mathbb{C}) & \xrightarrow{\widehat{+}} & \mathsf{EM}(\mathbb{C}) \\ {\scriptstyle F^{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F^{\mathbb{C}}} \\ \mathscr{A} \times \mathscr{A} & \xrightarrow{+} & \mathscr{A} \end{array}$$

However, one does not have to reprove the same theorem for operations of higher arity. Instead the unary Theorem 7.3.4 directly applies, using the fact that for an indexed collection of categories $\{\mathscr{A}_i\}_i$ and comonads $\{\mathbb{C}_i\}_i$ where, for i, $\mathbb{C}_i$ is a comonad on $\mathscr{A}_i$, we have that the product $\prod_i \mathbb{C}_i$ is a comonad on the product category $\prod_i \mathscr{A}_i$. Crucially, the cofree functor $F^{\prod_i \mathbb{C}_i}$ of the product comonad $\prod_i \mathbb{C}_i$ is computed as the product $\prod_i F^{\mathbb{C}_i}$ of the individual cofree functors.

In other words, an n-ary operation, i.e. a functor $H\colon \prod_i \mathscr{A}_i \to \mathscr{B}$ is, in fact, a unary operation from the product category. We obtain the following.

Corollary 7.4.1 *Given a functor $H\colon \prod_i \mathscr{A}_i \to \mathscr{B}$, if there is a Kleisli law $\mathbb{D} \circ H \Rightarrow H \circ \prod_i \mathbb{C}_i$ and $\mathsf{EM}(\mathbb{D})$ has equalisers of coreflexive pairs, then there is an $\widehat{H}\colon \prod_i \mathsf{EM}(\mathbb{C}_i) \to \mathsf{EM}(\mathbb{D})$ such that the following commutes.*

$$\begin{array}{ccc} \prod_i \mathsf{EM}(\mathbb{C}_i) & \xrightarrow{\widehat{H}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle \prod_i F^{\mathbb{C}_i}}\uparrow & & \uparrow{\scriptstyle F^{\mathbb{D}}} \\ \prod_i \mathscr{A}_i & \xrightarrow{H} & \mathscr{B} \end{array}$$

Example 7.4.2 For the coproduct (disjoint union) operation $+\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ on $\mathcal{S}tr(\sigma)$ and structures A_1, A_2, we define a mapping

$$\kappa_{A_1,A_2}\colon \mathbb{P}_k(A_1 + A_2) \to \mathbb{P}_k A_1 + \mathbb{P}_k A_2$$

which maps a sequence $s = [(p_1, a_1), \ldots, (p_n, a_n)]$ in $\mathbb{P}_k(A_1 + A_2)$ where $a_n \in A_i$ to its subsequence consisting of pairs (p_j, a_j) such that $a_j \in A_i$. Formally, $\kappa(s) = [(p_{i_1}, a_{i_1}), \ldots, (p_{i_m}, a_{i_m})] \in \mathbb{P}_k A_i$ where $i_1 \leq \ldots \leq i_m$ is the enumeration of all positions j such that $a_j \in A_i$.

One readily verifies that κ is a Kleisli law. Similar reasoning as in Example 7.2.4, and using Lemma 2.4.4, the following holds:

$$A_1 \equiv_{\mathsf{C}^k} B_1 \text{ and } A_2 \equiv_{\mathsf{C}^k} B_2 \quad \text{implies} \quad A_1 + A_2 \equiv_{\mathsf{C}^k} B_1 + B_2$$

A similar Kleisli law $\kappa_{A_1,A_2}\colon \mathbb{E}_k(A_1 + A_2) \to \mathbb{E}_k A_1 + \mathbb{E}_k A_2$ can be defined for $\mathbb{E}_k$ and other comonads important in finite model theory [66].

7.5 Heteromorphisms

In this section we assume that we have a Kleisli law $\kappa\colon \mathbb{D}H \Rightarrow H\mathbb{C}$, for a functor $H\colon \mathscr{A} \to \mathscr{B}$, which gives rise to an extended Kleisli lifting $\widehat{H}\colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$ as in Theorem 7.3.4:

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{\widehat{H}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle F^{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F^{\mathbb{D}}} \\ \mathscr{A} & \xrightarrow{H} & \mathscr{B} \end{array}$$

In Chapter 8 we study the question of when $X \rightleftharpoons Y$ implies $\widehat{H}X \rightleftharpoons \widehat{H}Y$ and for that we have to understand path embeddings of the form $P \rightarrowtail \widehat{H}X$. To this end, we give an elementary description of morphisms in $\mathsf{EM}(\mathbb{D})$ of the following form.

$$(B, \beta) \to \widehat{H}(A, \alpha) \tag{7.6}$$

In fact, we show that these morphisms have a simple description that can be stated using just the category $\mathscr{B}$.

Given (A, α) in $\mathsf{EM}(\mathbb{C})$ and (B, β) in $\mathsf{EM}(\mathbb{D})$, a morphism $f\colon B \to HA$ is said to be a **(Kleisli) heteromorphism**[7]

$$f\colon \beta \to_{\kappa} \alpha$$

if the following diagram commutes.

$$\begin{array}{ccccc} B & & \xrightarrow{\quad f \quad} & & HA \\ {\scriptstyle \beta}\downarrow & & & & \downarrow{\scriptstyle H\alpha} \\ \mathbb{D}B & \xrightarrow{\mathbb{D}f} & \mathbb{D}HA & \xrightarrow{\kappa} & H\mathbb{C}A \end{array}$$

In order to show that heteromorphisms are in a bijection with morphisms of the form (7.6), by the universal property of equalisers, it is enough to show that heteromorphisms are in a bijection with morphisms of $\mathbb{D}$-coalgebras $g\colon (B, \beta) \to F^{\mathbb{D}}(HA)$ which equalise (7.4), i.e. such that the following diagram commutes in $\mathsf{EM}(\mathbb{D})$.

$$(B, \beta) \xrightarrow{g} F^{\mathbb{D}}(HA) \underset{F^{\mathbb{D}}(H\alpha)}{\overset{F^{\mathbb{D}}(\kappa)\cdot\eta^{\mathbb{D}}}{\rightrightarrows}} F^{\mathbb{D}}(H\mathbb{C}A) \tag{7.7}$$

First, we show that we can translate between the two types of morphisms.

Lemma 7.5.1 *Given a morphism $g\colon (B, \beta) \to F^{\mathbb{D}}(HA)$ such that (7.7) commutes, the composite*

$$B \xrightarrow{U^{\mathbb{D}}(g)} \mathbb{D}(HA) \xrightarrow{\varepsilon} HA$$

in $\mathscr{B}$ is a heteromorphism of the type $\beta \to_{\kappa} \alpha$.

7: In fact, these can be seen as the duals of the categorical formalisation of bimorphisms, i.e. bilinear maps from linear algebra, see [65].

Proof. For simplicity we denote by g the underlying morphism of the coalgebra morphism g. Then:

$$\begin{aligned}
\kappa \cdot \mathbb{D}(\varepsilon\, g) \cdot \beta &= \kappa \cdot \mathbb{D}\varepsilon \cdot \delta \cdot g && (g \text{ is a coalgebra morphism}) \\
&= \kappa \cdot \varepsilon \cdot \delta \cdot g && (\text{a comonad law}) \\
&= \varepsilon \cdot \mathbb{D}\kappa \cdot \delta \cdot g && (\text{naturality of } \varepsilon) \\
&= \varepsilon \cdot \mathbb{D}(H\alpha) \cdot g && (\text{commutativity of (7.7)}) \\
&= H\alpha \cdot \varepsilon \cdot g && (\text{naturality of } \varepsilon)
\end{aligned}$$

□

Conversely, we translate in the opposite direction as follows.

Lemma 7.5.2 *Given a heteromorphism $f : \beta \to_\kappa \alpha$, the composite*

$$B \xrightarrow{\beta} \mathbb{D}B \xrightarrow{\mathbb{D}f} \mathbb{D}(HA)$$

is a morphism of coalgebras $(B, \beta) \to F^{\mathbb{D}}(HA)$ such that (7.7) commutes.

Proof. We first observe that $\mathbb{D}f \cdot \beta$ is a coalgebra morphism by:

$$\mathbb{D}^2 f \cdot \mathbb{D}\beta \cdot \beta = \mathbb{D}^2 f \cdot \delta \cdot \beta = \delta \cdot \mathbb{D}f \cdot \beta.$$

Further, it equalizes $F^{\mathbb{D}}(H\alpha)$ and $F^{\mathbb{D}}\kappa \cdot \delta$, as:

$$\begin{aligned}
\mathbb{D}(H\alpha) \cdot \mathbb{D}f \cdot \beta &= \mathbb{D}\kappa \cdot \mathbb{D}^2 f \cdot \mathbb{D}(\beta) \cdot \beta \\
&= \mathbb{D}\kappa \cdot \mathbb{D}^2 f \cdot \delta \cdot \beta \\
&= \mathbb{D}\kappa \cdot \delta \cdot \mathbb{D}f \cdot \beta
\end{aligned}$$

□

Finally, we check that these two translations are inverse to each other.

Proposition 7.5.3 *Operations $g \mapsto \varepsilon \cdot Ug$ and $f \mapsto \mathbb{D}f \cdot \beta$, from the previous two lemmas, establish a bijection between heteromorphisms $\beta \to_\kappa \alpha$ and coalgebra morphisms $(B, \beta) \to F^{\mathbb{D}}(HA)$ such that (7.7) commutes.*

Proof. Given a coalgebra morphism $g : (B, \beta) \to F^{\mathbb{D}}(HA)$ which makes (7.7) commute. Since it is a morphism of coalgebras we get that $\mathbb{D}g \cdot \beta = \delta^{\mathbb{D}}_{HA} \cdot g$ and so $\mathbb{D}(\varepsilon\, g)\beta = \mathbb{D}(\varepsilon)\delta\, g = g$. Conversely, given a heteromorphism $f : \beta \to_\kappa \alpha$, by naturality of ε, we have $\varepsilon \cdot \mathbb{D}f \cdot \beta = f \cdot \varepsilon \cdot \beta = f$. □

In order to obtain a bijection between heteromorphism and morphisms of the form (7.6), we introduce a special type of morphism. Given a $\mathbb{C}$-coalgebra (A, α), by u_α (or just u) denote the following composite.

$$u_\alpha = U^{\mathbb{D}}(\widehat{H}(A, \alpha)) \xrightarrow{U^{\mathbb{D}}(\iota_\alpha)} U^{\mathbb{D}}F^{\mathbb{D}}(HA) = \mathbb{D}HA \xrightarrow{\varepsilon^{\mathbb{D}}_{HA}} HA \qquad (7.8)$$

In other words, u_α is the conjugate $(\iota_\alpha)^\flat$ with respect to the adjunction $U^{\mathbb{D}} \dashv F^{\mathbb{D}}$.

Theorem 7.5.4 *There is a bijection between coalgebra morphisms*

$$(B,\beta) \to \widehat{H}(A,\alpha)$$

and heteromorphisms

$$\beta \to_\kappa \alpha$$

mapping a coalgebra homomorphism h to the heteromorphism $u_\beta \cdot U^{\mathbb{D}}(h)$.

Proof. By the universal property of equalisers, the mapping $h \mapsto \iota_\alpha \cdot U^{\mathbb{D}}(h)$ establishes a correspondence between coalgebra morphisms $(B,\beta) \to \widehat{H}(A,\alpha)$ and those of the form $(B,\beta) \to F^{\mathbb{D}}(HA)$ which make (7.7) commute.[8]

On the other hand, by Proposition 7.5.3, the mapping $g \mapsto \varepsilon\, g$ establishes a bijection between coalgebra morphisms $(B,\beta) \to F^{\mathbb{D}}(HA)$ which make (7.7) commute and heteromorphisms $\beta \to_\kappa \alpha$.

Therefore, the composition of the two, i.e. the mapping

$$h \mapsto \varepsilon \cdot U^{\mathbb{D}}(\iota_\alpha) \cdot U^{\mathbb{D}}(h) = u_\alpha \cdot U^{\mathbb{D}}(h)$$

establishes the desired bijection. □

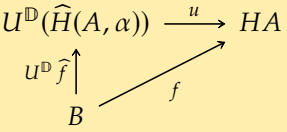
8: The converse to this mapping sends g to the unique $\overline{g}\colon (B,\beta) \to \widehat{H}(A,\alpha)$ such that $g = \iota \cdot \overline{g}$.

In fact, the above allows us to obtain the following interesting fact that explains the character of the morphisms u_α.

Proposition 7.5.5 (Universality of u_α) *For a $\mathbb{C}$-coalgebra (A,α), the morphism $u_\alpha\colon U^{\mathbb{D}}(\widehat{H}(A,\alpha)) \to HA$ is a heteromorphism*

$$\widehat{H}(\alpha) \to_\kappa \alpha.$$

Furthermore, for any heteromorphism $f\colon \beta \to_\kappa \alpha$ there is a unique coalgebra morphism $\widehat{f}\colon (B,\beta) \to \widehat{H}(A,\alpha)$ such that the following diagram commutes.



Proof. The fact that u_α is a heteromorphism follows from Lemma 7.5.1 since $u_\alpha = \varepsilon \cdot U^{\mathbb{D}}(\iota_\alpha)$ and $\iota_\alpha\colon \widehat{H}(A,\alpha) \to F^{\mathbb{D}}(HA)$ is a coalgebra morphism. The required universal property of u_α follows directly from Theorem 7.5.4. □

As the last result of this section we show that the morphism u_α for α which is a cofree coalgebra has a nice explicit description. This is absolutely essential later on.

Lemma 7.5.6 *For $\alpha = F^{\mathbb{D}}(HA)$, we have that $u_\alpha = \kappa_A$.*

Proof. Recall from (7.8) that $u_{F^{\mathbb{D}}(HA)} = \varepsilon_{H\mathbb{C}A} \cdot U^{\mathbb{D}}(\iota_{F^{\mathbb{D}}(HA)})$ and also, by Lemma 7.3.3, $\iota_{F^{\mathbb{D}}(HA)} = F^{\mathbb{D}}(\kappa) \cdot \eta^{\mathbb{D}}$. Therefore:[8]

$$\begin{aligned} u_{F^{\mathbb{D}}(HA)} &= \varepsilon_{H\mathbb{C}A} \cdot U^{\mathbb{D}}F^{\mathbb{D}}(\kappa_A) \cdot U^{\mathbb{D}}(\eta^{\mathbb{D}}) \\ &= \varepsilon_{H\mathbb{C}A} \cdot \mathbb{D}(\kappa_A) \cdot \delta_{HA} && \text{(by definitions)} \\ &= \kappa_A \cdot \varepsilon_{\mathbb{D}HA} \cdot \delta_{HA} && \text{(naturality of } \varepsilon^{\mathbb{D}}) \\ &= \kappa && \text{(a comonad law)} \end{aligned}$$

□

8: Alternatively, one obtains the same result by realising that u is the conjugate $(\iota_{F^{\mathbb{D}}(HA)})^\flat$ with respect to the adjunction $U^{\mathbb{D}} \dashv F^{\mathbb{D}}$. Then, simply $(F^{\mathbb{D}}(\kappa) \cdot \eta^{\mathbb{D}})^\flat = \kappa \cdot (\eta^{\mathbb{D}})^\flat = \kappa \cdot \mathrm{id} = \kappa$.

We can use heteromorphism to better understand extended Kleisli liftings arising in our examples.

Example 7.5.7 Consider the Kleisli law (comonad morphism) $\kappa \colon \mathbb{E}_k\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{P}_k$, mapping $[a_1, \ldots, a_n] \mapsto [(1, a_1), \ldots, (n, a_n)]$, as defined in Example 7.2.4. The corresponding extended Kleisli lifting is a functor $\widehat{\mathrm{Id}} \colon \mathsf{EM}(\mathbb{P}_k) \to \mathsf{EM}(\mathbb{E}_k)$. Note that, for a $\mathbb{P}_k$-coalgebra (A, α), the coalgebra $\widehat{\mathrm{Id}}(A, \alpha)$ is a subobject of $F^{\mathbb{E}_k}(A)$, by being an equaliser as in (7.4).

Recall from Example 2.10.1 that sequences $[a_1, \ldots, a_n] \in \mathbb{E}_k(A)$ are in one-to-one correspondence with path embeddings $e \colon (P, \pi) \rightarrowtail F^{\mathbb{E}_k}(A)$, for P with induced order $p_1 \prec_\pi \cdots \prec_\pi p_n$.

As follows from Proposition 7.5.3, $e \colon (P, \pi) \rightarrowtail F^{\mathbb{E}_k}(A)$ factors via the inclusion $\iota_\alpha \colon \widehat{\mathrm{Id}}(A, \alpha) \hookrightarrow F^{\mathbb{E}_k}(A)$ if and only if $\varepsilon \cdot e$ is a heteromorphism $\pi \to_\kappa \alpha$, i.e. iff the following commutes.

$$\begin{array}{ccccc} P & & \xrightarrow{\quad\varepsilon \cdot e\quad} & & A \\ {\scriptstyle\pi}\downarrow & & & & \downarrow{\scriptstyle\alpha} \\ \mathbb{E}_kP & \xrightarrow{\mathbb{E}_k(\varepsilon \cdot e)} & \mathbb{E}_kA & \xrightarrow{\kappa} & \mathbb{P}_kA \end{array}$$

Note that $\varepsilon \cdot e \colon P \to A$ is the mapping $p_i \mapsto a_i$ and so we ask that:

$$\begin{array}{ccccc} p_i & & \xmapsto{\quad\varepsilon \cdot e\quad} & & a_i \\ & & & & \downarrow{\scriptstyle\alpha} \\ {\scriptstyle\pi}\downarrow & & & & [(p_1, a'_1), \ldots, (p_m, a'_m)] \\ & & & & \| \\ [p_1, \ldots, p_i] & \xmapsto{\mathbb{E}_k(\varepsilon \cdot e)} & [a_1, \ldots, a_i] & \xmapsto{\kappa} & [(1, a_1), \ldots, (n, a_n)] \end{array}$$

In other words, if we view (A, α) as the forest-ordered structure $(A, \sqsubseteq, p)$ with a pebble function, as in Section 2.5, then $\widehat{\mathrm{Id}}(A, \alpha)$ consists of the sequences $[a_1, \ldots, a_n]$ from $F^{\mathbb{E}_k}(A)$ such that $a_i \sqsubseteq a_j$ iff $i \leq j$ and such that the pebble function satisfies $p(a_i) = i$. Alternatively, we can identify $\widehat{\mathrm{Id}}(A, \alpha)$ with the sub-forest of $(A, \sqsubseteq_\alpha)$ consisting of $x \in A$ such that $p(x) = \mathrm{ht}(x) + 1$.

In view of the above analysis, consider the Kleisli law $\mathbb{E}_k\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{E}_{k+1}$ from Example 7.2.4. We see that the corresponding extended Kleisli lifting $\widehat{\mathrm{Id}} \colon \mathsf{EM}(\mathbb{E}_{k+1}) \to \mathsf{EM}(\mathbb{E}_k)$ sends a forest-orderer structure of height $\leq k+1$ to a forest-orderer structure of height $\leq k$, simply by discarding the vertices of height $k+1$. Similarly, consider the Kleisli law $\mathbb{P}_k\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{P}_{k+1}$ which sends a forest-ordered structure with a $k+1$-pebble function to

that with a k-pebble function simply by restricting to those branches where the pebble $k+1$ does not appear.

Example 7.5.8 As in Example 7.2.5 we define a version of the Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k$ on $\mathcal{S}tr_*(\sigma)$ and then a Kleisli law $\kappa\colon \mathbb{M}_k\,\mathrm{Id} \Rightarrow \mathrm{Id}\,\mathbb{E}_k$. First, $\mathbb{E}_k(A, a_0)$ is defined as the substructure of $\mathbb{E}_k(A)$ on sequences $[a_1, \ldots, a_n]$ where $a_1 = a_0$ and $1 \leq n$. The designated point of $\mathbb{E}_k(A, a_0)$ is the length 1 sequence $[a_0]$. Then, $\kappa\colon \mathbb{M}_k\,\mathrm{Id} \Rightarrow \mathrm{Id}\,\mathbb{E}_k$ sends $a_0 \xrightarrow{R_1} a_1 \xrightarrow{R_2} a_2 \xrightarrow{R_3} \ldots \xrightarrow{R_n} a_n$ to the sequence $[a_0, \ldots, a_n]$.

As in Example 7.5.7, we obtain that $e\colon (P, \pi) \rightarrowtail F^{\mathbb{M}_k}(A)$ factors via the inclusion $\widehat{\mathrm{Id}}(A, \alpha) \hookrightarrow F^{\mathbb{M}_k}(A)$, for some $\mathbb{E}_k$-coalgebra (A, α), if and only if the following commutes.

$$\begin{array}{ccccc} P & & \xrightarrow{\quad\varepsilon \cdot e\quad} & & A \\ {\scriptstyle\pi}\downarrow & & & & \downarrow{\scriptstyle\alpha} \\ \mathbb{M}_k P & \xrightarrow{\mathbb{M}_k(\varepsilon \cdot e)} & \mathbb{M}_k A & \xrightarrow{\kappa} & \mathbb{E}_k A \end{array}$$

Now, we set (A, α) to be the path, described using the its forest-ordered structure $(A, \sqsubseteq)$ (cf. Proposition 7.5.3) as follows. We set $A = \{a_0, a_1, a_2\}$ where $a_i \sqsubseteq a_j$ iff $i \leq j$ and the relational structure in signature with two binary relations $\sigma = \langle R, S\rangle$ is shown on the left below.

$$(A, \alpha) = \quad \begin{array}{ccc} & a_2 & \\ & {\scriptstyle R}\uparrow & \\ & a_1 \circlearrowleft R & \\ {\scriptstyle R}\uparrow\uparrow{\scriptstyle S} & & \\ a_0 & & \end{array} \qquad\qquad \widehat{\mathrm{Id}}(A, \alpha) = \quad \begin{array}{ccccc} a_2 & & & & a_2 \\ {\scriptstyle R}\uparrow & & & & {\scriptstyle R}\uparrow \\ a_1 & & & & a_1 \\ & {\scriptstyle R}\nwarrow & & \nearrow{\scriptstyle S} & \\ & & a_0 & & \end{array}$$

Then, the $\mathbb{M}_k$-coalgebra $\widehat{\mathrm{Id}}(A, \alpha)$ is the structure on the right above. Remarkably, $\widehat{\mathrm{Id}}$ sends a path to a non-path.

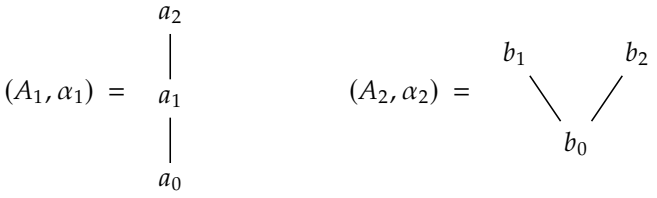
Example 7.5.9 Following a similar procedure as above for the Kleisli law $\kappa_{A_1,A_2}\colon \mathbb{E}_3(A_1 + A_2) \to \mathbb{E}_3 A_1 + \mathbb{E}_3 A_2$ from Example 7.4.2, and analysing path embeddings $(P, \pi) \rightarrowtail (A_1, \alpha_1) \mathbin{\widehat{+}} (A_2, \alpha_2)$ as heteromorphism

$$\begin{array}{ccccc} P & & \xrightarrow{\quad\varepsilon \cdot e\quad} & & A_1 + A_2 \\ {\scriptstyle\pi}\downarrow & & & & \downarrow{\scriptstyle\alpha_1+\alpha_2} \\ \mathbb{E}_3 P & \xrightarrow{\mathbb{E}_3(\varepsilon \cdot e)} & \mathbb{E}_3(A_1 + A_2) & \xrightarrow{\kappa} & \mathbb{E}_3 A_1 + \mathbb{E}_3 A_2 \end{array}$$

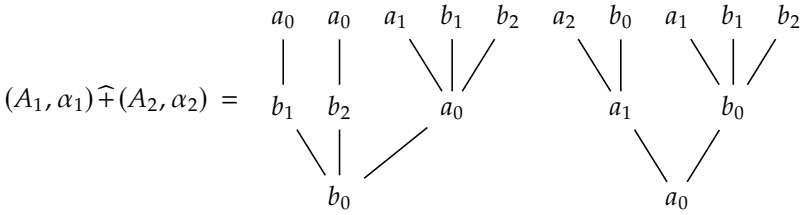
we see that $(A_1, \alpha_1) \mathbin{\widehat{+}} (A_2, \alpha_2)$ is the subtree of $F^{\mathbb{E}_3}(A_1 + A_2)$ consisting of sequences $w = [a_1, \ldots, a_n]$ such that the restriction of this sequence to $w_1 = [a_{i_1}, \ldots, a_{i_m}]$, where $i_1, \ldots, i_m$ is the enumeration of all indices i where $a_i \in A_1$, is a branch in (A_1, α_1) i.e. $w_1 = \alpha_1(a_{i_m})$ and, similarly, the restriction of w to w_2 defined dually must be a branch in (A_2, α_2). In other words, $(A_1, \alpha_1) \mathbin{\widehat{+}} (A_2, \alpha_2)$ is the interleaving of branches of both forests.

For example, for (A_1, α_1) and (A_2, α_2), without any relations, as shown

in the following diagram



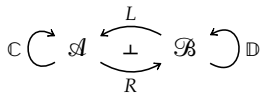
we obtain that:



7.6 Lifting adjunctions

It is very common in the theory of game comonads that we have a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ for H which has a left adjoint. It is a standard fact that such law induces an EM law in the opposite direction and, hence, a lifting of the adjoint functor to H. We study the consequences of this fact, with special focus on extended Kleisli liftings from the point of view heteromorphism.

Remark 7.6.1 In this section we often assume to have an adjunction with comonads on each end, as shown below.



Then, three different *counits* interact in our proofs. Therefore, we explicitly denote $\varepsilon^{\mathbb{C}} \colon \mathbb{C} \to \mathrm{Id}$ and $\varepsilon^{\mathbb{D}} \colon \mathbb{D} \to \mathrm{Id}$ the counits of the two comonads and write simply just $\varepsilon \colon LR \to \mathrm{Id}$ for the counit of the adjunction $L \dashv R$.

We start with the following observation which shows the correspondence between EM laws and Kleisli laws, in the presence of an adjunction. Note that item 2 and item 3 are the same statement, only rephrased in two different ways for ease of reference.

Proposition 7.6.2 (Theorem 9 in Street [118])

1. *If $\lambda \colon H\mathbb{C} \Rightarrow \mathbb{D}H$ is an invertible EM law, then the inverse $\lambda^{-1} \colon \mathbb{D}H \Rightarrow H\mathbb{C}$ is a Kleisli law.*

2. *If we have an adjunction $L \dashv R$, with unit $\eta \colon \mathrm{Id} \Rightarrow RL$ and counit $\varepsilon \colon LR \Rightarrow \mathrm{Id}$, $\lambda \colon L\mathbb{C} \Rightarrow \mathbb{D}L$ is an EM law if and only if its **mate**:*

$$\lambda^* \colon \mathbb{D}R \xrightarrow{\eta \mathbb{D} R} RL\mathbb{D}R \xrightarrow{R\lambda R} R\mathbb{C}LR \xrightarrow{R\mathbb{C}\varepsilon} R\mathbb{C}$$

is a Kleisli law.

3. *If we have an adjunction $L \dashv R$, with unit $\eta\colon \mathrm{Id} \Rightarrow RL$ and counit $\varepsilon\colon LR \Rightarrow \mathrm{Id}$, $\kappa\colon \mathbb{D}R \Rightarrow R\mathbb{C}$ is a Kleisli law if and only if its **mate**:*

$$\kappa^*\colon L\mathbb{D} \xrightarrow{L\mathbb{D}\eta} L\mathbb{D}RL \xrightarrow{L\kappa L} LR\mathbb{C}L \xrightarrow{\varepsilon\mathbb{C}L} \mathbb{C}L$$

is an EM law.

In the following lemma we show that there is a correspondence between heteromorphisms and morphisms from the images of the EM lifting.

For the following, fix an adjunction $L \dashv R$ and a Kleisli law $\kappa\colon \mathbb{D}R \Rightarrow R\mathbb{C}$. Recall that the EM lifting $L^{\mathsf{EM}}\colon \mathsf{EM}(\mathbb{D}) \to \mathsf{EM}(\mathbb{C})$ of $L\colon \mathscr{B} \to \mathscr{A}$, corresponding to the mate κ^* of κ (cf. Proposition 7.6.2), sends a $\mathbb{D}$-coalgebra (B, β) to $(LB, \kappa^* L\beta)$.

Lemma 7.6.3 *A morphism $f\colon B \to RA$ in $\mathscr{A}$ is a heteromorphism $\beta \to_\kappa \alpha$ iff $f^\flat\colon LB \to A$ is a coalgebra morphism $L^{\mathsf{EM}}(B, \beta) \to (A, \alpha)$.*

Proof. By unpacking the definitions, we ask that the diagram on the left below commutes if and only if the diagram on the right below commutes.

$$\begin{array}{ccccc} B & & \xrightarrow{\quad f \quad} & & RA \\ {\scriptstyle\beta}\downarrow & & & & \downarrow{\scriptstyle R\alpha} \\ \mathbb{D}B & \xrightarrow{\mathbb{D}f} & \mathbb{D}RA & \xrightarrow{\kappa} & R\mathbb{C}A \end{array} \qquad \begin{array}{ccc} LB & \xrightarrow{f^\flat} & A \\ {\scriptstyle L\beta}\downarrow & & \\ L\mathbb{D}B & & \downarrow{\scriptstyle \alpha} \\ {\scriptstyle \kappa^*}\downarrow & & \\ \mathbb{C}LB & \xrightarrow{\mathbb{C}f} & \mathbb{C}A \end{array}$$

Recall that by rules of conjugation, $(R\alpha \cdot f)^\flat = \alpha \cdot f^\flat$. Hence, for the equivalence, it is enough to show that $y = x^\#$ where

$$x = B \xrightarrow{\beta} \mathbb{D}B \xrightarrow{\mathbb{D}f} \mathbb{D}RA \xrightarrow{\kappa} R\mathbb{C}A$$

is the bottom-to-right composition in the left diagram above and

$$y = LB \xrightarrow{L\beta} L\mathbb{D}B \xrightarrow{\kappa^*} \mathbb{C}LB \xrightarrow{\mathbb{C}f} \mathbb{C}A$$

is the bottom-to-right composition in the right diagram above.

However, using the fact that $f = R(f^\flat) \cdot \eta$ and naturality of κ and ε, the morphism $x^\# = \varepsilon \cdot L(x)$ can be rewritten according to the following commutative diagram.

$$\begin{array}{ccccccccc} L(B) & \xrightarrow{L\beta} & L\mathbb{D}(B) & \xrightarrow{L\mathbb{D}\eta_B} & L\mathbb{D}RL(B) & \xrightarrow{L\kappa_{L(B)}} & LR\mathbb{C}L(B) & \xrightarrow{\varepsilon_{\mathbb{C}L(B)}} & \mathbb{C}L(B) \\ & \searrow^{L\mathbb{D}f} & \downarrow{\scriptstyle L\mathbb{D}Rf^\flat} & & \downarrow{\scriptstyle LR\mathbb{C}f^\flat} & & \downarrow{\scriptstyle \mathbb{C}f^\flat} & & \\ & & L\mathbb{D}R(A) & \xrightarrow{L\kappa_A} & LR\mathbb{C}(A) & \xrightarrow{\varepsilon_A} & \mathbb{C}(A) & & \end{array}$$

Taking into account that $\kappa^* = \varepsilon \cdot L\kappa L \cdot L\mathbb{D}\eta$, we see that $y = x^\#$, as requested. □

The following is the main result that relates the two types of lifts that arise in our setting. This theorem appeared first in Johnstone's note [69, Theorem 2]. However we state it differently to emphasise that what might appear as an ad-hoc construction is actually a natural consequence of the facts that we have established for heteromorphisms.

Theorem 7.6.4 *Assume we have an adjunction $L \dashv R$ and a Kleisli law $\kappa\colon \mathbb{D}R \Rightarrow R\mathbb{C}$. Further assume that $\mathsf{EM}(\mathbb{D})$ has equalisers of coreflexive pairs.*
Then, $L^{\mathsf{EM}} \dashv \widehat{R}$ where

- *L^{EM} is the EM lifting of the mate κ^* of κ, as in Theorem 7.1.1, and*
- *$\widehat{R}$ is the extended Kleisli lifting of κ, by Theorem 7.3.4.*

Proof. The result follows as a composition of natural bijections between

- coalgebra morphisms $L^{\mathsf{EM}}(\beta) \longrightarrow \alpha$,
- heteromorphisms $\beta \longrightarrow_{\kappa} \alpha$, and
- coalgebra morphisms $\beta \longrightarrow \widehat{R}(\alpha)$.

The bijection between the latter two types of morphisms is described in Theorem 7.5.4 and the former bijection is Lemma 7.6.3. □

Observe that item 1 of Proposition 7.6.2 also allows to lift left adjoints, which recovers an old theorem of Johnstone [69, Theorem 4].

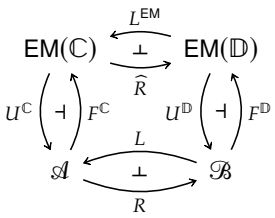
Theorem 7.6.5 *Let $\lambda\colon H\mathbb{C} \Rightarrow \mathbb{D}H$ be an invertible EM law of $H\colon \mathcal{A} \longrightarrow \mathcal{B}$, as in Theorem 7.1.1. If H has a right adjoint then so does the EM lifting H^{EM} of λ.*

See also [121] for a discussion when a lifted endofunctor is again a comonad, and related facts about tensors of (co)algebras.

On compositions of conjugations

In this section we prove a theorem that the author of this text believes to be folklore but could not find a reference for it in the literature. The statement says that all conjugations of adjoints induced by our extended Kleisli lifting and EM lifting commute with each other.

Let us fix an adjunction $L \dashv R$ and a Kleisli law $\kappa\colon \mathbb{D}R \Rightarrow R\mathbb{C}$ inducing an adjunction between the Kleisli categories according to Theorem 7.6.4, as shown in the diagram below.



(7.9)

Here, L^{EM} denotes the EM lifting of the mate $\kappa^*\colon L\mathbb{D} \Rightarrow \mathbb{C}L$ of our Kleisli law $\kappa\colon \mathbb{D}R \Rightarrow R\mathbb{C}$. The consequence of Theorem 7.1.1 and Theorem 7.3.4

is that the diagram made solely of the left adjoints commute and similarly for the right adjoints, that is,

$$U^{\mathbb{C}} \circ L^{\mathsf{EM}} = L \circ U^{\mathbb{D}} \quad \text{and} \quad F^{\mathbb{D}} \circ R = \widehat{R} \circ F^{\mathbb{C}}.$$

To further simplify our reasoning we fix a special notation for each type of conjugation that the four adjunctions introduce. Namely we have the following pairs of natural bijections:

- $(\text{-})^{\flat}$: $\hom(\text{-}, R(\text{-})) \to \hom(L(\text{-}), \text{-})$ and its inverse $(\text{-})^{\#}$
- $(\text{-})^{\widehat{\flat}}$: $\hom(\text{-}, \widehat{R}(\text{-})) \to \hom(L^{\mathsf{EM}}(\text{-}), \text{-})$ and its inverse $(\text{-})^{\widehat{\#}}$
- $(\text{-})^{\mathcal{A}}$: $\hom(\text{-}, F^{\mathbb{C}}(\text{-})) \to \hom(U^{\mathbb{C}}(\text{-}), \text{-})$ and its inverse $(\text{-})^{\mathbb{C}}$
- $(\text{-})^{\mathcal{B}}$: $\hom(\text{-}, F^{\mathbb{D}}(\text{-})) \to \hom(U^{\mathbb{D}}(\text{-}), \text{-})$ and its inverse $(\text{-})^{\mathbb{D}}$

Next we show that these commute with each other, which is an important fact in later chapters.

Theorem 7.6.6 *Given the situation as in* (7.9) *and a morphism* $f : T \to A$ *in* $\mathcal{A}$ *such that* $T = U^{\mathbb{C}}(L^{\mathsf{EM}}(X)) = L(U^{\mathbb{D}}(X))$ *for some* X *from* $\mathsf{EM}(\mathbb{D})$. *Then:*

$$(f^{\#})^{\mathbb{D}} = (f^{\mathbb{C}})^{\widehat{\#}}$$

Consequently, we also have the converse

$$(h^{\widehat{\flat}})^{\mathcal{A}} = (h^{\mathcal{B}})^{\flat}$$

for $h : X \to Y$ *in* $\mathsf{EM}(\mathbb{D})$ *such that* $Y = F^{\mathbb{D}}(R(A)) = \widehat{R}(F^{\mathbb{C}}(A))$ *for some* A *in* $\mathcal{A}$.

Proof. Given $X = (B, \beta)$ and $f : U^{\mathbb{C}}(L^{\mathsf{EM}}(X)) \to A$ we show that $f^{\mathbb{C}} = ((f^{\#})^{\mathbb{D}})^{\widehat{\flat}}$. First, we calculate $f^{\mathbb{C}}$. By the definition of the adjunction $U^{\mathbb{C}} \dashv F^{\mathbb{C}}$, the morphism $f^{\mathbb{C}}$ is calculated as

$$L^{\mathsf{EM}}(X) \xrightarrow{\eta^{\mathbb{C}}_{L^{\mathsf{EM}}(X)}} F^{\mathbb{C}}U^{\mathbb{C}}L^{\mathsf{EM}}(X) \xrightarrow{F^{\mathbb{C}}(f)} F^{\mathbb{C}}(A)$$

where the underlying morphism of the unit $\eta_{L^{\mathsf{EM}}(X)}$ of the adjunction $U^{\mathbb{C}} \dashv F^{\mathbb{C}}$ is given by the coalgebra structure of $L^{\mathsf{EM}}(X)$. By the definition of EM liftings, $\eta_{L^{\mathsf{EM}}(X)}$ is precisely the following morphism in $\mathcal{A}$.

$$\begin{aligned} & L(B) \xrightarrow{L\beta} L\mathbb{D}B \xrightarrow{\kappa^*} \mathbb{C}L(B) \\ = {} & L(B) \xrightarrow{L\beta} L\mathbb{D}(B) \xrightarrow{L\mathbb{D}\eta_B} L\mathbb{D}RL(B) \xrightarrow{L\kappa_{LB}} LR\mathbb{C}L(B) \xrightarrow{\varepsilon_{\mathbb{C}L(B)}} \mathbb{C}L(B) \end{aligned}$$

Here η and ε are the unit and counit of the adjunction $L \dashv R$, respectively.

We have the following simplification:

$$\begin{aligned} U^{\mathbb{C}}(f^{\mathbb{C}}) = {} & \mathbb{C}f \cdot \varepsilon_{\mathbb{C}L(B)} \cdot L\kappa_{LB} \cdot L\mathbb{D}\eta_B \cdot L\beta \\ = {} & \varepsilon_A \cdot LR\mathbb{C}f \cdot L\kappa_{LB} \cdot L\mathbb{D}\eta_B \cdot L\beta && \text{(naturality of } \varepsilon) \\ = {} & \varepsilon_A \cdot L\kappa_A \cdot L\mathbb{D}Rf \cdot L\mathbb{D}\eta_B \cdot L\beta && \text{(naturality of } \kappa) \\ = {} & \varepsilon_A \cdot L\kappa_A \cdot L\mathbb{D}(f^{\#}) \cdot L\beta && \text{(definition of } f^{\#}) \end{aligned}$$

Next, we calculate $((f^{\#})^{\mathbb{D}})^{\widehat{\flat}}$. Observe that the underlying morphism of $(f^{\#})^{\mathbb{D}} : X \to F^{\mathbb{D}}(RA)$ is simply just

$$B \xrightarrow{\beta} \mathbb{D}B \xrightarrow{\mathbb{D}f^{\#}} \mathbb{D}R(A).$$

The heteromorphism corresponding to this morphism is:

$$\begin{aligned} & B \xrightarrow{\beta} \mathbb{D}B \xrightarrow{\mathbb{D}f^{\#}} \mathbb{D}R(A) \xrightarrow{u_{F^{\mathbb{D}}(RA)}} R\mathbb{C}(A) \\ = \; & B \xrightarrow{\beta} \mathbb{D}B \xrightarrow{\mathbb{D}f^{\#}} \mathbb{D}R(A) \xrightarrow{\kappa_A} R\mathbb{C}(A) \qquad \text{(by Lemma 7.5.6)} \end{aligned}$$

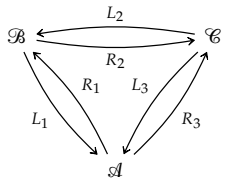
Finally, by Theorem 7.5.4 and Lemma 7.6.3, the underlying morphism of $((f^{\#})^{\mathbb{D}})^{\widehat{\flat}} : L^{\mathsf{EM}}(X) \to F^{\mathbb{C}}(A)$ is computed as the $(\text{-})^{\flat}$ conjugate of the above morphism, i.e.

$$U^{\mathbb{C}}(((f^{\#})^{\mathbb{D}})^{\widehat{\flat}}) \;=\; \varepsilon_A \cdot L(\kappa_A \cdot \mathbb{D}f^{\#} \cdot \beta) \;=\; \varepsilon_A \cdot L\kappa_A \cdot L\mathbb{D}f^{\#} \cdot L\beta$$

which is the same morphism as $f^{\mathbb{C}}$. The result follows by the fact that $(\text{-})^{\widehat{\#}}$ and $(\text{-})^{\widehat{\flat}}$ are inverse to each other. □

Remark 7.6.7 Note that the result of Theorem 7.6.6 simply says that the composition of the two adjunction induce the same adjoint structure on the composite of functors.[9]

9: More precisely, it is a standard fact that that whenever we have three adjunctions



and conjugation operations $b_i : \hom(L_i(\text{-}), \text{-}) \to \hom(\text{-}, R_i(\text{-}))$, for $i = 1,2,3$, then $b_3 = b_2 \cdot b_1$ iff $\eta^3 = R_2\eta^1_{L_2} \cdot \eta^2$.

This precisely corresponds to the situation that the adjunction $L_3 \dashv R_3$ is defined as the composition of the other two.

Right adjoints versus extended Kleisli liftings

An interesting observation made already by Meyer is that right adjoints to EM liftings are automatically the extended Kleisli liftings, in sense of Theorem 7.3.4.

Theorem 7.6.8 (Lemma 3.1 in [90]) *Assume $L \dashv R$ and $\lambda : L\mathbb{D} \Rightarrow \mathbb{C}L$ is an EM law. If L^{EM} has a right adjoint then the right adjoint is an extended Kleisli lifting $\widehat{R}$ of R.*

Furthermore, Meyer also observed that in most cases extended Kleisli liftings are given by the procedure we outlined above.

Theorem 7.6.9 (Theorem 3.4 in [90]) *Let $\overline{H} : \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$ be a lifting of $H : \mathscr{A} \to \mathscr{B}$ which commutes with cofree functors, as in Theorem 7.3.4. If the functor $\overline{H}$ preserves equalisers then it is given by the equaliser formula (7.4) for some Kleisli law $\kappa : \mathbb{D}H \Rightarrow H\mathbb{C}$.*

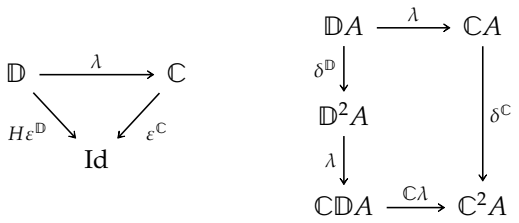
Note that in our usual situation when $\widehat{H}$ is the lifting of the identity functor, that is, our Kleisli law is in fact a comonad morphism $\lambda : \mathbb{D} \Rightarrow \mathbb{C}$, we have that $\widehat{H}$ automatically has a left adjoint and so $\widehat{H}$ preserves not only equalisers but in fact all limits.

Comonad morphisms

In this section we focus on the properties of liftings arising from comonad morphisms $\mathbb{D} \Rightarrow \mathbb{C}$ which, by definition, are both EM laws of the type

$\mathrm{Id}\mathbb{C} \Rightarrow \mathbb{D}\mathrm{Id}$ and Kleisli laws of the type $\mathbb{D}\mathrm{Id} \Rightarrow \mathrm{Id}\mathbb{C}$. Since this is a common situation, see e.g. Examples 7.1.3, 7.1.4, 7.2.4, etc., we wish to summarise the consequences of our findings for this particular situation here.

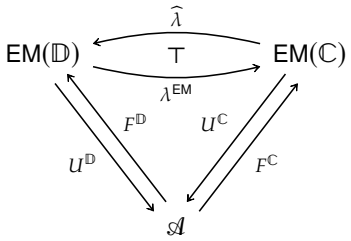
Recall from Section 6.5 that a **comonad morphism** $\lambda : \mathbb{D} \Rightarrow \mathbb{C}$, for comonads $\mathbb{D}$ and $\mathbb{C}$ over the same base category $\mathscr{A}$, is a natural transformation which satisfies:



By being both a Kleisli and EM law, λ induces both the Kleisli lifting λ^{Kl} (Theorem 7.2.1) and the EM lifting λ^{EM} (Theorem 7.1.1).[10]

10: In case when we are lifting the identity functor, we prefer to emphasise the name of the comonad morphism in the notation rather than the name of the functor.

If $\mathsf{EM}(\mathbb{D})$ has equalisers of coreflexive pairs, λ also induces the extended Kleisli lifting $\widehat{\lambda}$. Furthermore, since $\mathrm{Id} \dashv \mathrm{Id}$, the EM lifting and the extended Kleisli liftings are adjoint to each other (Theorem 7.6.4). In pictures we have the following situation.



(7.10)

Note that $\widehat{\lambda}$ commutes with the free functors and λ^{EM} commutes with the forgetful functors.

Furthermore, the following is a special case of Theorem 7.6.6.

Proposition 7.6.10 *In* (7.10)*, let* $f : T \to B$ *be a morphism in* $\mathscr{A}$*, for some* $X \in \mathsf{EM}(\mathbb{D})$ *and* $T = U^{\mathbb{D}}X = U^{\mathbb{C}}(\lambda^{\mathsf{EM}}X)$*. Then, the conjugation*

$$f \quad \mapsto \quad X \to F^{\mathbb{D}}B$$

and the double conjugation via $\mathsf{EM}(\mathbb{C})$

$$f \quad \mapsto \quad \lambda^{\mathsf{EM}}X \to F^{\mathbb{C}}B \quad \mapsto \quad X \to \widehat{\lambda}(F^{\mathbb{C}}B) = F^{\mathbb{D}}B$$

give the same results.

Proof. Follows simply by naturality of $\lambda : \mathbb{D} \Rightarrow \mathbb{C}$ as, in the notation of Theorem 7.6.6, $(f^{\mathbb{C}})^{\#} = u \cdot (f^{\#})^{\mathbb{D}} = \kappa \cdot (f^{\#})^{\mathbb{D}}$. □

Furthermore, we recall the following interesting and well-known fact.

Theorem 7.6.11 *The functor λ^{EM} from (7.10) is comonadic.*

Proof. Just observe that $U^{\mathbb{D}} = U^{\mathbb{C}} \circ \lambda^{\mathsf{EM}}$ and $U^{\mathbb{D}}$ and $U^{\mathbb{C}}$ are comonadic. Furthermore, we assumed that $\mathsf{EM}(\mathbb{D})$ has coequalisers of coreflexive pairs, which entails that it has all equalisers [82], therefore, [31, Corollary 4.5.7] applies. □

Preservation theorems and logic 8

8.1 Preservation of equivalence with respect to counting and positive existential fragments . . . 160
8.2 Preservations of the other fragments 161
8.3 Preservations of open morphisms 162
8.4 Preservations from minimal decompositions . . . 163
Preservation of the back-and-forth relation 164
Preservation of (open) pathwise-embeddings . . 166
Simplifying the axioms . . 166
8.5 Strong path functors . . . 167
Path embeddings and conjugations 167
All assumptions together 169
Preservation of paths . . . 170
8.6 Logic extensions and Kleisli liftings 171
8.7 Beyond unary operations 174
8.8 Product preservation theorems for free 175

In Chapter 7 we reviewed the theory of Eilenberg–Moore and Kleisli laws whilst presenting many examples of these in the setting of game comonads. In particular, we showed that Eilenberg–Moore liftings introduced by the former laws yield theorems of preservation of combinatorial properties (Section 7.1) and Kleisli liftings introduced by the latter lead to preservation of equivalence with respect to the counting fragments (Section 7.2).

In this chapter we focus specifically on the aspect of Kleisli laws as a tool for comparing logical fragments. Apart from the previously studied preservation of equivalence with respect to the counting fragment, we describe conditions for extended Kleisli liftings ensuring preservation of logical equivalences with respect the other studied fragments. The result is an essential tool in the study of game comonads which allows one to compare strengths of different logic fragments.

The motivating example is an old theorem of Mostowski [94], which shows that, for structures A, B:

$$A_1 \equiv_{\mathrm{F0}} B_1 \quad \text{and} \quad A_2 \equiv_{\mathrm{F0}} B_2 \quad \text{imply} \quad A_1 \times A_2 \equiv_{\mathrm{F0}} B_1 \times B_2$$

This result was later generalised by Feferman and Vaught [49] to coproducts (disjoint unions) and products of arbitrary arity. In particular, they showed that:

$$A_1 \equiv_{\mathrm{F0}} B_1 \quad \text{and} \quad A_2 \equiv_{\mathrm{F0}} B_2 \quad \text{imply} \quad A_1 + A_2 \equiv_{\mathrm{F0}} B_1 + B_2$$

In this section we prove a general type of Feferman–Vaught–Mostowski (FVM) *preservation theorems*, also known as the *composition method* in finite model theory. These theorems, for a given n-ary (functorial) operation H and logic fragments $\mathscr{L}_1, \dots, \mathscr{L}_{n+1}$, state that:

$$\begin{gathered} A_1 \equiv_{\mathscr{L}_1} B_1 \quad \dots \quad A_n \equiv_{\mathscr{L}_n} B_n \\ \text{imply} \\ H(A_1, \dots, A_n) \equiv_{\mathscr{L}_{n+1}} H(B_1, \dots, B_n) \end{gathered}$$

In fact, we mainly focus on the question of preservation of logical equivalence by unary operations since extending these to n-ary operations follows automatically, similarly to Section 7.4. Therefore, given a functor $H \colon \mathcal{A} \to \mathcal{B}$ between categories with comonads $\mathbb{C}$ and $\mathbb{D}$, we wish to establish conditions under which

$$\begin{aligned} A \equiv_{\#\mathbb{C}} B \quad &\text{implies} \quad H(A) \equiv_{\#\mathbb{D}} H(B), \\ A \Rrightarrow_{\exists^+\mathbb{C}} B \quad &\text{implies} \quad H(A) \Rrightarrow_{\exists^+\mathbb{D}} H(B), \\ A \Rrightarrow_{+\mathbb{C}} B \quad &\text{implies} \quad H(A) \Rrightarrow_{+\mathbb{D}} H(B), \\ A \Rrightarrow_{\exists\mathbb{C}} B \quad &\text{implies} \quad H(A) \Rrightarrow_{\exists\mathbb{D}} H(B), \quad \text{and} \\ A \equiv_{\mathbb{C}} B \quad &\text{implies} \quad H(A) \equiv_{\mathbb{D}} H(B). \end{aligned} \tag{8.1}$$

As our main tool we use the categorical toolkit from Chapter 7. It turns out that the applicability of our general preservation results goes way beyond just the original FVM theorems. Our general theorems are an essential tool in the setting of game comonads, e.g. for comparing logics and transferring logical equivalences and types. For the latter, see Chapter 9.

8.1 Preservation of equivalence with respect to counting and positive existential fragments

In Examples 7.2.4 and 7.2.5 we observed that the existence of a Kleisli law guarantees preservation of logical equivalence with respect to the counting fragments, simply because the Kleisli law induces a lifting which commutes with cofree functors.

The following statement is obtained by extracting the general method from these examples to arbitrary comonads. As it turns out, functoriality also implies preservation of the positive existential fragments.

Corollary 8.1.1 *If $H : \mathcal{A} \to \mathcal{B}$ admits a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ then*

$$A \Rightarrow_{\exists^+\mathbb{C}} B \quad \textit{implies} \quad H(A) \Rightarrow_{\exists^+\mathbb{D}} H(B)$$

and

$$A \equiv_{\#\mathbb{C}} B \quad \textit{implies} \quad H(A) \equiv_{\#\mathbb{D}} H(B).$$

Proof. By the existence of a Kleisli law and Theorem 7.2.1, there exists a Kleisli lifting, as shown below.

$$\begin{array}{ccc} \mathsf{Kl}(\mathbb{C}) & \xrightarrow{H^{\mathsf{Kl}}} & \mathsf{Kl}(\mathbb{D}) \\ {\scriptstyle F_{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F_{\mathbb{D}}} \\ \mathcal{A} & \xrightarrow{H} & \mathcal{B} \end{array}$$

Assume that $A \Rightarrow_{\exists^+\mathbb{C}} B$, i.e. that some $f : F_{\mathbb{C}}(A) \to F_{\mathbb{C}}(B)$ exists. Then, by commutativity of the above diagram, we have that $H^{\mathsf{Kl}}(f) : H^{\mathsf{Kl}}(F_{\mathbb{C}}(A)) \to H^{\mathsf{Kl}}(F_{\mathbb{C}}(B))$ is a morphism $F_{\mathbb{D}}(H(A)) \to F_{\mathbb{D}}(H(B))$, i.e. a witness for $H(A) \Rightarrow_{\exists^+\mathbb{C}} H(B)$.

Similarly, if we have that $A \equiv_{\#\mathbb{C}} B$, i.e. that $F_{\mathbb{C}}(A) \cong F_{\mathbb{C}}(B)$, then $H^{\mathsf{Kl}}(F_{\mathbb{C}}(A)) \cong H^{\mathsf{Kl}}(F_{\mathbb{C}}(B))$ which gives us that $H(A) \equiv_{\#\mathbb{D}} H(B)$ by functoriality of H^{Kl} since $F_{\mathbb{D}}(H(A)) = H^{\mathsf{Kl}}(F_{\mathbb{C}}(A)) \cong H^{\mathsf{Kl}}(F_{\mathbb{C}}(B)) = F_{\mathbb{D}}(H(B))$. □

Remark 8.1.2 Observe that for the former implication in Corollary 8.1.1 above we only need a collection of morphisms $\{\mathbb{D}HA \to H\mathbb{C}A\}_{A \in \mathrm{Ob}(\mathcal{A})}$, without necessarily requiring naturality or the other conditions of Kleisli laws.

Example 8.1.3 Recall from Example 6.7.6 that the comonad $\mathtt{Cos}$ on graphs, defined as the discrete density comonad $\mathbb{D}_\Delta$ of the class Δ of cycles, expresses cospectrality with the equivalence $\equiv_{\#\mathtt{Cos}}$. It is not difficult to observe that $\mathbb{P}_k$ restricts from the category $\mathcal{S}tr(\sigma)$, where σ contains one binary relation, to the category of graphs, i.e. σ-structures which

are anti-symmetric and loopless. Moreover, we can define a comonad morphism $\mathtt{Cos} \Rightarrow \mathbb{P}_3$ in graphs which, by Corollary 8.1.1, recovers a result in [43] showing that equivalence in 3-variable counting logic implies cospectrality. For details, see [66, Example 4.7].

8.2 Preservations of the other fragments

Recall from Section 3.1 that all but the first two logical refinements and equivalences mentioned in (8.1) are expressed using the EM categories of coalgebras. We show that, in order for a functor $H\colon \mathscr{A} \to \mathscr{B}$ between categories with comonads $\mathbb{C}$ and $\mathbb{D}$ to preserve these logical refinements and equivalences, it is enough if H has an extended Kleisli lifting[1] $\widehat{H}$, making the following diagram commute,

1: Cf. Theorem 7.3.4.

$$\begin{array}{ccc} \mathsf{EM}(\mathbb{C}) & \xrightarrow{\widehat{H}} & \mathsf{EM}(\mathbb{D}) \\ {\scriptstyle F^{\mathbb{C}}}\uparrow & & \uparrow{\scriptstyle F^{\mathbb{D}}} \\ \mathscr{A} & \xrightarrow{H} & \mathscr{B} \end{array}$$

and, moreover, the lifting $\widehat{H}$ has to preserve the structure that defines these logical refinements and equivalences. Furthermore, in the case of the standard fragments, we can lean on the fact that $\equiv_{\mathbb{C}}$ has many different characterisations, cf. Section 4.8.

Theorem 8.2.1 *Assume that* $\mathsf{EM}(\mathbb{C})$ *and* $\mathsf{EM}(\mathbb{D})$ *above are elementary path categories. We have the following:*

1. *If* $\widehat{H}$ *preserves pathwise-embeddings then*

$$A \Rrightarrow_{\exists\mathbb{C}} B \quad \textit{implies} \quad H(A) \Rrightarrow_{\exists\mathbb{D}} H(B). \tag{8.2}$$

2. *If* $\widehat{H}$ *preserves tree bijections and open pathwise-embeddings, then*

$$A \Rrightarrow_{+\mathbb{C}} B \quad \textit{implies} \quad H(A) \Rrightarrow_{+\mathbb{D}} H(B). \tag{8.3}$$

3. *If* $\widehat{H}$ *preserves the back-and-forth equivalence* $\rightleftharpoons$ *then*

$$A \equiv_{\mathbb{C}} B \quad \textit{implies} \quad H(A) \equiv_{\mathbb{D}} H(B). \tag{8.4}$$

In addition, assume that $\mathsf{EM}(\mathbb{C})$ *and* $\mathsf{EM}(\mathbb{D})$ *are concrete arboreal categories with binary products, then we have:*

4. *If* $\widehat{H}$ *preserves open pathwise-embeddings, then* (8.4) *holds.*
5. *If* $\mathsf{EM}(\mathbb{C})$ *and* $\mathsf{EM}(\mathbb{D})$ *have pullbacks of embeddings and* $\widehat{H}$ *preserves open morphisms, then* (8.4) *holds.*
   If, moreover, $\widehat{H}$ *preserves tree bijections then* (8.3) *holds too.*

Proof. Items 1–3 follow by the definition of relations $\Rrightarrow_{\exists\mathbb{C}}$, $\Rrightarrow_{+\mathbb{C}}$ and $\equiv_{\mathbb{C}}$. For example, assume that $\widehat{H}$ preserves pathwise-embeddings. Then, $A \Rrightarrow_{\exists\mathbb{C}} B$ implies that there is a pathwise-embedding $f\colon F^{\mathbb{C}}(A) \to F^{\mathbb{C}}(B)$. By our assumption $\widehat{H}(f)\colon \widehat{H}(F^{\mathbb{C}}(A)) \to \widehat{H}(F^{\mathbb{C}}(B))$ is a pathwise-embedding too. Consequently, $H(A) \Rrightarrow_{\exists\mathbb{D}} H(B)$ since $\widehat{H}(F^{\mathbb{C}}(A)) = F^{\mathbb{D}}(H(A))$ and similarly for B.

The penultimate item follows from the fact that $X \rightleftharpoons Y$ iff $X \sim Y$ in $\mathsf{EM}(\mathbb{C})$, by Proposition 4.8.2. Then, since $\widehat{H}$ preserves open pathwise-embedding, $\widehat{H}(X) \sim \widehat{H}(Y)$ which implies that $\widehat{H}(X) \rightleftharpoons \widehat{H}(Y)$ by Proposition 3.4.4, i.e. $H(A) \equiv_{\mathbb{D}} H(B)$.

The last item follows from the penultimate and Theorem 3.4.8 together with Theorem 4.3.2. □

As we see, it is imperative to understand when a functor $\mathcal{X} \to \mathcal{Y}$ between elementary path categories $\mathcal{X}$ and $\mathcal{Y}$ satisfies the assumptions for $\widehat{H}$ in the above theorem, i.e. when it preserves the b&f equivalence, open morphisms, open pathwise-embeddings, etc. We explicitly focus on this question in the following sections.

Notice that extended Kleisli liftings $\widehat{H}$ often have a left adjoint, in particular if they arise from a comonad morphism, cf. Subsection"Comonad morphisms" on page 156. Therefore, special care is given to this situation which leads us to the notion of *strong path functors*, in Section 8.5 below.

8.3 Preservations of open morphisms

We focus on the situation when a functor between elementary path categories preserves *open* morphisms,[2] which is one of the essential assumptions appearing in Theorem 8.2.1.

2: Cf. Subsection"Open morphisms à la Joyal–Nielsen–Winskel" on page 58.

We rephrase an observation due to Colin Riba that shows that preservation of open morphisms is automatic in the presence of left adjoints which preserve paths.[3]

3: Note that Riba's observation is inspired, in turn, by the notion of *strong functors*, which we describe in Section 8.5.

Proposition 8.3.1 *Let $L \dashv R$ be a pair of adjoint functors between elementary path categories. If L sends paths to paths then R preserves open morphisms.*

Proof. Given a commutative square where P, Q are paths and f is open.

$$\begin{array}{ccc} P & \longrightarrow & RX \\ \downarrow & & \downarrow{\scriptstyle Rf} \\ Q & \longrightarrow & RY \end{array}$$

By conjugation, we obtain the following commutative diagram.

$$\begin{array}{ccc} LP & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ LQ & \longrightarrow & Y \end{array}$$

Now, since L preserves paths and f is open, there is a diagonal filler $d\colon LQ \to X$. The conjugate $Q \to RX$ of d is then a diagonal filler of the original diagram. □

The above and Theorem 3.4.8 immediately imply that the right adjoint preserves bisimilarity and behavioural equivalence. Furthermore, if R is a functor between arboreal categories then it also preserves b&f equivalence because that is in fact the same as bisimilarity (cf. Proposition 4.8.2).

Corollary 8.3.2 *Let $L \dashv R$ be a pair of adjoint functors between elementary path categories with (M_{pb}) and $(P_{\twoheadrightarrow})$ such that L sends paths to paths. Then,*

1. *$X \sim Y$ implies $RX \sim RY$*
2. *$X \leftrightarrows Y$ implies $RX \leftrightarrows RY$*

Furthermore, if $L \dashv R$ is a pair of adjoints between concrete arboreal categories with binary products,

1. *$X \rightleftharpoons Y$ implies $RX \rightleftharpoons RY$.*

The above situation is quite common in our setting and allows us to go very far. In fact, checking that L (i.e. some EM lifting, Theorem 7.1.1) preserves paths is usually quite simple. Recall that L is computed simply as the composition of the coalgebra map with the EM law.

Example 8.3.3 Observe that the EM lifting arising from the comonad morphisms described in Examples 7.1.3, 7.1.4 and 7.2.5 preserves paths. For example, the comonad morphism $\lambda : \mathbb{E}_k \Rightarrow \mathbb{P}_k$ sending $[a_1, \ldots, a_n]$ to $[(1, a_n), \ldots, (n, a_n)]$ induces an EM lifting λ^{EM} which sends a path $(P, \leq)$ of $\mathsf{EM}(\mathbb{E}_k)$, in the induced forest order (cf. Section 2.5), to the path $(P, \leq, p)$ of $\mathsf{EM}(\mathbb{P}_k)$, with the pebble function given by $p(x) = \mathrm{ht}(x) + 1$.

8.4 Preservations from minimal decompositions

In some cases our extended Kleisli lifting might not have a left adjoint, see e.g. Example 7.2.6 or situations where H corresponds to an n-ary operation (Section 7.4). To this end, we introduce weaker axioms for a functor between elementary path categories to still ensure that it preserves the b&f equivalence, to fulfil the assumptions of Theorem 8.2.1.

In the following we fix a functor

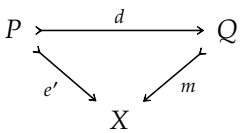
$$H : \mathcal{X} \to \mathcal{Y}$$

between elementary path categories. Furthermore, we assume that the following two conditions are met.

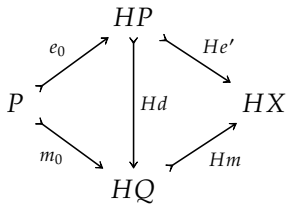
(S1) H preserves embeddings.

(S2) Any path embedding $e : P \rightarrowtail HX$ has a **minimal decomposition** as $e_0 : P \to HP'$ followed by $He' : HP' \to HX$, for some path embeddings $e' : P' \rightarrowtail X$.

Where the property of being minimal means that whenever e decomposes as $m_0 : P \to H(Q)$ followed by $H(m) : H(Q) \to HX$, for some path embedding $m : Q \rightarrowtail X$ then there is a (necessarily unique embedding) $d : P \to Q$ such that:



Observe that from this we also get that the following diagram commutes by (S1) and the fact that embeddings are monos.



Remark 8.4.1 Observe that all morphisms in the above diagram are automatically embeddings.

We postpone the question of how axioms (S1) and (S2) can be checked until Subsection"Simplifying the axioms" on page 166 where we develop a transparent method in the language of heteromorphisms.

Preservation of the back-and-forth relation

In this subsection we show that axioms (S1) and (S2) guarantee preservation of the b&f equivalence $\rightleftharpoons$. We fix some $X, Y \in \mathfrak{X}$ and a b&f system $\mathcal{S}$ between X and Y in $\mathfrak{X}$, witnessing $X \rightleftharpoons Y$.

In order to construct a b&f system $\mathcal{S}'$ between HX and HY in $\mathcal{Y}$, witnessing $HX \rightleftharpoons HY$, we first define $\widetilde{\mathcal{S}}$ as the collection of minimal decompositions of path embeddings $HX \hookleftarrow P \rightarrowtail HY$ where the corresponding pair $X \hookleftarrow P_1 \rightarrowtail Y$ from the minimal decomposition is in $\mathcal{S}$.

Formally, $\widetilde{\mathcal{S}}$ is the collection of tuples (e_0, e_1, m_0, m_1) such that for some path P in $\mathcal{Y}$ and path P_1 in $\mathfrak{X}$

(i) $e_0 \colon P \rightarrowtail HP_1$, $m_0 \colon P \rightarrowtail HP_1$ are embeddings in $\mathcal{Y}$,
(ii) the pair of $e_1 \colon P_1 \rightarrowtail X$ and $m_1 \colon P_1 \rightarrowtail Y$ belongs to $\mathcal{S}$,
(iii) e_0 and e_1, and m_0 and m_1 constitute the minimal decompositions of $H(e_1)e_0$ and $H(m_1)m_0$, respectively.

Then, the desired b&f system is defined as:

$$\mathcal{S}' = \{(H(e_1)e_0,\ H(m_1)m_0) \mid (e_0, e_1, m_0, m_1) \in \widetilde{\mathcal{S}}\}$$

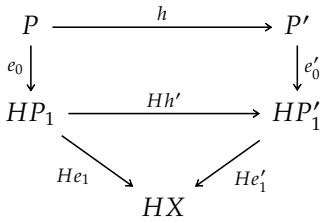
Observe that by (S1) this is a collection of path embeddings of the right type. Next we check that it also satisfies the axioms of b&f systems.

Lemma 8.4.2 *$\mathcal{S}'$ is a b&f system.*

Proof. We only check (forth) and not (back) since it follows the same lines. Fix a pair $(e, m) \in \mathcal{S}'$ and e' such that $e \prec e'$. We need to find some m' such that $m \prec m'$ and $(e', m') \in \mathcal{S}'$.

From $(e, m) \in \mathcal{S}'$ there is some (e_0, e_1, m_0, m_1) and P, P_1 satisfying (i)–(iii) above. Then, by (S2), e' splits as $He_1' \cdot e_0'$ for some minimal $e_1' \colon P_1' \rightarrowtail X$.

By minimality of $He_1 \cdot e$ there is some $h' \colon P_1 \to P'_1$ such that $e_1 = e'_1 h'$ such that



where $h \colon P \to P'$ is the morphisms witnessing $e \prec e'$. Since $(e_1, m_1) \in \mathcal{S}$ and $e_1 \leq e'_1$, there is some m'_1 such that $m_1 \leq m'_1$ and $(e'_1, m'_1) \in \mathcal{S}$.

We set

$$m' := \quad P' \xrightarrow{e'_0} HP'_1 \xrightarrow{Hm'_1} HY \tag{8.5}$$

and check that e'_0, m'_1 are the minimal decomposition of m'. Observe that this is enough for $(e'_0, e'_1, e'_0, m'_1) \in \widetilde{\mathcal{S}}$, to validate that $(e', m') \in \mathcal{S}'$.

Let $g_0 \colon P' \rightarrowtail HQ$ and $g_1 \colon Q \rightarrowtail Y$ be the minimal decomposition of (8.5). In particular $m' = Hg_1 \cdot g_0$. Minimality implies that there is some $r \colon Q \to P'_1$ such that $g_1 = m'_1 r$.

Since $e'_0 = Hr \cdot g_0$, as Hm'_1 is a monomorphism, we can consider a different decomposition of e' given by:

$$P' \xrightarrow{g_0} HQ \xrightarrow{Hr} HP'_1 \xrightarrow{He'_1} HX$$

Then, by minimality of e'_0, e'_1, there is some $s \colon P'_1 \rightarrowtail Q$ such that

$$e'_1 = e'_1 r s$$

from which it follows that $rs = \mathrm{id}$ since e'_1 is a mono. But then

$$m'_1 = m'_1 r s = g_1 s$$

which gives us minimality of (8.5).

Finally, we check (forth-init) and, again, leave out the symmetric case of (back-init). Assume there is some $e \colon P \rightarrowtail HX$ and take its minimal decomposition $e = He_1 \cdot e$ for some $e_1 \colon P_1 \rightarrowtail X$. By (forth-init) for $\mathcal{S}$, there is some $m_1 \colon P_1 \rightarrowtail Y$ such that $(e_1, m_1) \in \mathcal{S}$. Finally, we verify that $(e_0, e_1, e_0, m_1) \in \widetilde{\mathcal{S}}$ following exactly the same lines as above, which gives us that $(H(e_1)e_0,\ H(m_1)e_0)$ is in $\mathcal{S}'$. □

Since the choice of X and Y above was completely arbitrary, we obtain the following result.

Theorem 8.4.3 *Assuming that $H \colon \mathcal{X} \to \mathcal{Y}$ satisfies (S1), (S2) then, for any $X, Y \in \mathcal{X}$,*

$$X \rightleftharpoons Y \quad \textit{implies} \quad HX \rightleftharpoons HY.$$

Note that this theorem, when applied to the extended Kleisli lifting, implies the preservation of the $\equiv_{\mathbb{C}}$ relation, by item 3 of Theorem 8.2.1.

Preservation of (open) pathwise-embeddings

It turns out that the same axioms also guarantee preservation of $\Rrightarrow_{\exists\mathbb{C}}$ and $\Rrightarrow_{+\mathbb{C}}$, in case tree bijections are preserved too.

Theorem 8.4.4 *Any functor between elementary path categories satisfying (S1) and (S2) preserves pathwise-embeddings and open pathwise-embeddings.*

Proof. The proof of the first part follows precisely the same steps as [67, Lemma 5.15] and the second part is proved as in [67, Theorem 5.17]. The proofs therein are written for functors arising as extended Kleisli liftings but this assumption is never used. □

In a similar vein to Corollary 8.3.2, we also have the following consequence of Theorem 8.4.4 and Theorem 8.4.3.

Corollary 8.4.5 *Let H be a functor between elementary path categories satisfying (S1) and (S2). Then,*

1. *$X \sim Y$ implies $HX \sim HY$,*
2. *$X \leftrightarrows Y$ implies $HX \leftrightarrows HY$, and*
3. *$X \rightleftharpoons Y$ implies $HX \rightleftharpoons HY$.*

Simplifying the axioms

Axioms (S1) and (S2) give us sufficient conditions for functors between arbitrary path categories to preserve open pathwise-embeddings and b&f equivalences. In our case the elementary path categories of interests are categories of coalgebras for our game comonads and the functor in question is typically an extended Kleisli lifting $\widehat{H}$ of some functor between the base categories. In this subsection we describe more elementary conditions, written in terms of heteromorphisms, that imply axioms (S1) and (S2).

As in Section 8.2, we assume to have a functor $H\colon \mathcal{A} \to \mathcal{B}$ between the base categories which has an extended Kleisli lifting $\widehat{H}\colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$.

Proposition 8.4.6 ([67, Proposition 6.5]) *If H preserves embeddings then so does $\widehat{H}$.*

Finally, we state a condition ensuring that (S2) holds, which can be checked directly in the base categories, since it is specified using heteromorphisms.

Proposition 8.4.7 ([67, Proposition 6.7]) *The axiom (S2) is implied by the following axiom:*

(S2′) *For a path (P, π) in $\mathsf{EM}(\mathbb{D})$ and coalgebra (A, α) in $\mathsf{EM}(\mathbb{C})$, every heteromorphism $\pi \to_\kappa \alpha$ has a minimal decomposition through*

$$H(e)\colon H(P') \to H(A)$$

for some path embedding $e : (P', \pi') \rightarrowtail (A, \alpha)$.

Example 8.4.8 Recall from Example 7.2.6, the Kleisli law $\kappa_A : \mathbb{E}_k(A \times B) \to (\mathbb{E}_k A) \times B$ for the operation $H(A) = A \times B$. It is clear that the functor H preserves embeddings of $\mathcal{S}tr(\sigma)$. What is left to check is (S2′). Let (P, π) be a path and (A, α) some coalgebra in $\mathsf{EM}(\mathbb{E}_k)$, i.e. these are forest-ordered structures with respect to the induced forest order $\sqsubseteq_\alpha$, cf. Section 2.5.

Assume that we have a heteromorphism $\pi \to_\kappa \alpha$, as depicted below.

$$\begin{array}{ccccc} P & & \xrightarrow{\quad f \quad} & & A \times B \\ {\scriptstyle\pi}\downarrow & & & & \downarrow{\scriptstyle \alpha \times \mathrm{id}_B} \\ \mathbb{E}_k P & \xrightarrow{\mathbb{E}_k f} & \mathbb{E}_k(A \times B) & \xrightarrow{\kappa} & (\mathbb{E}_k A) \times B \end{array}$$

If P is of length n, i.e. $P = (x_1 \prec_\pi \cdots \prec_\pi x_n)$, we denote by $(a_i, b_i) = f(x_i)$ the pairs in the image by f. Since $\pi(x_i) = [x_1, \ldots, x_i]$, the bottom-to-right composition gives us $\kappa(\mathbb{E}_k(\pi(x_i))) = ([a_1, \ldots, a_i], b_i)$ where $[a_1, \ldots, a_i]$ is some branch of the forest $(A, \sqsubseteq_\alpha)$. We see that f factorises via $H(e)$ of the path embedding $e : (P', \pi') \rightarrowtail (A, \alpha)$ where P' represents the branch $a_1 \prec_\alpha \cdots \prec_\alpha a_n$ in $(A, \sqsubseteq_\alpha)$.

Example 8.4.9 The same strategy applies to show that our significant examples (such as those arising from comonad morphisms, cf. Subsection"Comonad morphisms" on page 156) and more satisfy (S1) and (S2) too, see e.g. [64] and [66, 67].

8.5 Strong path functors

In this section we look at a special type of functors $\mathcal{X} \to \mathcal{Y}$ which combine features of the functors studied in Sections 8.3 and 8.4. We dub this type of functors *strong functors*, and we view them as the notion of morphism between elementary path categories which guarantees strong preservation properties. Strong functors often arise from comonad morphisms between game comonads, as discussed in Subsection"Comonad morphisms" on page 156.

Path embeddings and conjugations

To this end, let us fix an adjunction between elementary path categories, as shown below.

$$\mathcal{Y} \underset{L}{\overset{R}{\underset{\top}{\rightleftarrows}}} \mathcal{X}$$

Recall that any morphism $f : Y \to RX$ decomposes via the unit $\eta_Y : Y \to RLY$ and the R-image of the conjugation $f^\flat : LY \to X$.

As we show next, this decomposition validates (S2) assuming **conjugation** $(\text{-})^\flat$ **preserves path embeddings**, i.e. whenever $f : P \rightarrowtail RX$ is a path embedding then so is $f^\flat$.[4]

4: Observe that the last condition requires two things at the same time. It requires that LP is a path and, also, that the transposed morphism $e^\flat$ is an embedding.
Note that in [67] it is shown that functors satisfying (S1) and (S2) are a type of parametric right adjoints and so, having an actual left adjoint is indeed a strengthening of the situation.

Proposition 8.5.1 *If, for every path embedding $e : P \rightarrowtail RX$, the conjugate $e^\flat : LP \to X$ is a path embedding then R satisfies (S2).*

Proof. We first show that a decomposition exists. Let $e : P \rightarrowtail R(X)$ be a path embedding. By our assumptions $e^\flat : L(P) \rightarrowtail X$ is a path embedding too. Then,

$$e = e^{\flat\#} = (L(P) \xrightarrow{\mathrm{id}} L(P) \xrightarrow{e^\flat})^\# = P \xrightarrow{\mathrm{id}^\#} R(L(P)) \xrightarrow{R(e^\flat)} R(X)$$

which proves that e factors through $R(e^\flat)$.

Next, we show that the decomposition above is minimal. Let $f : Q \rightarrowtail X$ be a path embedding such that e equals to $R(f)f_0$ for some $f_0 : P \rightarrowtail RQ$ and $f_0 : Q \rightarrowtail X$. We need to show that $e^\flat$ factors through f, i.e. $e^\flat = fg$ for some $g : LP \to Q$. This is satisfied for $g = f_0^\flat$ since

$$(L(P) \xrightarrow{f_0^\flat} Q \xrightarrow{f} X)^\# = P \xrightarrow{f_0} R(Q) \xrightarrow{Rf} R(X) = e$$

and, since $(\text{-})^\#$ and $(\text{-})^\flat$ are inverse to each other, $f f_0^\flat = e^\flat$. □

The next natural question is when the assumption that conjugation preserves embeddings is attained in our examples. In fact, we can often see this directly by checking the following simpler condition on the base category of our comonads.

Lemma 8.5.2 *Let $\mathbb{C}$ and $\mathbb{D}$ be regular comonads on categories $\mathcal{A}$ and $\mathcal{B}$ with proper factorisation systems, respectively. Furthermore, let there be an adjunction $L \dashv R : \mathcal{A} \leftrightarrows \mathcal{B}$ and a Kleisli law $\kappa : \mathbb{D}R \Rightarrow R\mathbb{C}$, such that*

1. *L^{EM} sends paths to paths, and*
2. *the conjugation $(B \xrightarrow{f} RA) \mapsto (LB \xrightarrow{f^\flat} A)$ preserves embeddings, and*
3. *κ is a collection of embeddings.*

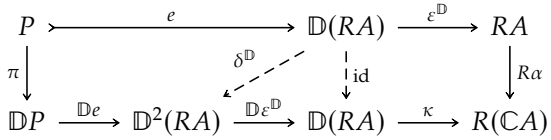
If the extended Kleisli lifting $\widehat{R}$ exists (as in Theorem 7.6.4) then, for every path embedding $\pi \rightarrowtail \widehat{R}(\alpha)$, its conjugate $L^{\mathsf{EM}}(\pi) \to \alpha$ (with respect to $L^{\mathsf{EM}} \dashv \widehat{R}$) is a path embedding.

Proof. Consider a path embedding $m : \pi \rightarrowtail \widehat{R}(\alpha)$, for some $\mathbb{C}$-coalgebra (A, α). Recall from the proof of Theorem 7.6.4 that the underlying morphism of $\widehat{m^\flat} : L^{\mathsf{EM}}(\pi) \to \alpha$ is computed as

$$\left(P \xrightarrow{m} \widehat{R}(\alpha) \xrightarrow{\iota_\alpha} \mathbb{D}(RA) \xrightarrow{\varepsilon^{\mathbb{D}}_{RA}} RA \right)^\flat$$

For the following we denote by e the composite of the first two morphisms, i.e. $e = \iota_\alpha \cdot m$. Recall that ι_α is an embedding, by Lemma 2.9.3 for the lifted factorisation system (Lemma 2.9.6), since it is an equaliser, and so e is an embedding too.

Furthermore, recall that $\varepsilon^{\mathbb{D}} \cdot e = u_\alpha \cdot m$ is a heteromorphism $\pi \to_\kappa \alpha$, hence the solid morphisms in the diagram below commute.



Since e is a composite of two $\mathbb{D}$-coalgebra morphisms, the left trapezium in the diagram above commutes. Therefore, we have that

$$R\alpha \cdot \varepsilon^{\mathbb{D}} \cdot e = \kappa \cdot \mathbb{D}\varepsilon^{\mathbb{D}} \cdot \mathbb{D}e \cdot \pi = \kappa \cdot \mathbb{D}\varepsilon^{\mathbb{D}} \cdot \delta^{\mathbb{D}} \cdot e \overset{(2.5)}{=} \kappa \cdot e.$$

Using that κ is an embedding we obtain that the right-hand side is an embedding. Consequently, also $\varepsilon^{\mathbb{D}} \cdot e = \varepsilon^{\mathbb{D}} \cdot \iota_\alpha \cdot m$ is an embedding too. Finally, we use our first two assumptions, to obtain that $\widehat{m^{\flat}}$ is a path embedding. □

Remark 8.5.3 Observe, in the proof above, the assumption that κ is a collection of embeddings could be replaced by the requirement that, for every path embedding $e : \pi \rightarrowtail F^{\mathbb{D}}(RA)$, the composite $\varepsilon^{\mathbb{D}} \cdot e$ is an embedding, and the same proof would still go through.

This observation together with Proposition 8.5.1 allows us to quickly infer that our extended Kleisli lifting satisfies (S2) in many of our examples.

Example 8.5.4 Checking that L^{EM} in our examples[5] sends paths to paths is usually immediate since its action on coalgebras is computed as a simple composition $L^{\mathsf{EM}}(\pi) = \lambda \cdot L\pi$ where λ is the mate of κ according to Proposition 7.6.2. See e.g. Example 7.1.3, Example 7.1.4 and Example 7.1.5.

5: And also λ^{EM}, in case of a comonad morphisms λ, cf. Subsection"Comonad morphisms" on page 156.

What is left to discuss is when the components of our comonad morphisms are embeddings. The following are examples of comonad morphisms or Kleisli laws where this is the case:

- The comonad morphisms $\mathbb{E}_k \Rightarrow \mathbb{E}_{k+1}$, $\mathbb{P}_k \Rightarrow \mathbb{P}_{k+1}$, $\mathbb{M}_k \Rightarrow \mathbb{M}_{k+1}$.
- The comonad morphism $\mathbb{E}_k \Rightarrow \mathbb{P}_k$.
- The Kleisli law $\mathbb{E}_k U \Rightarrow U\mathbb{E}_k^{\mathtt{MSO}}$, where $\mathbb{E}_k^{\mathtt{MSO}}$ is the comonad for interpreting MSOL, on two-sorted structures, from Example 2.8.1 and $U : \mathcal{S}tr(\sigma^+) \to \mathcal{S}tr(\sigma)$ the forgetful functor, which throws away the second sort and the extra relations I and e. It is easy to see that U has a left adjoint.

On the other hand, neither the comonad morphism $\mathbb{M}_k \Rightarrow \mathbb{P}_2$ nor $\mathbb{M}_k \Rightarrow \mathbb{E}_k$ from Example 7.2.5 and Example 7.5.8 satisfy the assumptions of Lemma 8.5.2 above since these are merely collections of monomorphisms, not embeddings.

All assumptions together

Apart from satisfying the assumptions of Lemma 8.5.2 above, we notice that in our examples one further property is true. We often (but not always[6]) have that the extended Kleisli lifting preserves paths, cf. Example 7.5.7. Therefore, we consider this as our final assumption when specifying strong functors.

6: Cf. Example 7.5.8.

We call a functor $R\colon \mathcal{X} \to \mathcal{Y}$ between elementary path categories a **strong path functor** if

- R preserves embeddings,[7]
- R sends paths to paths,
- R has a left adjoint L such that $(\text{-})^\flat$ preserves path embeddings.

7: Equivalently, it satisfies (S1).

These functors play a crucial role for preservation of equivalences of types, cf. Section 9.5 below. Since (S1) and (S2) are automatically satisfied for strong path functors, by Proposition 8.5.1, the following variant of Corollary 8.4.5 also holds for strong path functors.

Corollary 8.5.5 *Let R be a strong path functor between elementary path categories, then*

1. *$X \sim Y$ implies $RX \sim RY$,*
2. *$X \leftrightarrows Y$ implies $RX \leftrightarrows RY$, and*
3. *$X \rightleftharpoons Y$ implies $RX \rightleftharpoons RY$.*

Preservation of paths

We finish our detour of laying down the foundations for strong path functors by observing that the requirement that they preserve paths could be checked by examining properties of their left adjoint.

Lemma 8.5.6 *Let $L \dashv R$ be an adjunction $\mathcal{X} \leftrightarrows \mathcal{Y}$ between elementary path categories such that $(\text{-})^\flat$ preserves path embeddings. If $L\colon \mathcal{Y} \to \mathcal{X}$ is a full functor and if paths in $\mathcal{X}$ are given implicitly[8] then R sends paths to paths.*

8: Cf. Section 4.1.

Proof. We need to show that if $e\colon P \rightarrowtail RQ$ and $m\colon S \rightarrowtail RQ$ are path embeddings then either $e \leq m$ or $m \leq e$. Consider the conjugates $e^\flat\colon LP \rightarrowtail Q$ and $m^\flat\colon LS \rightarrowtail Q$. Since both are path embeddings and Q is a path, its subobjects are linearly ordered. Therefore, either $e^\flat \leq m^\flat$ or $m^\flat \leq e^\flat$. Without loss of generality we assume the former. Then, there is some $h\colon LP \to LQ$ such that $e^\flat = m^\flat h$. Since L is full there is some $f\colon P \to Q$ such that $Lf = h$. Then, $e = (e^\flat)^\# = (m^\flat \cdot Lf)^\# = m f$, which shows that $e \leq m$. □

Furthermore, achieving that the left adjoint is a full functor is again quite simple, since this is usually an EM lifting.

Lemma 8.5.7 *Let $\lambda\colon H\mathbb{D} \Rightarrow \mathbb{C}H$ be an EM law. If all components of λ are monomorphisms and H is fully faithful then the EM lifting H^{EM} of λ is fully faithful.*

Proof. Recall that for EM coalgebras $\alpha\colon A \to \mathbb{D}A$ and $\beta\colon B \to \mathbb{D}B$, $H^{\mathsf{EM}}(A,\alpha)$ is defined as $(A, \lambda H\alpha)$. If we have a morphism of coalgebras $f\colon H^{\mathsf{EM}}(A,\alpha) \to H^{\mathsf{EM}}(B,\beta)$ then, since H is fully faithful, there is a unique

$h\colon A \to B$ such that $H(h) = f$. Furthermore, since f is a coalgebra morphism, the following diagram of solid arrows commutes.

$$\begin{array}{ccc} HA & \xrightarrow{Hh} & HB \\ {\scriptstyle H\alpha}\downarrow & & \downarrow{\scriptstyle H\beta} \\ H\mathbb{D}A & \overset{H\mathbb{D}h}{\dashrightarrow} & H\mathbb{D}B \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ \mathbb{C}HA & \xrightarrow{\mathbb{C}Hh} & \mathbb{C}HB \end{array}$$

From the fact that λ is a natural transformation, the bottom square in the above diagram commutes. Now, because $\lambda\colon \mathbb{D}B \to \mathbb{C}B$ is a monomorphism also the top diagram commutes. Finally, since H is faithful, $H(\beta \cdot h) = H\beta \cdot Hh = H\mathbb{D}h \cdot H\alpha = H(\mathbb{D}h \cdot \alpha)$ implies $\beta \cdot h = \mathbb{D}h \cdot \alpha$, i.e. h is a morphism of coalgebras $(A, \alpha) \to (B, \beta)$. □

Remark 8.5.8 A corollary of the above lemma is that if $L \dashv R$ is a coreflection and the mate $L\mathbb{D} \Rightarrow \mathbb{C}L$ of a Kleisli law $\mathbb{D}R \Rightarrow R\mathbb{C}$ consists of monos then $L^{\mathsf{EM}} \dashv \widehat{R}$ is a coreflection too. In particular, this follows anytime when we have a monomorphic comonad morphisms $\mathbb{D} \Rightarrow \mathbb{C}$.

Recall that extended Kleisli liftings of operations which preserve embeddings also preserve embeddings, by Proposition 8.4.6. This holds, in particular for extended Kleisli liftings of the identity functor by comonad morphisms. With that in mind, our results in this section give us the following.

Theorem 8.5.9 *Let $\lambda\colon \mathbb{D} \Rightarrow \mathbb{C}$ be a morphism of regular comonads on a category with a proper factorisation system. If*

- *λ^{EM} sends paths to paths,*
- *λ consists of embeddings, and*
- *paths in $\mathsf{EM}(\mathbb{C})$ are given implicitly,*

then the extended Kleisli lifting $\widehat{\lambda}$ (if it exists) is a strong path functor.

Proof. First, we observe that the assumptions of Lemma 8.5.2 are satisfied when instantiated with the adjunction $\mathrm{Id} \dashv \mathrm{Id}$. In particular, the conjugation of the adjunction $\mathrm{Id} \dashv \mathrm{Id}$ trivially preserves embeddings. Therefore, the conjugation of the lifted adjunction[9] $\lambda^{\mathsf{EM}} \dashv \widehat{\lambda}$ preserves path embeddings.

9: Cf. Subsection "Comonad morphisms" on page 156.

Next, by Lemma 8.5.7, λ^{EM} is fully faithful, since all embeddings are monos. Therefore, by Lemma 8.5.6, $\widehat{\lambda}$ sends paths to paths. Finally, the fact that $\widehat{\lambda}$ preserves embeddings follows from Proposition 8.4.6 since Id trivially preserves embeddings. □

8.6 Logic extensions and Kleisli liftings

Recall from Section 2.8 that a powerful way to capture further logic fragments is by using translation functors. From this point of view a logic

is given by a pair $(\mathbb{C}, \mathsf{t})$ shown below

$$\mathscr{C} \xrightarrow{\mathsf{t}} \mathscr{A} \circlearrowright \mathbb{C}$$

where $\mathsf{t}\colon \mathscr{C} \to \mathscr{A}$ is a (translation) functor and $\mathbb{C}$ a comonad on $\mathscr{A}$.

In this chapter we have described the conditions under which a functor between base categories $H\colon \mathscr{A} \to \mathscr{B}$ preserves logical equivalences and refinements, i.e. satisfies assumptions of Corollary 8.1.1 and Theorem 8.2.1. Next we take a look at how to adapt these results so that they still apply to logics parameterised by a translation functor.

Preserving $\equiv_{\#\mathbb{C}}$ and $\Rrightarrow_{\exists^+\mathbb{C}}$. As discussed in Section 8.1, without the presence of translation functors, we only require a Kleisli law between the two comonads which, by functoriality of their Kleisli lifting, gives us the preservation theorems.

Given logics specified by pairs $(\mathbb{C}, \mathsf{t})$ and $(\mathbb{C}', \mathsf{t}')$, observe that if there is a pair of functors F, H such that the diagram of solid arrows below commutes

$$\begin{array}{ccccc} \mathscr{C} & \xrightarrow{\mathsf{t}} & \mathscr{A} & \xrightarrow{F_{\mathbb{C}}} & \mathsf{Kl}(\mathbb{C}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H^{\mathsf{Kl}}} \\ \mathscr{C}' & \xrightarrow{\mathsf{t}'} & \mathscr{A}' & \xrightarrow{F_{\mathbb{D}}} & \mathsf{Kl}(\mathbb{D}) \end{array}$$

and such that there is a Kleisli law $\mathbb{C}'H \Rightarrow H\mathbb{C}$ then

$$\mathsf{t}A \equiv_{\#\mathbb{C}} \mathsf{t}B \quad \text{implies} \quad \mathsf{t}'F(A) \equiv_{\#\mathbb{D}} \mathsf{t}'F(B)$$

or, equivalently, that

$$F_{\mathbb{C}}(\mathsf{t}A) \cong F_{\mathbb{C}}(\mathsf{t}B) \quad \text{implies} \quad F_{\mathbb{D}}(\mathsf{t}'F(A)) \cong F_{\mathbb{D}}(\mathsf{t}'F(B)).$$

Indeed, by Theorem 7.2.1, there is a Kleisli lifting $H^{\mathsf{Kl}}\colon \mathsf{Kl}(\mathbb{C}) \to \mathsf{Kl}(\mathbb{D})$ giving us $H^{\mathsf{Kl}}(F_{\mathbb{C}}(\mathsf{t}A)) = F_{\mathbb{D}}(\mathsf{t}'F(A))$ and similarly for B.

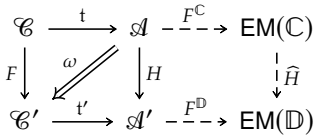
The argument for preservation of $\Rrightarrow_{\exists^+\mathbb{C}}$ follows similar lines.

Example 8.6.1 Consider the functor $V\colon \mathcal{S}tr(\sigma \cup \tau) \to \mathcal{S}tr(\sigma)$ which sends a $\sigma \cup \tau$-structure to its σ-reduct. It is immediate to see that the collection of identity functions induces a Kleisli law $\kappa\colon \mathbb{E}_k V \Rightarrow V\mathbb{E}_k$ where the first instance of $\mathbb{E}_k$ is with respect to σ-structures and the second with respect to $\sigma \cup \tau$-structures. Consider the following situation

$$\begin{array}{ccccc} \mathcal{S}tr(\sigma \cup \tau) & \xrightarrow{(\text{-})^I} & \mathcal{S}tr(\sigma^I \cup \tau) & \xrightarrow{F_{\mathbb{E}_k}} & \mathsf{Kl}(\mathbb{E}_k) \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle W^{\mathsf{Kl}}} \\ \mathcal{S}tr(\sigma) & \xrightarrow{(\text{-})^I} & \mathcal{S}tr(\sigma^I) & \xrightarrow{F_{\mathbb{E}_k}} & \mathsf{Kl}(\mathbb{E}_k) \end{array}$$

where $(\text{-})^I$ are the 'translation functors' extending the signature with a fresh binary relation, which they interpret as equality, as in Section 2.8, and W computes the σ^I-reduct out of a $\sigma^I \cup \tau$-structure. As a consequence, we obtain the trivial fact that $A \equiv_{\exists^+\mathrm{FO}^=_k} B$ implies $VA \equiv_{\exists^+\mathrm{FO}^=_k} VB$.

Preserving $\equiv_{\mathbb{C}}$. When considering preservation of the standard fragments, the situation adapts accordingly. In fact, we can weaken some of the assumptions slightly by assuming a natural transformation ω, as shown in the diagram below, instead of strict commutativity.



In order to apply our results from Section 8.4 assume that we have a Kleisli law $\kappa\colon \mathbb{D}H \Rightarrow H\mathbb{C}$ such that the extended Kleisli lifting $\widehat{H}$ exists and that it satisfies (S1) and (S2).

Observe that, if $F^{\mathbb{D}}(\omega_A)$ is an open pathwise-embedding for every $A \in \mathscr{A}$, then we recover that

$$\mathrm{t}A \equiv_{\mathbb{C}} \mathrm{t}B \quad \text{implies} \quad \mathrm{t}'F(A) \equiv_{\mathbb{D}} \mathrm{t}'F(B)$$

or, equivalently, that

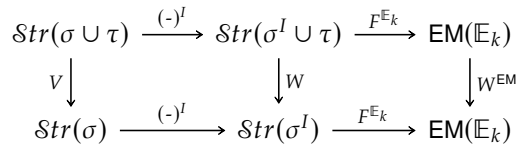
$$F^{\mathbb{C}}(\mathrm{t}A) \rightleftharpoons F^{\mathbb{C}}(\mathrm{t}B) \quad \text{implies} \quad F^{\mathbb{D}}(\mathrm{t}'F(A)) \rightleftharpoons F^{\mathbb{D}}(\mathrm{t}'F(B)).$$

To see that, recall that $\widehat{H}$ preserves b&f equivalence $\rightleftharpoons$, by Theorem 8.4.3. Therefore, from $\mathrm{t}A \equiv_{\mathbb{C}} \mathrm{t}B$ we have the following situation

$$\begin{array}{ccccc} F^{\mathbb{D}}(H(\mathrm{t}A)) & = & \widehat{H}(F^{\mathbb{C}}(\mathrm{t}A)) & \rightleftharpoons & \widehat{H}(F^{\mathbb{C}}(\mathrm{t}B)) \;=\; F^{\mathbb{D}}(H(\mathrm{t}B)) \\ {\scriptstyle F^{\mathbb{D}}(\omega_A)}\downarrow & & & & \downarrow{\scriptstyle F^{\mathbb{D}}(\omega_B)} \\ F^{\mathbb{D}}(\mathrm{t}'F(A)) & & & & F^{\mathbb{D}}(\mathrm{t}'F(B)) \end{array}$$

where the vertical morphisms are open pathwise-embeddings by our assumptions. Then, by Lemma 3.4.2, each of these morphisms induces a b&f equivalence witnessing $F^{\mathbb{D}}(\mathrm{t}'F(A)) \rightleftharpoons F^{\mathbb{D}}(H(\mathrm{t}A))$ and similarly for B. Since b&f equivalences compose (Lemma 3.3.4) we obtain the required fact that $F^{\mathbb{D}}(\mathrm{t}'F(A))$ is b&f equivalent to $F^{\mathbb{D}}(\mathrm{t}'F(B))$.

Example 8.6.2 Building on Example 8.6.1, since the extended Kleisli lifting W^{EM} of W exists, we have the following commutative diagram.



Checking axioms (S1) and (S2) using tools from Subsection"Simplifying the axioms" on page 166 is immediate, which gives us the other trivial fact that $A \equiv_{\mathtt{FO}^{=}_k} B$ implies $VA \equiv_{\mathtt{FO}^{=}_k} VB$.

A similar approach can be used to show that $\equiv_{\mathtt{MSO}^{=}_k}$ implies $\equiv_{\mathtt{FO}^{=}_k}$ using the Kleisli law $\mathbb{E}_k U \Rightarrow U\mathbb{E}_k^{\mathtt{MSO}}$ constructed in Example 8.5.4.

8.7 Beyond unary operations

10: In fact, the arity of H need not even be finite.

Recall from Section 7.4 that an n-ary operation[10] $H\colon \prod_i \mathcal{A}_i \to \mathcal{B}$ is the same as an unary operation from the product category. Therefore, the following is just a rephrasing of Corollary 8.1.1.

Corollary 8.7.1 *If $H\colon \prod_i \mathcal{A}_i \to \mathcal{B}$ admits a Kleisli law $\mathbb{D}H \Rightarrow H \circ \prod_i \mathbb{C}_i$ then*

$$\big(\forall i \in I,\ \ A_i \Rrightarrow_{\exists^+\mathbb{C}_i} B_i\big) \quad \textit{implies} \quad H(\vec{A}_i) \Rrightarrow_{\exists^+\mathbb{D}} H(\vec{B}_i)$$

and

$$\big(\forall i \in I,\ \ A_i \equiv_{\#\mathbb{C}_i} B_i\big) \quad \textit{implies} \quad H(\vec{A}_i) \equiv_{\#\mathbb{D}} H(\vec{B}_i).$$

In case $H\colon \prod_i \mathcal{X}_i \to \mathcal{Y}$ is a functor where each $\mathcal{X}_i$ and $\mathcal{Y}$ are elementary path categories, in order to be able to reuse the conditions for unary operations under which functors preserves the back-and-forth equivalence, we must specify the elementary path structure of product categories.

To this end, define the embeddings of $\prod_i \mathcal{X}_i$ to be the tuples of morphisms $(f_i)_i$ where each $f_i \in \mathcal{X}_i$ is an embedding and define paths of $\prod_i \mathcal{X}_i$ to be tuples of objects $(A_i)_i$ such that each A_i is a path in $\mathcal{X}_i$.[11]

11: Observe that even if the paths in each $\mathcal{X}_i$ are given implicitly then this is no longer the case for $\prod_i \mathcal{X}_i$.

Unpacking Theorem 8.4.3 for operations of arbitrary arity yields the following, where we implicitly assume that categories $\mathcal{X}_i$ and $\mathcal{Y}$ are elementary path categories.

Corollary 8.7.2 *Assume that $H\colon \prod_i \mathcal{X}_i \to \mathcal{Y}$ satisfies*

- *If $\vec{f} = (f_i)_i$ is a collection of embeddings then so is $H(\vec{f})$ and*
- *Any path embedding $e\colon P \rightarrowtail H(\vec{X})$ has a minimal decomposition as $e_0\colon P \to H(\vec{P})$ followed by $H(\vec{e})\colon H(\vec{P}) \to H(\vec{X})$ for a collection of path embeddings $\vec{e} = (e_i\colon P_i \rightarrowtail X_i)_i$.*

Then, for any $\vec{X}, \vec{Y} \in \prod_i \mathcal{X}_i$,

$$(\forall i \in I,\ \ X_i \rightleftharpoons Y_i) \quad \textit{implies} \quad H(\vec{X}) \rightleftharpoons H(\vec{Y}).$$

Finally, Corollaries 8.7.2 and 7.4.1 together with Propositions 8.4.6 and 8.4.7 give the following conditions which are easy to check. As before, we implicitly assume that the categories of coalgebras $\mathsf{EM}(\mathbb{C}_i)$ and $\mathsf{EM}(\mathbb{D})$ are elementary path categories.

Corollary 8.7.3 ([67, Theorem 6.9]) *Assume that $H\colon \prod_i \mathcal{A}_i \to \mathcal{B}$ admits a Kleisli law $\mathbb{D}H \Rightarrow H \circ \prod_i \mathbb{C}_i$ such that the following holds.*

- *$\mathsf{EM}(\mathbb{D})$ has equalisers of coreflexive pairs.*
- *If $\vec{f} = (f_i)_i$ is a collection of embeddings then so is $H(\vec{f})$.*
- *For any path (P, π) in $\mathsf{EM}(\mathbb{D})$ and coalgebras (A_i, α_i) in $\mathsf{EM}(\mathbb{C}_i)$ (for every i), any heteromorphism $\pi \to_\kappa \vec{\alpha}$ has a minimal decomposition through*

$$H(\vec{e})\colon H(\vec{\pi}) \to H(\vec{A}),$$

for a collection of path embeddings $\vec{e} = (e_i\colon (P_i, \pi_i) \rightarrowtail (A_i, \alpha_i))_i$.

Then,

$$\big(\forall i \in I,\ \ A_i \equiv_{\mathbb{C}_i} B_i\big) \quad \textit{implies} \quad H(\vec{A}_i) \equiv_{\mathbb{D}} H(\vec{B}_i).$$

Example 8.7.4 We use Corollary 8.7.3 to recover the original Feferman–Vaught theorem about coproducts, which we rephrase for the fragment of restricted quantifier rank:

$$A_1 \equiv_{\mathrm{FO}_k} B_1 \quad \text{and} \quad A_2 \equiv_{\mathrm{FO}_k} B_2 \quad \text{implies} \quad A_1 + A_2 \equiv_{\mathrm{FO}_k} B_1 + B_2$$

To this end, denote by $+\colon \mathcal{S}tr(\sigma) \times \mathcal{S}tr(\sigma) \to \mathcal{S}tr(\sigma)$ the operation of taking the binary coproduct (disjoint union) of σ-structures. Recall from Example 7.4.2 that we have a Kleisli law $\kappa_{A_1,A_2}\colon \mathbb{E}_k(A_1 + A_2) \to \mathbb{E}_k A_1 + \mathbb{E}_k A_2$ sending a word $\overline{a} = [a_1, \ldots, a_n]$ to the subword $[a_{i_1}, \ldots, a_{i_m}]$ where i_j's list all the positions of letters in $\overline{a}$ that are from the same side of the coproduct $A_1 + A_2$ as a_n.

We know that $\mathsf{EM}(\mathbb{E}_k)$ has equalisers of all pairs (in fact, it is complete) and it is immediate to observe that the operation $+$ preserves embeddings. What is left to check that morphisms from paths decompose via minimal path embeddings. Consider a bimorphism $\pi \to_\kappa (\alpha_1, \alpha_2)$ as shown below.

$$\begin{array}{ccccc} P & & \xrightarrow{\qquad\qquad f \qquad\qquad} & & A_1 + A_2 \\ {\scriptstyle\pi}\downarrow & & & & \downarrow{\scriptstyle\alpha_1+\alpha_2} \\ \mathbb{E}_k P & \xrightarrow{\mathbb{E}_k f} & \mathbb{E}_k(A_1 + A_2) & \xrightarrow{\kappa} & (\mathbb{E}_k A_1) + (\mathbb{E}_k A_2) \end{array}$$

As in Example 7.5.9 we see that, if P is a chain $(x_1 \prec \cdots \prec x_n)$ then f selects out some sequence $\overline{a} = [a_1, \ldots, a_n]$ where the subwords $[a_{i_1}, \ldots, a_{i_m}]$ and $[a_{j_1}, \ldots, a_{j_r}]$ of $\overline{a}$ restricted to the positions where a letter is either from A_1 or A_2, respectively, form branches of $(A_1, \sqsubseteq_{\alpha_1})$ and $(A_2, \sqsubseteq_{\alpha_2})$, respectively. Hence, f factors via $e_1 + e_2$ where each $e_1\colon (P_1, \pi_1) \rightarrowtail (A_1, \alpha_1)$ is the path inclusion of the branch given by $[a_{i_1}, \ldots, a_{i_m}]$ and, similarly, for $e_2\colon (P_2, \pi_2) \rightarrowtail (A_2, \alpha_2)$ and $[a_{j_1}, \ldots, a_{j_r}]$.

8.8 Product preservation theorems for free

For checking of preservation of products (and hence also the original Mostowski theorem) we device an even simpler procedure.

Fix an arbitrary comonad $\mathbb{C}$ on a category $\mathcal{A}$ with products. Observe that, for a family of objects $\{A_i\}_i$ of $\mathcal{A}$, indexed by a set I, we have morphisms $\mathbb{C}(\pi_i)\colon \mathbb{C}(\prod_i A_i) \to \mathbb{C}(A_i)$ for each $i \in I$ and, by the universal property of products, also the morphism

$$\kappa_{\vec{A}}\colon \mathbb{C}(\textstyle\prod_i A_i) \to \prod_i \mathbb{C}(A_i).$$

One can show that κ is a Kleisli law [67, Lemma 8.2], hence we automatically have the following by Corollary 8.1.1.

Corollary 8.8.1 *For collections of σ-structures $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$, indexed*

by a common set I, and a comonad $\mathbb{C}$ on a category with products,

$$\big(\forall i \in I,\ \ A_i \Rrightarrow_{\exists^+\mathbb{C}_i} B_i\big) \quad \textit{implies} \quad \prod_i A_i \Rrightarrow_{\exists^+\mathbb{C}_i} \prod_i B_i$$

and

$$(\forall i \in I,\ \ A_i \equiv_{\#\mathbb{C}} B_i) \quad \textit{implies} \quad \prod_i A_i \equiv_{\#\mathbb{C}} \prod_i B_i.$$

One can check that products preserve embeddings [67, Lemma 8.4] and, furthermore, that also (S2′) is automatically satisfied if we have ($\mathrm{P}_{\twoheadrightarrow}$). Hence, we obtain the following generalisation of Mostowski's theorem.

Theorem 8.8.2 ([67, Theorem 8.6]) *Let $\mathbb{C}$ be a comonad on a category with products such that $\mathsf{EM}(\mathbb{C})$ is an elementary path category. If $\mathsf{EM}(\mathbb{C})$ has equalisers and satisfies ($\mathrm{P}_{\twoheadrightarrow}$) then*

$$(\forall i \in I,\ \ A_i \equiv_{\mathbb{C}} B_i) \quad \textit{implies} \quad \prod_i A_i \equiv_{\mathbb{C}} \prod_i B_i.$$

Recall that if a comonad $\mathbb{C}$ on $\mathcal{S}tr(\sigma)$ or $\mathcal{S}tr_*(\sigma)$ preserves embeddings then $\mathsf{EM}(\mathbb{C})$ is complete (Proposition 2.9.9). Moreover, any time $\mathsf{EM}(\mathbb{C})$ is arboreal then it also satisfies ($\mathrm{P}_{\twoheadrightarrow}$) (Theorem 4.3.2). The following are examples of this situation.

Example 8.8.3 We have that

$$(\forall i \in I,\ \ A_i \equiv_{\mathscr{L}} B_i) \quad \text{implies} \quad \prod_i A_i \equiv_{\mathscr{L}} \prod_i B_i.$$

for $\mathscr{L} = \mathtt{FO}_k, \mathscr{L}^k, \mathtt{FO}^k_n, \mathtt{ML}_k$ by instantiating the above theorem with comonads $\mathbb{E}_k, \mathbb{P}_k, \mathbb{P}_{k,n}, \mathbb{M}_k$ where $\mathtt{FO}^k_n$ is the fragment of $\mathtt{FO}_n$ consisting of k-variable formulas of quantifier rank $\leq n$.

9 Model-Theoretic Types

In this chapter we give a categorical treatment of model theoretic types, using the machinery we built in the previous chapters. To do so we define an abstract notion of variable assignment which allows us to lift comonads to structures with constants, define types, formulas and prove preservation theorems for logical equivalences of assignments, similar to the theorems Theorem 8.4.3. Our approach relies on the machinery of strong path functors introduced in Section 8.5.

9.1 Motivation 177
9.2 Assignments 178
The coreflection 179
Elementary path category structure of $\mathfrak{X}_P$ 181
Related work 182
9.3 Structured and unstructured assignments 182
Equivalence of types . . . 183
Adjunctions and comonads for structures with constants 183
9.4 Formulas and quantification 185
9.5 Preservation for assignments 186
Comparing standard logic fragments and their formulas 191

9.1 Motivation

Recall that, for a fragment $\mathcal{L}$ of first-order *formulas* (and not just sentences), and for a structure A and an n-tuple $\overline{a} \in A$, the set of formulas $\mathtt{tp}(\overline{a}) = \{\varphi(\overline{x}) \in \mathcal{L} \mid A \models \varphi(\overline{a})\}$ is the n-**type** of $\overline{a}$ (with respect to $\mathcal{L}$). Types are an important notion not only in finite model theory [47, 80] but also in classical model theory [36, 60].

For example, let $\mathcal{L} = \mathtt{FO}^{=}_{k}[n]$ be the set of formulas (with equality) of quantifier rank $\leq k$, with free variables among $x_1, \ldots, x_n$, then the n-type $\mathtt{tp}(\overline{a})$ encodes all properties of $\overline{a}$ expressible with formulas from $\mathtt{FO}^{=}_{k}[n]$. Now consider a different n-tuple $\overline{b}$ of some structure B. When is $\mathtt{tp}(\overline{a}) = \mathtt{tp}(\overline{b})$? This is precisely when the structures A, B extended with constants[1] assigned to $\overline{a}$ and $\overline{b}$, respectively, are $\mathtt{FO}^{=}_{k}[n]$-equivalent, i.e. whenever $(A, \overline{a}) \equiv_{\mathtt{FO}^{=}_{k}[n]} (B, \overline{b})$. Using the same reasoning as in the proof of Theorem 2.1.1, the latter is precisely when Duplicator has a winning strategy in the k-round Ehrenfeucht–Fraïssé game where, after playing the additional k moves $a_{n+1}, \ldots, a_{n+k}$ and $b_{n+1}, \ldots, b_{n+k}$, the mapping $a_i \mapsto b_i$ (for $i \in \{1, \ldots, n+k\}$) defines a partial isomorphism.

We see that the equivalence of types of $\overline{a}$ and $\overline{b}$ can be expressed as the b&f equivalence of subtrees of the Ehrenfeucht–Fraïssé comonad $\mathbb{E}_{k+n}$. Indeed, the n-tuple $\overline{a}$ can be viewed as an element of $\mathbb{E}_{k+n}(A)$, as the sequence $[a_1, \ldots, a_n]$. Then, the possible moves $a_{n+1}, \ldots, a_{n+k}$ in the game correspond to extensions $\overline{a}'$ of $\overline{a}$, i.e. sequences of the form $[a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}]$ of $\mathbb{E}_{k+n}(A)$. To this end, we view $\mathtt{tp}(\overline{a})$ as a subtree of $\mathbb{E}_{k+n}(A)$, in the prefix order, consisting of all extensions of $\overline{a}$. Notice that $\mathtt{tp}(\overline{a})$ with the induced structure from $\mathbb{E}_{k+n}(A)$ is forest-ordered structure and, hence, is an object of $\mathsf{EM}(\mathbb{E}_{k+n})$.[2]

We see that the n-types of $\overline{a}$ and $\overline{b}$ are equal, in the model-theoretic sense, iff we have $\mathtt{tp}(\overline{a}) \rightleftharpoons \mathtt{tp}(\overline{b})$ in $\mathsf{EM}(\mathbb{E}_{k+n})$ or, equivalently, using the simplification operation $(\text{-})/_{\sim}$ on $\mathsf{EM}(\mathbb{E}_{k+n})$ from Theorem 4.8.8, iff $\mathtt{tp}(\overline{a})/_{\sim} \cong \mathtt{tp}(\overline{b})/_{\sim}$.

In fact, one can view $\mathtt{tp}(\overline{a})/_{\sim}$ as the genuine representation of the model-theoretic n-type of $\overline{a}$, since it fully encodes the information about the type without any redundancy. However, in the setting of elementary path or arboreal categories we prefer to work with the full subtree of all extensions of $\overline{a}$ as this is technically easier.

1: More formally, for a $\varphi \in \mathtt{FO}_k[n]$, $(A, \overline{a}) \models \varphi$ iff $A \models \varphi(\overline{a})$.

2: Assuming the concrete representation of $\mathbb{E}_{k+n}$-coalgebras from Section 2.5.

9.2 Assignments

In this section we abstract away from the concrete representation of types as sequences and, instead, we define types in any path category $\mathfrak{X}$ which satisfies ($\mathrm{PE}_{\mathrm{mn}}$) and ($\mathrm{PE}_{\vee}$).

Observe that the role of sequences $\overline{a} = [a_1, \ldots, a_n]$ in Section 9.1 is to represent *assignments* for free variables $x_1, \ldots, x_n$, i.e. mappings $v\colon \{x_1, \ldots, v_n\} \to A$. In case of $\mathbb{E}_{k+n}$-coalgebras, picking such sequence $\overline{a}$ is equivalent to picking a morphism $P \to F^{\mathbb{E}_k}(A)$ from a path P of height n. However, sequences of length n are in a bijection with morphism $P \to F^{\mathbb{E}}_{k+n}(A)$ in $\mathsf{EM}(\mathbb{E}_{k+n})$ if and only if P is discrete as a σ-structure. Indeed, if e.g. $R^P(x_3, x_1, x_5)$ holds in P, then morphisms $P \to F^{\mathbb{E}_k}(A)$ correspond to the assignments $v\colon \{x_1, \ldots, v_n\} \to A$ where $R^A(v(x_3), v(x_1), v(x_5))$ in A.

Although it is possible to specify paths that are discrete within an elementary path category,[3] it bears no technical advantage for us here. Therefore, we parametrise assignments by an arbitrary path P in $\mathfrak{X}$.

3: E.g. we say that P is *discrete* if it is projective with respect to quotients, i.e. whenever $f\colon P \to Y$ and there is a quotient $q\colon X \twoheadrightarrow Y$ then there is some $f'\colon P \to X$ such that $f = q \cdot f'$.

An **assignment** is a morphism

$$x\colon P \to X$$

in $\mathfrak{X}$ from a path P, representing a sequence of free variables.[4] By ($\mathrm{PE}_{\vee}$) we define the **type** given by the assignment x as the subobject (embedding)

$$\mathtt{tp}(x) \xrightarrow{\mathtt{i}_x} X$$

where $\mathtt{tp}(x)$ is the colimit of path embeddings $P' \rightarrowtail X$ such that x factorises as a composition $P \to P' \rightarrowtail X$.

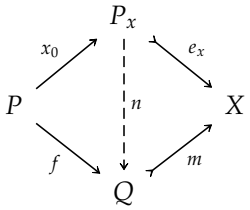
4: Notice that we do not require x to be an embedding nor do we require anything about X, although in our usual examples X is of the form $F^{\mathbb{C}}(A)$ for some A and some comonad $\mathbb{C}$.

Extensions of sequences. To see that $\mathtt{tp}(x)$ indeed represents the subtree of X consisting of the paths extending x, as in Section 9.1, consider the minimal path-embedding factorisation of x, by ($\mathrm{PE}_{\mathrm{mn}}$).

$$x \;=\; P \xrightarrow{x_0} P_x \xrightarrow{e_x} X \tag{9.1}$$

The path embedding e_x above can be viewed as the position in the back-and-forth game, induced by the assignment x.

We see that $e_x\colon P_x \rightarrowtail X$ is one of the path embeddings in the diagram defining $\mathtt{tp}(x)$. Moreover, e_x is the least such path embedding. Consider any other decomposition of x into some $f\colon P \to Q$ followed by an embedding $m\colon Q \rightarrowtail P$. Then, by minimality of e_x, there is a unique (embedding) $n\colon P_x \to P'$ such that the triangle on the right below and, therefore, also the diagram on the left below, commutes.



(9.2)

Remark 9.2.1 The above shows that $\mathtt{tp}(x)$ can be equivalently defined as the colimit of path embeddings $P' \rightarrowtail X$ such that e_x factorises as a composition $P_x \rightarrowtail P' \rightarrowtail X$.

The coreflection

Observe that there is a morphism $x^{\uparrow} : P \to \mathtt{tp}(x)$, intuitively representing the enumeration of the variables given by P in the type $\mathtt{tp}(x)$. The morphism $x^{\uparrow}$ is defined as the composition of $x_0 : P \to P_x$ from (9.1) with the colimit inclusion $i : P_x \to \mathtt{tp}(x)$ corresponding to $e_x : P_x \rightarrowtail X$. In other words, the inclusion i is the unique morphism such that $e_x = \mathtt{i}_x \cdot i$. Consequently, we obtain that x also factorises as follows.

$$x \;=\; P \xrightarrow{x^{\uparrow}} \mathtt{tp}(x) \xrightarrowtail{\mathtt{i}_x} X \tag{9.3}$$

The equation (9.3) exposes the coreflective nature of the construction of types. We can view the construction of types as an operation on the comma category $P \downarrow \mathfrak{X}$. On objects, this operation sends an assignment $x : P \to X$ to a new assignment $x^{\uparrow}$. Furthermore, $\mathtt{i}_x$ is a morphism $x^{\uparrow} \to x$ in $P \downarrow \mathfrak{X}$. In the following lines we show that this is a coreflection.

First, we show that the operation $(\text{-})^{\uparrow}$ is idempotent, i.e. that applying the same construction twice yields an isomorphic object in $P \downarrow \mathfrak{X}$.

Lemma 9.2.2 *Given an assignment x, the morphism $\mathtt{i}_{x^{\uparrow}} : \mathtt{tp}(x^{\uparrow}) \rightarrowtail \mathtt{tp}(x)$ is an isomorphism.*

Proof. Formally, the diagram $\mathscr{D}_1$ that defines $\mathtt{tp}(x)$ consists of triples (f, Q, m), with Q a path and m an embedding, such that x decomposes as follows.

$$P \xrightarrow{f} Q \xrightarrowtail{m} X$$

Also, morphisms of triples $(f, Q, m) \to (f', Q', m')$ are morphisms $h : Q \to Q'$ such that $m = m'h$, which also implies that $f' = hf$ since m is a monomorphism and $mf = x = m'f'$.

By definition, $\mathtt{tp}(x)$ is the colimit for the functor that projects out the path $(f, Q, m) \mapsto Q$. The definition of the diagram $\mathscr{D}_2$ that defines $\mathtt{tp}(x^{\uparrow})$ is similar, it consists of triples (f, Q, m) where

$$x^{\uparrow} \;=\; P \xrightarrow{f} Q \xrightarrowtail{m} \mathtt{tp}(x).$$

We show that the two diagrams defining $\mathtt{tp}(x)$ and $\mathtt{tp}(x^{\uparrow})$ are the same. As the map $\mathscr{D}_1 \to \mathscr{D}_2$ we send (f, Q, m) to (f, Q, v_m) where $v_m : Q \to \mathtt{tp}(x)$ is defined as the colimit inclusion, satisfying the following.

$$\begin{array}{ccc} Q & \xrightarrowtail{m} & X \\ & \searrow^{v_m} & \uparrow{\scriptstyle \mathtt{i}_x} \\ & & \mathtt{tp}(x) \end{array}$$

The commutativity of the above diagram gives us that v_m is an embedding. Therefore, the mapping $\mathcal{D}_1 \to \mathcal{D}_2$ is well-defined. The fact that this is functorial follows from the fact that inclusion morphisms form a cocone.

Conversely, we define a mapping $\mathcal{D}_2 \to \mathcal{D}_1$ by mapping (f, Q, m) to $(f, Q, \mathtt{i}_x\, m)$. This is clearly functorial by definition.

We now check that these two maps are inverse to each other. Starting from (f, Q, m) in $\mathcal{D}_1$, by mapping it to $\mathcal{D}_2$ and back to $\mathcal{D}_1$, we obtain $(f, Q, \mathtt{i}_x\, v_m)$ which is equal to (f, Q, m) by the above triangle.

Conversely, for (f, Q, m) in $\mathcal{D}_2$ and mapping it to $\mathcal{D}_1$ and back, we obtain $(f, Q, v_{\mathtt{i}_x\, m})$. The commutativity of the above triangle for $v_{\mathtt{i}_x\, m}$ gives us that $\mathtt{i}_x \cdot v_{\mathtt{i}_x\, m} = \mathtt{i}_x\, m$ which by the fact that $\mathtt{i}_x$ is a monomorphism gives us the intended equality, i.e. that $v_{\mathtt{i}_x\, m} = m$. □

Finally, we define

$$\mathcal{X}_P$$

as the full subcategory of $P \downarrow \mathcal{X}$ on assignments x which satisfy that $\mathtt{i}_x$ is an isomorphism.

Theorem 9.2.3 *Assume that $\mathcal{X}$ is an elementary path category with (PE_{mn}) and $(PE_{\vee})$. Then, $(\text{-})^{\uparrow}$ is a coreflection of $P \downarrow \mathcal{X}$ onto $\mathcal{X}_P$, with the counit of the coreflection given by $\mathtt{i}_{(\text{-})}$.*

Proof. We want to show that the inclusion of $\mathcal{X}_P$ into $P \downarrow \mathcal{X}$ has a left adjoint. Given an assignment $x : P \to X$, i.e. an object of $P \downarrow \mathcal{X}$, the morphism $\mathtt{i}_x$ is a morphism $x^{\uparrow} \to x$. Moreover, the previous lemma implies that $\mathtt{i}_{x^{\uparrow}}$ is an isomorphism, giving us that $x^{\uparrow}$ is in $\mathcal{X}_P$.

What is left to show is that for any morphism $f : x \to y$ in $P \downarrow \mathcal{X}$, with $x \in \mathcal{X}_P$, there is a unique morphism $f^* : x \to y^{\uparrow}$ such that $\mathtt{i}_y\, f^* = f$.

Let us assume that the codomain of x is X and the codomain of y is Y. We define f^* by the universal property of $\mathtt{i}_x$. Assume that there is a factorisation of x via a path embedding as follows.

$$P \xrightarrow{g} Q \xrightarrow{m} X$$

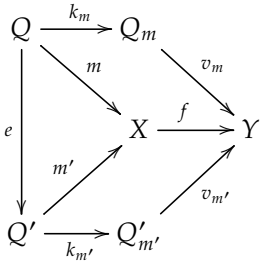
where Q is a path. We take the least factorisation of $f\, m$, by (PE_{mn}), as follows

$$Q \xrightarrow{k_m} Q_m \xrightarrow{v_m} Y$$

where v_m is an embedding. Since we have a morphism $k_m\, g$ of type $P \to Q_m$, by the universal property of $\mathtt{tp}(y)$, there is a morphism $\rho_m : Q_m \rightarrowtail \mathtt{tp}(y)$ such that $v_m = \mathtt{i}_y\, \rho_m$.

Assume that there is another path embedding $m' : Q' \rightarrowtail X$ and $g' : P \to Q'$ and a morphism $e : Q \rightarrowtail Q'$ such that $m = m'e$. Then, we have

a commutative diagram

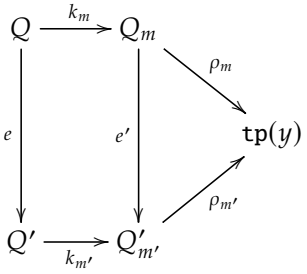


where $v_{m'}$ is an embedding. Therefore, by minimality of v_m, there is an $e' : Q_m \to Q'_{m'}$ making the arising square and triangle commute.

Next, we observe that e' commutes with the embeddings into $\mathtt{tp}(y)$, that is, $\rho_{m'} e' = \rho_m$. This follows by a simple diagram chase and the fact that $\mathtt{i}_y$ is a mono. Indeed, we have:

$$\mathtt{i}_y \rho_m = m = m' e = \mathtt{i}_y \rho_{m'} e'$$

As a result, we have the following commutative diagram.



Since $\mathtt{i}_x : \mathtt{tp}(x) \rightarrowtail X$ is an isomorphism, X is the colimit of the diagram of $m : Q \rightarrowtail X$ through which x factors. Furthermore, we showed above that this diagram is a cocone over $\mathtt{tp}(y)$ as well. Therefore, there is a unique $f^* : X \to \mathtt{tp}(y)$ such that $\rho_m k_m = f^* m$.

What is left to show is that $\mathtt{i}_y f^* = f$ and that f^* is a morphism $x \to y^{\uparrow}$ in $P \downarrow \mathcal{X}$. The latter is immediate by construction and for the former, observe that for every $m : Q \rightarrowtail X$ through which x factors, we have

$$\mathtt{i}_y f^* m = \mathtt{i}_y \rho_m k_m = v_m k_m = f m$$

and so $\mathtt{i}_y f^* = x$ by the fact that $\mathtt{i}_y$ is an isomorphism. □

Elementary path category structure of $\mathcal{X}_P$

In Subsection"Equivalence of types" on page 183 below we use the category $\mathcal{X}_P$ as the main tool to express logical properties of structures with constants. In order to be able to talk about b&f equivalences, bisimulations and so on, in the sense of Chapter 3, we need to equip $\mathcal{X}_P$ with the structure of an elementary path category. To this end, we define the paths of $\mathcal{X}_P$ to be the assignments $x : P \to X$ where X is a path. Similarly, embeddings of $\mathcal{X}_P$ are the morphisms $f : x \to y$ in $P \downarrow \mathcal{X}$ whose underlying morphism is an embedding of $\mathcal{X}$.

The following observation due to Luca Reggio is easy to verify.

Lemma 9.2.4 *The category $\mathfrak{X}_P$ is an elementary path category. Moreover, if $\mathfrak{X}$ is a representable arboreal category then so is $\mathfrak{X}_P$.*

Proof. The fact that $\mathfrak{X}_P$ is an elementary path category is immediate. Let $\mathfrak{X} = \mathsf{Ar}(\Phi)$ for some $\Phi\colon \mathbb{N}^{op} \to \mathbf{Pos}$, by Theorem 4.5.26. Given $x\colon (P,p) \to (T,t)$ in $\mathsf{Ar}(\Phi)$. Observe that $\mathtt{tp}(x)$ is given as the subtree of T on the set $\mathrm{im}(x) \cup \{y \mid \forall i \in P\ x(i) \leq y\}$, with the labelling $t'\colon \mathtt{tp}(x) \to \Phi$ given as the restriction of t to $\mathtt{tp}(x)$. If $\mathrm{ht}(P) = n$ and the top element of P is $\top$, it follows that $\mathfrak{X}_P \cong \mathsf{Ar}(\Phi')$ where

$$\Phi'(m) = \{z \in \Phi(n+m) \mid \Phi(n \leq (n+m))(z) = x(\top)\}. \qquad \square$$

Related work

The following result, which also exposes the coreflective nature of types, has been shown independently in the setting of arboreal categories.

Lemma 9.2.5 ([13, Lemma 5.12]) *Given path embeddings $m : P \rightarrowtail X$ and $n : P \rightarrowtail Y$ in an arboreal category, if there is some $f : X \to Y$ such that $fm = n$ (i.e. there is a morphism $m \to n$ in the comma category $P \downarrow \mathfrak{X}$) then*

1. *there is a unique $g : \mathtt{tp}(m) \to \mathtt{tp}(n)$ such that*

$$\begin{array}{ccc} \mathtt{tp}(m) & \xrightarrow{g} & \mathtt{tp}(n) \\ \downarrow{\scriptstyle \mathtt{i}_m} & & \downarrow{\scriptstyle \mathtt{i}_n} \\ X & \xrightarrow{f} & Y \end{array}$$

   and, moreover,
2. *$g\,m^{\uparrow} = n^{\uparrow}$, i.e. g is a morphism $m^{\uparrow} \to n^{\uparrow}$ in the comma category $P \downarrow \mathfrak{X}$.*

Notably, this construction is merely a technical tool in [13], without exploiting the relationship with model-theoretic types. Moreover, the same paper then constructs $\mathtt{tp}(x)/_{\sim}$, in the sense of our Section 4.8, see the proof of Lemma 7.2 therein. Also, the space of 0-types $F^{\mathbb{E}_k}(A)/_{\sim}$ appeared as a main technical tool in [30, Section 7] (under a different name), when showing FPT decidability of FO formulas on classes of bounded twin-width.

9.3 Structured and unstructured assignments

Typically, in applications, we have a path adjunction between some base category $\mathscr{A}$ and an elementary path category $\mathfrak{X}$:

$$\mathscr{A} \underset{F}{\overset{U}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathfrak{X} \tag{9.4}$$

We call any morphism of the form

$$x\colon U(P) \to A.$$

where P is a path in $\mathfrak{X}$ an **unstructured assignment**. This name is justified since its conjugate with respect to the adjunction $U \dashv F$, i.e. the morphism

$$x^{\#}\colon P \to FA$$

is an assignment, in the sense of Section 9.2. We refer to assignments of the type $P \to FA$ as **structured assignments**. Recall that these precisely describe the situation that inspires our categorical description of types, cf. Section 9.1. Moreover, since conjugation is a bijection, the two types of assignments represent the same information.

Equivalence of types

We say that assignments $x : P \to X$ and $y : P \to Y$ are **of the same type**, and write

$$x \rightleftharpoons y$$

if $\mathtt{tp}(x) \rightleftharpoons \mathtt{tp}(y)$ in $\mathfrak{X}$, which represents that the model-theoretic types of x and y are the same, cf. Section 9.1. In fact, in an arboreal category $x \rightleftharpoons y$ implies that $P_x \cong P_y$.

Then, given unstructured assignments $x\colon U(P) \to A$ and $y\colon U(P) \to B$ in $\mathcal{A}$, we write

$$x \equiv_{\mathfrak{X}} y$$

whenever $x^{\#} \rightleftharpoons y^{\#}$.

Example 9.3.1 For $\mathfrak{X} = \mathsf{EM}(\mathbb{E}_{k+n})$, we set P to be the discrete chain $(1 \prec \cdots \prec n)$ of length n. Due to discreteness of P, any choice of an n-tuple $\overline{a} = [a_1, \ldots, a_n]$ corresponds to a choice of a homomorphism $x\colon U(P) \to A$ where $x(i) = a_i$, and vice versa.

Now, given n-tuples $\overline{a}$ and $\overline{b}$ in A and B, respectively, let $x\colon U(P) \to A$ be $x(i) = a_i$ and, similarly, let $y\colon U(P) \to B$ be $y(i) = b_i$. Then, $x \equiv_{\mathfrak{X}} y$ iff $x^{\#} \rightleftharpoons y^{\#}$ iff $\mathtt{tp}(x) \rightleftharpoons \mathtt{tp}(y)$ iff $(A, \overline{a}) \equiv_{\mathtt{FO}_k[n]} (B, \overline{b})$, i.e. if $\overline{a}$ and $\overline{b}$ realise the same formulas with free variables $x_1, \ldots, x_n$.

Adjunctions and comonads for structures with constants

The fact that $x \equiv_{\mathfrak{X}} y$ corresponds to the equivalence in the fragment of logic with free variables, where x and y interpret these variables as constants appears repeatedly throughout this chapter.

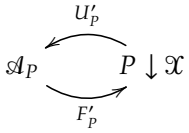
In this section we make it more precise. We show how one obtains a path adjunction for structures with constants (given by unstructured assignments) from the original path adjunction (3.2).

First, we define the **category of unstructured P-assignments**

$$\mathcal{A}_P$$

to be the comma category $U(P) \downarrow \mathcal{A}$. For example, when $\mathcal{A} = \mathcal{S}tr(\sigma)$ and $U(P)$ is discrete then $\mathcal{A}_P$ is isomorphic with the category of pairs $(A, \overline{a})$ where $A \in \mathcal{S}tr(\sigma)$ and $\overline{a}$ is an interpretation of constants.

We have functors:



The first functor is defined by adjointness

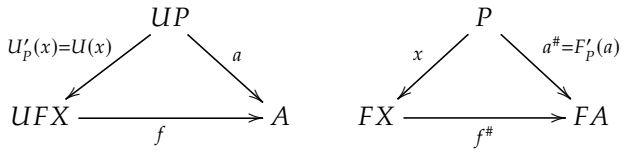
$$F'_P(a : U(P) \to A) = a^{\#} : P \to F(A)$$

and the second is simply just the application of U

$$U'_P(x : P \to FA) = U(x) : U(P) \to UF(A)$$

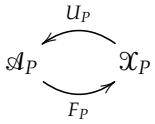
Lemma 9.3.2 *These two functors define an adjunction, with U'_P on the left and F'_P on the right.*

Proof. We need to check that there is a bijection between morphisms $U'_P(x) \to a$ and morphisms $x \to F'_P(a)$. However, this follows from the rules for conjugation (cf. Subsection"Adjunctions" on page 20) and the fact that $(\text{-})^{\#}$ establishes a bijection between $UX \to A$ and $X \to FA$. Indeed, a morphism $U'_P(x) \to a$ gives us a triangle on the left below



and the triangle on the right above is obtained from the one on the left by conjugation, since $a^{\#} = (f \cdot U(x))^{\#} = f^{\#} \cdot x$. □

As a result, we obtain the adjunction



as the composition of the adjunction $U'_P \dashv F'_P$ from Lemma 9.3.2 and the adjunction between $P \downarrow \mathcal{X}$ and $\mathcal{X}_P$, arising from the coreflection from Theorem 9.2.3.

Moreover, this adjunction gives rise to the comonad

$$\mathbb{C}_P : \mathcal{A}_P \to \mathcal{A}_P$$

on $\mathcal{A}_P$, i.e. on the abstraction for a category of structures with constants. The comonad $\mathbb{C}_P$ is obtained as the composition of F'_P, with the coreflection which defines $\texttt{tp}(\text{-})$, followed by U'_P, i.e.[5]

$$\mathbb{C}_P(a : UP \to A) = U((a^{\#})^{\uparrow}).$$

5: We leave it for future work to investigate when U_P is comonadic, which is also a common property in our examples.

Example 9.3.3 We start from a comonadic adjunction, e.g. $\mathscr{A} \leftrightarrows \mathscr{X}$ where $\mathscr{A} = \mathcal{S}tr(\sigma)$ and $\mathscr{X} = \mathsf{EM}(\mathbb{E}_{k+n})$ and P is a discrete $\mathbb{E}_{k+n}$-path of height n, with elements $\{1, \dots, n\}$. Then, as in Example 9.3.1, we see that $\mathscr{A}_P$ is equivalent to the category of σ-structures with n constants and $\mathscr{X}_P$ is the category of trees of height $\leq n + k$ where each of their branches extends P. Concretely, given an assignment $\overline{a} : U(P) \to A$, corresponding to a choice of a word $w = [a_1, \dots, a_n]$, by the above description of $\mathbb{C}_P(A, \overline{a})$, we have $(\overline{a})^{\#} \colon P \to F^{\mathbb{E}_k}(A)$ i.e. the mapping $i \mapsto [a_1, \dots, a_i]$. Then, $\mathtt{i}_{(\overline{a})^{\#}} \colon \mathtt{tp}(\overline{a}^{\#}) \rightarrowtail F^{\mathbb{E}_k}(A)$ restricts the cofree coalgebra to the lists comparable with w. Finally, $U'_P(F'_P(A, a)) \colon U(P) \to U(\mathtt{tp}(a^{\#}))$ can be viewed as the structure with universe $\mathtt{tp}(a^{\#})$ and constants $[a_1], [a_1, a_2], \dots, w$, i.e. a list representation of $\overline{a}$. We see that, in this case, the induced adjunction $\mathscr{A}_P \leftrightarrows \mathscr{X}_P$ is comonadic since $\mathscr{X}_P \cong \mathsf{EM}(\mathbb{C}_P)$.

A similar construction can be used to obtain a version of $\mathbb{P}_k$ for structures with constants and the fragment $\mathscr{L}^k_{\infty\omega}[n]$ of the k-variable infinitary logic of formulas with free variables among the fixed variables $x_1, \dots, x_n$ of $\mathscr{L}^k_{\infty\omega}[n]$.[6]

6: Note that the free variables $x_1, \dots, x_n$ are a subset of the k-variables used in $\mathscr{L}^k_{\infty\omega}[n]$ for quantification. Therefore, when quantifying with $\exists x_i$ we lose access to the constant originally assigned to x_i.

9.4 Formulas and quantification

Motivation. Let φ be a first-order sentence in signature σ. Observe that the class of models $\mathbf{Mod}(\varphi) = \{A \in \mathcal{S}tr(\sigma) \mid A \models \varphi\}$ of φ is closed under $\equiv_{\mathtt{FO}}$. In fact, it is closed under $\equiv_{\mathtt{FO}_k}$ where k is the quantifier rank of φ.

Conversely, let $\Delta \subseteq \mathcal{S}tr(\sigma)$ be a class closed under $\equiv_{\mathtt{FO}_k}$. Since, up to equivalence, $\mathtt{FO}_k$ consists of only finitely many sentences, $\mathcal{S}tr(\sigma) = \Delta_1 \cup \cdots \cup \Delta_r$ is a *finite* disjoint union of equivalence classes of $\equiv_{\mathtt{FO}_k}$. Therefore, for each Δ_i there is some ψ_i such that $\Delta_i = \mathbf{Mod}(\psi_i)$. Indeed, ψ_i is the conjunction of (representatives of equivalence classes of) χ such that $\Delta_i \models \chi$. Finally, we set φ to be the disjunction of those ψ_i such that $\Delta_i \subseteq \Delta$, and obtain that $\Delta = \mathbf{Mod}(\varphi)$.

This all shows that first-order sentences correspond to classes of structures $\Delta \subseteq \mathcal{S}tr(\sigma)$ closed under $\equiv_{\mathtt{FO}_k}$ for some k.

Using the same reasoning, we obtain a representation of formulas as classes of structures with constants. Namely, there is a correspondence between first-order formulas $\varphi(\overline{x})$ with n free variables $\overline{x}$ and quantifier rank k and classes Δ of structures with n constants $(A, \overline{a})$ which are closed under $\equiv_{\mathtt{FO}_k[n]}$.

Synthetic reformulation. We use the above intuition to represent formulas. To this end, we fix a path adjunction $U \dashv F$, as in (9.4).

Given a path P in $\mathscr{X}$, we define a **P-formula** Δ (with respect to $U \dashv F$) to be a class of unstructured assignments

$$\Delta \subseteq U(P) \downarrow \mathscr{A}$$

such that if $x \in \Delta$, for some $x \colon UP \to A$ and $\mathtt{tp}(x^{\#}) \rightleftharpoons \mathtt{tp}(y^{\#})$ for some $y \colon UP \to B$, then $y \in \Delta$. Equivalently, Δ is a P-formula if $x \in \Delta$ and $x \equiv_{\mathscr{X}} y$ implies $y \in \Delta$.

Similarly, we say that $\Delta \subseteq \mathscr{A}$ is a **sentence** (with respect to $U \dashv F$) if $A \in \Delta$ and $A \equiv_{\mathscr{X}} B$ implies $B \in \Delta$.[7]

7: Note that if $\mathscr{X}$ satisfies (P_0), i.e. it has the initial object **0**, then being a sentence is the same as being a **0**-formula.

The following follows directly from definitions.

Lemma 9.4.1 *Given a class $\Delta \subseteq U(P) \downarrow \mathscr{A}$, it is a P-formula with respect to $U \dashv F$ if and only if it is a sentence with respect to the adjunction $U_P \dashv F_P$ from Section 9.3.*

Example 9.4.2 If P is a discrete chain of length n in $\mathbb{E}_{k+n}$ then a class Δ is a P-formula iff it is a class of structures with n constants which is closed under logical equivalence $\equiv_{\mathtt{FO}_k[n]}$.

Example 9.4.3 We say that a P-formula Δ is **homomorphism closed** if it is a *sub-functor*

$$\Delta \subseteq \hom(U(P), \text{-})$$

or, equivalently, whenever $x : UP \to A$ is in Δ and $f : A \to B$ then $f \cdot x \in \Delta$.

Recall that each unstructured assignment $x : UP \to A$ corresponds to a structure with constants $(A, \overline{a})$. Then, being homomorphism closed means that if $(A, \overline{a}) \in \Delta$ and $f : (A, \overline{a}) \to (B, \overline{b})$ is a homomorphism of structures with constants, i.e. $f(\overline{a}) = \overline{b}$, then $(B, \overline{b}) \in \Delta$.

The Homomorphism Preservation Theorem (HPT) says that any sentence which is homomorphism closed is equivalent to an existential positive sentence. Then Abramsky and Reggio [13] show, using a similar representation of formulas as above, that the existential positive formula obtained this way has the same quantifier rank as the original formula.

Example 9.4.4 Let Δ be a P-formula and let $e : Q \rightarrowtail P$ be an embedding of discrete paths in $\mathsf{EM}(\mathbb{E}_k)$ whose height differs by one. For example, the height of Q and P is n and $n+1$, respectively.

Define the **existential closure** $\Delta^{\exists}$ by

$$\Delta^{\exists} = \{f \cdot U(e) \mid f \in \Delta\}.$$

Then, Δ corresponds to a class of structures with $n+1$ constants and $\Delta^{\exists}$ consists of such structures with n constants $(A, \overline{a})$ such that there exists $a' \in A$ such that $(A, \overline{a}a')$ is in Δ. By the definition of $\rightleftharpoons$ it immediately follows that $\Delta^{\exists}$ is a Q-formula too.

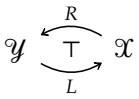
9.5 Preservation for assignments

In the following we wish to obtain preservation theorems for assignments along strong path functors, akin to the preservation theorem (Corollary 8.5.5) for bisimulations of objects. As a direct application, we obtain a preservation theorem for formulas, using the definition of formulas from Section 9.4.

8: Cf. Section 8.5.

In the following we fix a strong path functor[8] $R : \mathscr{X} \to \mathscr{Y}$ between

elementary path categories. In particular, this implies that we have an adjoint situation



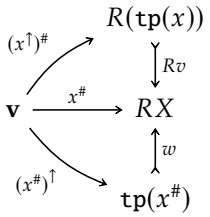
where L is the left adjoint to R. We further assume that $\mathcal{X}$ and $\mathcal{Y}$ satisfy ($\mathrm{PE}_{\mathrm{mn}}$) and ($\mathrm{PE}_{\vee}$), in order to be able to define $\mathtt{tp}(\text{-})$ and therefore also $\rightleftharpoons$ for assignments. Finally, we also assume that $\mathcal{X}$ and $\mathcal{Y}$ satisfy ($\mathrm{PE}_{\mathrm{cp}}$).

We also fix a path $\mathbf{v}$ in $\mathcal{Y}$, as our abstraction for a vector of variables,[9] and assignments $x\colon L\mathbf{v} \to X$ and $y\colon L\mathbf{v} \to Y$ in $\mathcal{X}$ or, equivalently, $x^{\#}\colon \mathbf{v} \to RX$ and $y^{\#}\colon \mathbf{v} \to RY$ in $\mathcal{Y}$. In the following we show that $x \rightleftharpoons y$ in $\mathcal{X}$ implies $x^{\#} \rightleftharpoons y^{\#}$ in $\mathcal{Y}$. This amounts to showing that $\mathtt{tp}(x) \rightleftharpoons \mathtt{tp}(y)$ implies $\mathtt{tp}(x^{\#}) \rightleftharpoons \mathtt{tp}(y^{\#})$. Recall that x decomposes as:

$$L\mathbf{v} \xrightarrow{x^{\uparrow}} \mathtt{tp}(x) \rightarrowtail[\mathtt{i}_x] X$$

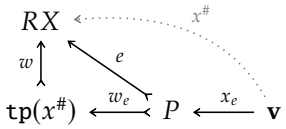
9: This time we use $\mathbf{v}$ instead of P to denote the domain of our assignments, to emphasise its different role from other paths arising in the arguments below.

In order to transfer the b&f system witnessing $x \rightleftharpoons y$ from $\mathcal{X}$ to $\mathcal{Y}$, we relate $\mathtt{tp}(x^{\#})$ and $R(\mathtt{tp}(x))$ via an open pathwise-embedding. Consider the following situation

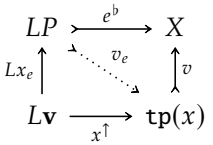


where, for simplicity, we denote $\mathtt{i}_x$ by v and $\mathtt{i}_{x^{\#}}$ by w.

Construction of $\mathtt{tp}(x^{\#}) \to R(\mathtt{tp}(x))$. We look at the cocone defining $\mathtt{tp}(x^{\#})$ as its colimit. Let $e\colon P \rightarrowtail RX$ be a path embedding such that $x^{\#}$ is equal to ex_e for some $x_e\colon \mathbf{v} \to P$. By the definition of $\mathtt{tp}(x^{\#})$ there is an inclusion morphism w_e from P into the colimit $\mathtt{tp}(x^{\#})$. We obtain the following commutative diagram.

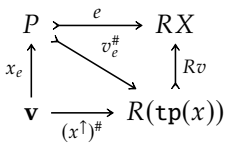


Next, we conjugate parts of this diagram and obtain the following diagram of solid arrows



where x is equal to both the right-to-top composition and top-to-right compositions. The diagonal morphism $v_e\colon LP \to \mathtt{tp}(x)$ is the inclusion into the colimit $\mathtt{tp}(x)$, which exists by fact that x factors through the path

embedding $e^\flat$. Transposing the above diagram gives us



(9.5)

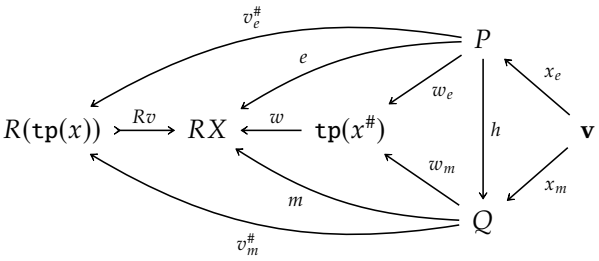
where $v_e^\#$ is an embedding since e is.

The construction we just described works for any e of the required type. We obtain a collection of embeddings:

$$\{P \overset{v_e^\#}{\rightarrowtail} R(\mathtt{tp}(x)) \mid \mathbf{v} \xrightarrow{x_e} P \overset{e}{\rightarrowtail} RX\} \tag{9.6}$$

In the next lines we show that this collection forms a cocone.

Let $e\colon P \rightarrowtail RX$ and $m\colon Q \rightarrowtail RX$ through which $x^\#$ factors and such that $e = mh$ for some $h\colon P \rightarrowtail Q$. The relevant commutative diagram with arrows arising as described above is the following.



where the upper and lower left-most triangles commute by (9.5). By diagram chasing we obtain that

$$Rv \cdot v_m^\# \cdot h = m \cdot h = e = Rv \cdot v_e^\#.$$

Finally, since Rv is an embedding, $v_m^\# \cdot h = v_e^\#$ and hence (9.6) is a cocone. As a result, by the universal property of $\mathtt{tp}(x^\#)$, there is a unique morphism

$$z\colon \mathtt{tp}(x^\#) \longrightarrow R(\mathtt{tp}(x))$$

such that, for every $e\colon P \rightarrowtail RX$ through which $x^\#$ factors, $v_e^\# = z\, w_e$.

Lemma 9.5.1 *$w = Rv \cdot z$ or, equivalently, $\mathtt{i}_{x^\#} = R(\mathtt{i}_x) \cdot z$.*

Proof. We use the universal property of $\mathtt{tp}(x^\#)$. Let $e\colon P \rightarrowtail RX$ be some path embedding through which $x^\#$ factors. By the defining property of inclusion morphisms $w \cdot w_e = e$. On the other hand, $e = Rv \cdot v_e^\# = Rv \cdot z \cdot w_e$ where the first equality holds by (9.5) and the second equality follows from the universal property of z, as stated above. □

The above lemma allows us to obtain a fundamental fact about the morphism z constructed above.

Proposition 9.5.2 *If R is a strong path functor, the induced morphism $z\colon \mathtt{tp}(x^\#) \longrightarrow R(\mathtt{tp}(x))$ is an open pathwise-embedding.*

Proof. We first show that it is a pathwise-embedding. Let $e\colon P \rightarrowtail \mathtt{tp}(x^\#)$ be an arbitrary path embedding. By ($\mathrm{PE}_{\mathrm{cp}}$) for $\mathcal{Y}$, $e = w_{e'}\, h$ for some $w_{e'}\colon P' \rightarrowtail \mathtt{tp}(x^\#)$, with $e'\colon P' \rightarrowtail RX$ such that $x^\#$ factors through it. Then, by the defining equations of z, stated above,

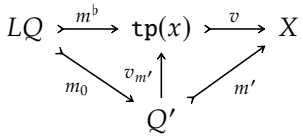
$$ze = z\, w_{e'}\, h = v^\#_{e'} h.$$

Observe that the right-hand side is a composition of two embeddings: (1) h is an embedding since $e = w_{e'}\, h$ and e is an embedding and (2) $v_{e'}$ is an embedding since $e' = Rv \cdot v_{e'}$ and e' is an embedding. Consequently, ze is an embedding too.

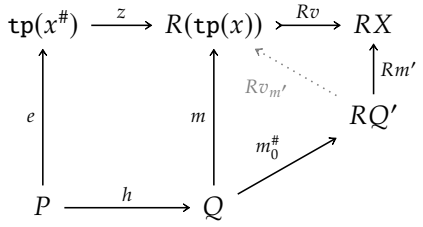
Next, we show that z is open. Consider the following commutative square of path embeddings.

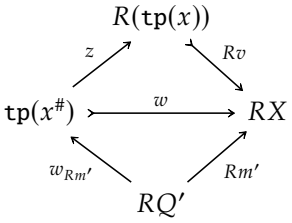
$$\begin{array}{ccc} \mathtt{tp}(x^\#) & \xrightarrow{\ z\ } & R(\mathtt{tp}(x)) \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle m} \\ P & \xrightarrow{\ h\ } & Q \end{array} \tag{9.7}$$

By ($\mathrm{PE}_{\mathrm{cp}}$) for $\mathcal{X}$, we find a decomposition of the conjugate $m^\flat$ into $m_0\colon LQ \rightarrowtail Q'$ and $v_{m'}\colon Q' \rightarrowtail \mathtt{tp}(x)$ for some $m'\colon Q' \rightarrowtail X$ through which $x\colon L\mathbf{v} \to X$ decomposes. We obtain the following picture.



Conjugating the whole diagram back and glueing it onto (9.7) gives us the following diagram on the left below.





Furthermore, Lemma 9.5.1 gives us the commuting top triangle in the diagram on the right above. Finally, the fact that R is a strong path functor implies that RQ' is a path. Therefore, since $x\colon L\mathbf{v} \to X$ decomposes as $m' \cdot m_0$, also $x^\#\colon \mathbf{v} \to RX$ decomposes as $(m' \cdot m_0)^\# = Rm' \cdot m_0^\#$ where Rm' is an embedding since R is a strong path functor. Therefore, there is some $w_{Rm'}$ making the bottom triangle in the diagram on the right above commute.

Finally, upon setting $d\colon Q \to \mathtt{tp}(x^\#)$ to $w_{Rm'} \cdot m_0^\#$, we obtain that

$$w \cdot e = Rv \cdot z \cdot e = Rm' \cdot m_0^\# \cdot h = w \cdot d \cdot h$$

and, since w is an embedding, $e = dh$ as required. For the second triangle to commute we see that

$$Rv \cdot m = Rm' \cdot m_0^\# = w \cdot w_{Rm'} \cdot m_0^\# = Rv \cdot z \cdot d$$

and, again, since Rv is an embedding, $m = zd$, which is the last condition we needed to check. □

Preservation of $\rightleftharpoons$ for assignments. Assume that $x \rightleftharpoons y$ in $\mathcal{X}$, i.e. that $\mathtt{tp}(x) \rightleftharpoons \mathtt{tp}(y)$ in $\mathcal{X}$. Since R is strong, by Corollary 8.5.5, we also have that $R\mathtt{tp}(x) \rightleftharpoons R\mathtt{tp}(y)$ in $\mathcal{Y}$. Therefore, by Proposition 9.5.2, we have the following situation.

$$\begin{array}{ccc} \mathtt{tp}(x^\#) & & \mathtt{tp}(y^\#) \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle z} \\ R(\mathtt{tp}(x)) & \rightleftharpoons & R(\mathtt{tp}(y)) \end{array}$$

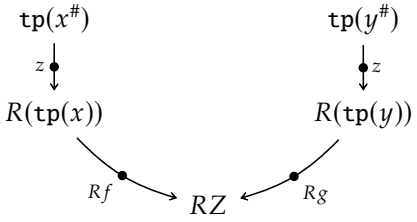
Since OPEs induce b&f equivalences (Proposition 3.4.4) and these compose (Lemma 3.3.4), we have that $\mathtt{tp}(x^\#) \rightleftharpoons \mathtt{tp}(y^\#)$, i.e. that $x^\# \rightleftharpoons y^\#$ in $\mathcal{Y}$.

The following theorem summarises what we showed in this section.

Theorem 9.5.3 *Let $R\colon \mathcal{X} \to \mathcal{Y}$ be a strong path functor between elementary path categories satisfying (PE_{mn}), ($PE_\vee$), (PE_{cp}) and (P_0). Then, given assignments $x\colon L\mathbf{v} \to X$ and $y\colon L\mathbf{v} \to Y$ in $\mathcal{Y}$,*

$$x \rightleftharpoons y \quad \textit{implies} \quad x^\# \rightleftharpoons y^\#.$$

Preservation of $\leftrightarrows$ for assignments. Alternatively, for assignments $x\colon L\mathbf{v} \to X$ and $y\colon L\mathbf{v} \to Y$, assume that $\mathtt{tp}(x) \leftrightarrows \mathtt{tp}(y)$, i.e. there is a cospan of OPEs $\mathtt{tp}(x) \twoheadrightarrow Z \twoheadleftarrow \mathtt{tp}(y)$. Then, by the fact that morphisms z from Proposition 9.5.2 are OPEs and the fact that strong path functors preserve this property (cf. Proposition 8.5.1 and Theorem 8.4.4), we have the following situation



where all depicted morphisms are OPEs. Since OPEs compose (Lemma 3.4.1) the above situation is a cospan of OPEs between $\mathtt{tp}(x^\#)$ and $\mathtt{tp}(y^\#)$, i.e. $\mathtt{tp}(x^\#) \leftrightarrows \mathtt{tp}(y^\#)$.

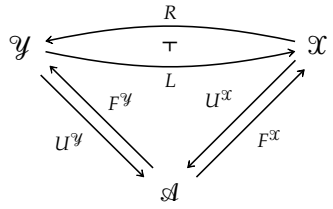
We summarise this in the following theorem, which is a version of Theorem 9.5.3 with slightly milder assumptions, where b&f equivalence of types is replaced by their behavioural equivalence.

Theorem 9.5.4 *Let $R\colon \mathcal{X} \to \mathcal{Y}$ be a strong path functor between elementary path categories satisfying (PE_{mn}), ($PE_\vee$) and (PE_{cp}). Then, given assignments $x\colon L\mathbf{v} \to X$ and $y\colon L\mathbf{v} \to Y$ in $\mathcal{Y}$,*

$$\mathtt{tp}(x) \leftrightarrows \mathtt{tp}(y) \quad \textit{implies} \quad \mathtt{tp}(x^\#) \leftrightarrows \mathtt{tp}(y^\#).$$

Comparing standard logic fragments and their formulas

Next, we consider the following familiar adjoint situation arising, for example, from comonad morphisms, as in (7.10).



Preserving sentences. Let $\Delta \subseteq \mathcal{A}$ be a sentence with respect to the path adjunction $U^{\mathcal{Y}} \dashv F^{\mathcal{Y}}$, as in Section 9.4. Observe that Δ is also a sentence with respect to $U^{\mathcal{X}} \dashv F^{\mathcal{X}}$. Indeed, given $A \in \Delta$ and $A \equiv_{\mathcal{X}} B$, i.e. $F^{\mathcal{X}}A \rightleftharpoons F^{\mathcal{X}}B$. Since R is a strong path functor, by Corollary 8.5.5,

$$F^{\mathcal{Y}}A = R(F^{\mathcal{X}}A) \rightleftharpoons R(F^{\mathcal{X}}B) = F^{\mathcal{Y}}B.$$

Consequently, from the fact that Δ is a sentence with respect to $U^{\mathcal{Y}} \dashv F^{\mathcal{Y}}$ we obtain that $B \in \Delta$.

Example 9.5.5 Given the comonad morphism $\mathbb{E}_k \Rightarrow \mathbb{P}_k$ from Example 7.1.4, if $\Delta = \mathbf{Mod}(\varphi)$ for some $\mathtt{FO}_k$ sentence φ then $\Delta = \mathbf{Mod}(\varphi')$ for some $\mathcal{L}^k_{\infty\omega}$ sentence φ'. In other words, this gives us the trivial fact that

$$\mathtt{FO}_k \subseteq \mathcal{L}^k_{\infty\omega}.$$

The same statement holds for the logics related by comonad morphisms which induces a strong path functor, as discussed in Section 8.5.

Preserving formulas. Theorem 9.5.3 allows us to extend the same reasoning to formulas. Fix a path $\mathbf{v}$ in $\mathcal{Y}$ and a formula $\Delta \subseteq U^{\mathcal{Y}}(\mathbf{v}) \downarrow \mathcal{A}$ with respect to $U^{\mathcal{Y}} \dashv F^{\mathcal{Y}}$ as in Section 9.4, i.e. Δ is a class of objects in the comma category such that $x \in \Delta$ and $x \equiv_{\mathcal{Y}} y$ implies $y \in \Delta$.

To this end, observe that, since $U^{\mathcal{Y}}(\mathbf{v}) = U^{\mathcal{X}}(L\mathbf{v})$, we automatically have that also $\Delta \subseteq U^{\mathcal{X}}(L\mathbf{v}) \downarrow \mathcal{A}$. Therefore, in order to show that Δ is a formula with respect to $U^{\mathcal{X}} \dashv F^{\mathcal{X}}$, we need to show that it is closed under $\equiv_{\mathcal{X}}$. To this end, take assignments $x, y \colon U^{\mathcal{X}}(L\mathbf{v}) \to A$ where $x \in \Delta$ and $x \equiv_{\mathcal{X}} y$, that is, $x^{\mathcal{X}} \rightleftharpoons y^{\mathcal{X}}$ in $\mathcal{X}$ where $x^{\mathcal{X}} \colon L\mathbf{v} \to F^{\mathcal{X}}A$ denotes the conjugate of x with respect to $U^{\mathcal{X}} \dashv F^{\mathcal{X}}$ and similarly for $y^{\mathcal{X}}$ and y. By Theorem 9.5.3 we have that $(x^{\mathcal{X}})^{\#} \rightleftharpoons (y^{\mathcal{X}})^{\#}$, for the conjugates with respect to the $L \dashv R$ adjunction. Finally, since $x^{\mathcal{Y}} = (x^{\mathcal{X}})^{\#}$ where $\mathcal{Y}$ is the conjugation of $x \colon U^{\mathcal{Y}}(\mathbf{v}) = U^{\mathcal{X}}(L\mathbf{v}) \to A$ of x with respect to $U^{\mathcal{Y}} \dashv F^{\mathcal{Y}}$, we see that $x^{\mathcal{Y}} \rightleftharpoons y^{\mathcal{Y}}$, i.e. that $x \equiv_{\mathcal{Y}} y$. Therefore, since Δ is a formula with respect to $U^{\mathcal{Y}} \dashv F^{\mathcal{Y}}$ this gives us that $y \in \Delta$.

Example 9.5.6 Using the comonad morphism $\mathbb{E}_k \Rightarrow \mathbb{P}_k$, as in Example 9.5.5, we obtain that every FO formula $\varphi(\overline{x})$ with n free variables $\overline{x}$ of quantifier rank k is also a formula in the $(k+n)$-variable FO logic, i.e.

$$\mathtt{FO}_k[n] \subseteq \mathcal{L}^k_{\infty\omega}[n].$$

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Glossary of Symbols

$(\text{-})^{\#}$	Left-to-right conjugation operation of an adjunction. 20
$(\text{-})^{\flat}$	Right-to-left conjugation operation of an adjunction. 20
ε	Counit of an adjunction. 20
η	Unit of an adjunction. 20
$\mathscr{C} \downarrow C$	Comma category. 50
$\mathscr{P} \downarrow X$	Comma category of path embeddings. 51
Pos	Category of posets and monotone functions. 73
Set	Category of sets and functions. 28
σ	Finite and finitary relational signature. 9
σ^I	Signature σ extended with binary $I(\text{-},\text{-})$. 12
$\mathscr{F}$	Category of forests. 66
$\mathcal{S}tr_*(\sigma)$	Category of pointed σ-structures. 23
$\mathbf{Set}_f$	Category of finite sets. 102
$\mathcal{S}tr(\sigma)$	Category of σ-structures. 23
$\mathcal{S}tr_f(\sigma)$	Category of finite σ-structures. 102
$\mathscr{A}_P$	Category of unstructured assignments. 183
$\mathscr{X}_P$	A coreflexive subcategory of assignments. 180
$\mathscr{T}$	Category of trees. 65
$\mathbb{C}, \mathbb{D}$, etc.	Comonads. 19
$\sqsubseteq_\alpha$	Induced forest order of a coalgebra. 30
$(-)^*$	Coextension operation of a comonad. 19
δ	Comultiplication of a comonad. 19
ε	Counit of a comonad. 19
$\mathsf{EM}(\mathbb{C})$	Eilenberg–Moore category of coalgebras. 22
δ	Cofree coalgebra of a comonad. 22
$U^{\mathbb{C}}$	Left adjoint from an EM category. 22
$F^{\mathbb{C}}$	Right adjoint into an EM category. 22
$\mathtt{FO}$	First-Order logic without equality. 10
$\mathtt{FO}^{=}$	First-Order logic with equality. 10
$\mathscr{L}_{\infty\omega}$	Infinitary First-Order logic without equality. 10
$\mathscr{L}^{=}_{\infty\omega}$	Infinitary First-Order logic with equality. 10
$\mathtt{C}$	Counting extension of $\mathscr{L}_{\infty\omega}$. 14
$\mathtt{C}_k$	Restriction of $\mathtt{C}$ to sentences of quantifier rank n. 14
$\mathtt{C}^k_n$	Restriction of $\mathtt{C}$ to k-variable sentences of quantifier rank n. 107
$\mathscr{L}^k$	k-variable fragment of $\mathscr{L}$. 15
$\exists^+\mathscr{L}$	Positive existential fragment of $\mathscr{L}$. 13
0_X	Least subpath. 54
$\rightarrowtail$	Embedding. 36
$\mathscr{M}$	Embeddings. 35
$\Rrightarrow_{\mathscr{L}}$	Logical refinement relation. 13

$\mathscr{P}_{\mathfrak{X}}$	Class of paths of $\mathfrak{X}$. 71
$\mathscr{P}$	Paths of an elementary path category. 47
PDiag_X	Canonical path diagram. 52
$\prec$	Covering relation. 28
$\twoheadrightarrow$	Quotient. 36
$\mathbb{Q}$	Quotients. 35
$q \perp m$	q is orthogonal to m. 36
$X/_\sim$	Simplification of X. 93
$\mathsf{Sub}(X)$	Poset of subobjects. 63
TDiag_X	Canonical tree diagram. 71
$\mathbf{0}$	Initial object. 49
$\mathsf{Kl}(\mathbb{C})$	Kleisli category. 21
$K_{\mathbb{C}}$	Comparison functor. 23
$\bullet$	Kleisli composition. 21
$U_{\mathbb{C}}$	Left adjoint from a Kleisli category. 21
$F_{\mathbb{C}}$	Right adjoint into a Kleisli category. 21
$\rightsquigarrow$	Kleisli morphism. 21
H^{EM}	Eilenberg–Moore (EM) lifting. 137
H^{Kl}	Kleisli lifting. 139
$\widehat{H}$	Extended Kleisli lifting. 143
λ^{EM}	Eilenberg–Moore (EM) lifting of a comonad morphism. 157
$\widehat{\lambda}$	Extended Kleisli lifting of a comonad morphism. 157
$\rightarrow_\kappa$	Heteromorphism. 147
$\mathbb{C}_P$	Comonad on the category $\mathscr{A}_P$, which is an abstraction for a category of structures with constants. 184
$\mathbb{D}_M$	Density comonad. 113
$\mathbb{E}_k$	Ehrenfeucht–Fraïssé comonad. 24
$\mathbb{M}_k$	Modal comonad. 25
$\mathbb{P}_k$	Pebble comonad. 25
$\mathbb{P}_{k,n}$	Pebble comonad of restricted depth. 107
$\rightleftharpoons$	Back-and-forth equivalence. 31, 53
$\leftrightarrows$	There exist morphisms in both directions. 48
$\leftrightarrows$	Behavioural equivalence. 57
$\sim$	Bisimilarity. 57
$\equiv_{\mathfrak{X}}$	Equivalence $\rightleftharpoons$ relative to $F\colon \mathscr{A} \to \mathfrak{X}$. 48
$\equiv_{\#\mathfrak{X}}$	Equivalence $\cong$ relative to $F\colon \mathscr{A} \to \mathfrak{X}$. 48
$\equiv_{\exists^+\mathfrak{X}}$	Equivalence $\leftrightarrows$ relative to $F\colon \mathscr{A} \to \mathfrak{X}$. 48
$\Rrightarrow_{\exists^+\mathfrak{X}}$	There exists a morphism relative to $F\colon \mathscr{A} \to \mathfrak{X}$. 48
$\Rrightarrow_{\exists\mathfrak{X}}$	There exists a pathwise-embedding relative to $F\colon \mathscr{A} \to \mathfrak{X}$. 48
$\Rrightarrow_{+\mathfrak{X}}$	A refinement relation for positive fragments. 49
$\equiv_{\mathbb{C}}$	The same as $\equiv_{\mathsf{EM}(\mathbb{C})}$. 49
$\equiv_{\#\mathbb{C}}$	The same as $\equiv_{\#\mathsf{EM}(\mathbb{C})}$. 49
$\equiv_{\exists^+\mathbb{C}}$	The same as $\equiv_{\exists^+\mathsf{EM}(\mathbb{C})}$. 49
$\Rrightarrow_{\exists^+\mathbb{C}}$	The same as $\Rrightarrow_{\exists^+\mathsf{EM}(\mathbb{C})}$. 49
$\Rrightarrow_{\exists\mathbb{C}}$	The same as $\Rrightarrow_{\exists\mathsf{EM}(\mathbb{C})}$. 49
$\Rrightarrow_{+\mathbb{C}}$	The same as $\Rrightarrow_{+\mathsf{EM}(\mathbb{C})}$. 49
$\mathsf{Ar}(\Phi)$	Arboreal category from a functor. 74

$\mathbb{P}$	Path functor. 64
$\Phi_{\mathfrak{X}}$	Functor of paths. 73
$\exists_f$	Embedding part of a factorisation. 65
$\mathtt{tp}(x)$	Subobject representing the model-theoretic type of an assignment. 178
$x^{\uparrow}$	Factorisation of an assignment via its type. 179

Index

σ-structure, 9
  forest-ordered, 30
  forest-ordered with pebble function, 30
  homomorphism, 23
  pointed, 16
  pointed Kripke frames, 16
  strong homomorphism, 36
2-out-of-3 condition, 65

adjunction
  conjugation, 20
  counit, 20
  path, 48
  unit, 20
assignment, 178
  of the same type, 183
  structured, 183
  unstructured, 183

basic proof rules, 131

category
  $\mathscr{M}$-well-powered, 40, 64
  arboreal, 68
  component-based, 117
  combinatorial, 102
  comma, 50
  componental, 116
  dense, 69
  discrete, 115
  Eilenberg–Moore (EM), 22
  elementary path, 47
  Kleisli, 21
  locally finite, 102
  locally finitely presentable (lfp), 41
  of trees, 65
  representable wooded path, 71
  with a set of paths, 70
  wooded path, 65
class
  component-based, 118
  homomorphism closed, 186
  monotone, 120
  nowhere dense, 126
  somewhere dense, 126
coalgebra, 22
  admitting, 31
  cofree, 22
  concrete description, 30
  morphism, 22
  number, 122
colimit
  directed, 40
  tree, 67
comonad
  classifies a class, 115
  coextension, 19
  coextension form, 19
  comonoid form, 19
  comultiplication, 19
  counit, 19
  density, 113
  discrete density, 115
  graded, 122
  idempotent, 30
  morphism, 121, 157
  pointwise density, 114
  regular, 38
coreflexive pair, 143
coSOPE, 57

decomposition
  k-pebble forest-decomposition, 30, 107
  depth, 107
  forest-decomposition, 30
dense subcategory, 69
designated point, 16
diagonal filler, 36, 56
diagram
  canonical path, 52
  canonical tree, 71
  path, 52
  pushout square, 102
  tree, 67

element
  degenerate, 103
  generic, 103
essentially in a subcategory, 116

factorisation system
  orthogonal, 35
  proper, 36
  stable, 37
factorises, 35
forest, 28
  morphism, 28
  root, 28

formula, 185
  quantifier rank, 11
functor
  comonadic, 23
  comparison, 23
  crates colimits, 35
  dense, 130
  finitary, 40
  finite rank, 40
  preserves heights, 82
  strong path, 170

Gaifman graph, 34
game
  back-and-forth k-pebble, 15
  back-and-forth Ehrenfeucht–Fraïssé, 9
  bijective k-pebble, 15
  bijective bisimulation, 17
  bijective Ehrenfeucht–Fraïssé, 14
  bisimulation, 17
  forth-only Ehrenfeucht–Fraïssé, 13
  relaxed Ehrenfeucht–Fraïssé, 12
  simulation, 17
  unbounded, 12
graded modalities, 18
graph parameter, 122
  standard, 122
graph property, 123

height
  of a forest, 28
  of an element, 28
  of forest-ordering, 30
  of path, 71

law
  Eilenberg–Moore (EM), 137
  Kleisli, 139
lifting
  Eilenberg–Moore (EM), 137
  extended Kleisli, 143
  Kleisli, 139
logic
  bounded modal depth, 18
  bounded quantifier rank, 11
  bounded variable count, 15
  counting extension, 14
  counting modal, 18
  finitary first-order (without equality), 10
  finitary first-order with equality, 10
  infinitary first-order (without equality), 10, 12
  infinitary first-order with equality, 10
  modal, 17
  positive existential fragment, 13
  standard fragment, 27

mate, 152, 153
minimal decomposition, 163
morphism
  embedding, 36, 47
  embedding extension, 51
  embedding-open, 55
  heteromorphism, 147
  lax, 74
  open, 58
  open pathwise-embeddings (OPE), 56
  orthogonal, 36
  path embedding, 50, 51
  pathwise-embedding, 55
  quotient, 36
  semi-open, 58
  strict, 75
  strict quotient, 103
  tree bijection, 67

object
  component, 116
  connected, 116
  finitely presentable, 40
  initial, 49
  path, 42, 47
  path generated, 67
  rigid, 54
  tree-connected, 67

partial
  correspondence, 12
  homomorphism, 13
  isomorphism, 10
paths given implicitly, 63
pebble function, 30
primitive positive sentences, 131

relation
  back-and-forth equivalence, 31, 53
  back-and-forth system, 53
  behavioural equivalence, 57
  bisimilarity, 57
  homomorphism indistinguishability, 101
  logical refinement, 13
  strong b&f equivalence, 92

sentence, 186
sequence
  prefix, 24
signature
  modal, 16
  relational, 9

simplification, 93
SOPE, 57

tree, 28
    labelling, 74
    morphism, 28
    root, 28
    synchronisation, 30
type, 177, 178

winning strategy, 10