An Invitation to Game Comonads, day 5: Advanced Topics [a]

Tomáš Jakl

8 August 2025

ESSLLI 2025, Bochum

[a] This project has received funding from the EU's Horizon Europe research and innovation programme under the Marie Skłodowska-Curie grant agreement No 101111373.

Funded by
the European Union

Outlook

Summary of previous days

$$\begin{tikzcd} \mathbf{Str}(\sigma) \arrow[loop above, "G"] \arrow[r, "\mathbf{t}"] & \mathbf{Str}(\sigma^I) \arrow[loop above, "G^I"] \arrow[r, "F^{G^I}"] & \mathbf{EM}(G^I) \end{tikzcd}$$

Game comonads express logical properties of structures:

- $G(A) \to B$ iff $A \Rrightarrow^{\mathscr{L} \cap \mathrm{PP}} B$
- $F^G(\mathbf{t}(A))$ and $F^G(\mathbf{t}(B))$ are bisimilar iff $A \equiv^{\mathscr{L}} B$,

As well as combinatorial properties:

$$\begin{array}{ccc} A \text{ satisfies property } \Delta & & A \text{ admits a coalgebra} \\ (\text{for some } \Delta \subseteq \mathbf{Str}(\sigma)) & \iff & A \to G(A) \end{array}$$

"To be done"

1. Designing more comonads
2. Categorifying known results
3. Building the abstract theory

Designing more comonads

Logic fragments

- local fragments
- modal μ-calculus
- dependence logic
- rank logics ($\Rightarrow$ algebraic games)
  - probably impossible, see (Lichter–Pago–Seppelt, 2024)
- ...

Combinatorial properties

- local tree-width, local tree-depth
- twin-width
- shrub-depth
- ...

Categorifying known results

Identify the corresponding categorical notions in

1. preservation theorems (van Benthem-Rosen, Rossman, ...) ✓
2. decomposition methods (FVM theorems) ✓
3. homomorphism-counting theorems ✓
4. 0–1 laws ??
5. locality methods (Gaifman & Hanf locality) ✓ [TJ, 2023+]
6. Beth definability ??
7. ... (follow Libkin's *Elements of Finite Model Theory*)

Building the abstract theory

- Develop the theory of arboreal categories.
- Study connections with Universal Coalgebra.
  (coinductive definitions, coinductive proof principle, ...)
- Connection of arboreal categories and categorical logic.
- A theory of presentations of comonads is still missing!

Lovász' homomorphism-counting theorems

(j.w.w. Anuj Dawar and Luca Reggio)

Counting fragments

A **counting quantifier** has the form

$$\exists^{\geq d} x \ \varphi$$

and expresses that $A \vDash \exists^{\geq d} x \ \varphi(x, \overline{b})$ iff there are at least d different $a \in A$, $A \vDash \varphi(a, \overline{b})$.

Counting fragments

A **counting quantifier** has the form

$$\exists^{\geq d} x \ \varphi$$

and expresses that $A \models \exists^{\geq d} x \ \varphi(x, \overline{b})$ iff there are at least d different $a \in A$, $A \models \varphi(a, \overline{b})$.

We write

$$\#\mathrm{FO}$$

for the extension of FO with counting quantifiers $\exists^{\geq d}$, for every natural number d. Also, write

$$\#\mathrm{FO}_k \quad \text{and} \quad \#\mathrm{FO}^k$$

for the quantifier rank k and k-variable fragments of $\#\mathrm{FO}$, respectively.

Graded modalities

Graded modality

$$\Diamond_R^d \varphi$$

expresses that $A, a \vDash \Diamond_R^d \varphi$ iff there exists at least d different $b \in A$ such that $(a, b) \in R^A$ and $A, b \vDash \varphi$.

Graded modalities

Graded modality

$$\Diamond_R^d \varphi$$

expresses that $A, a \models \Diamond_R^d \varphi$ iff there exists at least d different $b \in A$ such that $(a, b) \in R^A$ and $A, b \models \varphi$.

We write

$$\#\mathrm{ML}$$

for the extension of ML with graded modalities $\Diamond_R^d$.

Also, write

$$\#\mathrm{ML}_k$$

for the restriction of $\#\mathrm{ML}$ to formulas of modal depth k.

Theorem (Lovász, 1967)

For finite σ-structures A, B,

$$A \cong B \iff |\hom(C, A)| = |\hom(C, B)| \quad \forall \text{ finite } C$$

Lovász-type theorems

Theorem (Lovász, 1967)

For finite σ-structures A, B,

$$A \cong B \iff |\,\mathsf{hom}(C, A)| = |\,\mathsf{hom}(C, B)| \quad \forall \textit{ finite } C$$

Theorem (Dvořák, 2010)

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}^k} B \iff |\,\mathsf{hom}(C, A)| = |\,\mathsf{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{tw} < k$$

Theorem (Grohe, 2020)

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \iff |\,\mathsf{hom}(C, A)| = |\,\mathsf{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{td} \leq k$$

Theorem (Hella, 1996)

For finite A, B, we have $A \equiv^{\#\mathrm{FO}_k} B$ iff Duplicator has a winning strategy in the k-round bijection Ehrenfeucht–Fraïssé game:

- *In round n, Duplicator chooses a bijection $f : A \to B$.*
- *Spoiler chooses a_n in A and Duplicator $b_n = f(a_n)$ in B.*
- *Duplicator wins round n if $\{(a_i, b_i) \mid i \leq n\}$ is a partial isomorphism.*

Remark: If the cardinalities of A and B differ, Duplicator loses immediately.

k-pebble bijection game

Theorem (Hella, 1996)

For finite A, B, we have $A \equiv^{\#\mathrm{FO}^k} B$ iff Duplicator has a winning strategy in the k-pebble bijection game:

- *In round n, Spoiler chooses a pebble p and then Duplicator chooses a bijection $f : A \to B$.*
- *Spoiler puts pebble p on a_p in A and Duplicator puts pebble p on $b_p = f(a_p)$.*
- *Duplicator wins round n if $\{(a_p, b_p) \mid p \in S\}$ is a partial isomorphism, where S is the set of pebbles played so far.*

k-round graded bisimulation game

Theorem (Rijke, 2000)

For finite (A, a) and (B, b), we have $(A, a) \equiv^{\#\mathrm{ML}_k} (B, b)$ iff Duplicator has a winning strategy in the k-round <u>graded</u> bisimulation game:

- *In round n, Spoiler chooses a binary $R \in \sigma$ and then Duplicator chooses a bijection*

$$f\colon \{x \in A \mid (a_{n-1}, x) \in R^A\} \to \{y \in B \mid (b_{n-1}, y) \in R^B\}$$

- *Spoiler chooses x in A such that $(a_{n-1}, x) \in R^A$ and Duplicator chooses $b_n = f(a_n)$.*
- *Duplicator wins round n if $\{(a_i, b_i) \mid i \leq n\}$ is a partial isomorphism.*

Comonadic formulations

Theorem (Abramsky–Dawar–Wang & Abramsky–Shah)
For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \iff F^{\mathbb{E}_k}(\mathbf{t}(A)) \cong F^{\mathbb{E}_k}(\mathbf{t}(B))$$

$$A \equiv^{\#\mathrm{FO}^k} B \iff F^{\mathbb{P}_k}(\mathbf{t}(A)) \cong F^{\mathbb{P}_k}(\mathbf{t}(B))$$

For finite pointed σ-structures (A, a) and (B, b) in a modal signature σ,

$$(A, a) \equiv^{\#\mathrm{ML}_k} (B, b) \iff F^{\mathbb{M}_k}(A, a) \cong F^{\mathbb{M}_k}(B, b)$$

Our strategy

(1) Equivalence in $\#\mathscr{L}$ is an isomorphism of cofree coalgebras.

$\Rightarrow$ Show that isomorphism in $\mathbf{EM}(G)$ is determined by counting.

(2) Combinatorial property Δ given by coalgebras:

$$A \text{ has property } \Delta \iff \exists \text{ coalgebra } A \to G(A)$$

$\Rightarrow$ exploit the interaction between forgetting the coalgebra structure and creating the cofree coalgebra structure.

Given functors

$$\begin{tikzcd} \mathscr{D} \arrow[d, bend right, "L"'] \\ \mathscr{C} \arrow[u, bend right, "R"'] \end{tikzcd}$$

We say that L and R are **adjoint**, with L to the left and R to the right, written $L \dashv R$, if there is a bijection

$$b_{A,B} : \mathscr{C}(L(A), B) \xrightarrow{\cong} \mathscr{D}(A, R(B))$$

for every $A \in \mathscr{D}$ and $B \in \mathscr{C}$, such that

$$b_{A,B'}(L(A) \xrightarrow{h} B \xrightarrow{h'} B') \;=\; A \xrightarrow{b_{A,B}(h)} R(B) \xrightarrow{R(h')} R(B')$$

$$b^{-1}_{A',B}(A' \xrightarrow{f'} A \xrightarrow{f} R(B)) \;=\; L(A') \xrightarrow{L(f')} L(A) \xrightarrow{b^{-1}_{A,B}(f)} B$$

Example adjunction 1

Recall functors

$$
\begin{tikzcd}
\mathbf{Str}(\sigma) \arrow[d, bend right, "L"'] \\
\mathbf{Set} \arrow[u, bend right, "R"']
\end{tikzcd}
$$

defined on objects by

- $L(A, R_1^A, \ldots, R_n^A) = A$ and
- $R(X) = (X, R_1^X, \ldots, R_n^X)$ where $R_i^X = X^n$ (if R_i is n-ary)

Then, we have

$$\mathbf{Set}(L(A, R_1^A, \ldots, R_n^A), X) \quad \cong \quad \mathbf{Str}(\sigma)((A, R_1^A, \ldots, R_n^A), R(X))$$

Example adjunction 2

For a comonad $(G, \varepsilon, (\cdot)^*)$ on category $\mathscr{C}$, we have

$$\begin{tikzcd} \mathbf{EM}(G) \arrow[d, bend right, "U^G"'] \\ \mathscr{C} \arrow[u, bend right, "F^G"'] \end{tikzcd}$$

Where

$$F^G \colon \mathscr{C} \to \mathbf{EM}(G), \qquad A \mapsto (G(A), \delta_A)$$

and

$$U^G \colon \mathbf{EM}(G) \to \mathscr{C}, \qquad (A, \alpha) \mapsto A$$

These functors are adjoint to each other!

We have $U^G \dashv F^G$ that is, for $(A, \alpha) \in \mathbf{EM}(G)$ and $B \in \mathscr{C}$,

$$\mathscr{C}(\underbrace{U(A,\alpha)}_{A}, B) \quad \cong \quad \mathbf{EM}(G)((A,\alpha), \underbrace{F(B)}_{(G(B),\delta_B)})$$

- Given $f\colon A \to B$ compute $f^{\#}\colon (A,\alpha) \to (G(B), \delta_B)$ by setting $G(f) \circ \alpha$.
- Given $g\colon (A,\alpha) \to (G(B), \delta_B)$, set $g^{\flat}\colon A \to B$ as $\varepsilon_B \circ g$.

Exercise

Show that $(\cdot)^{\flat}$ and $(\cdot)^{\#}$ are well-defined, inverse to each other and witness that $U^G \dashv F^G$.

A category $\mathscr{C}$ is **combinatorial** if

$$A \cong B \iff |\mathscr{C}(C, A)| = |\mathscr{C}(C, B)| \quad \forall C \in \mathscr{C}$$

Examples: sets, finite graphs or σ-structures, finite groups, finite Boolean algebras, finite inverse semigroups, ...

Abstract Lovász' theorem, first steps

Lemma

For a comonad G on a category $\mathscr{C}$, if $\mathbf{EM}(G)$ is combinatorial then

$$F^G(A) \cong F^G(B) \iff |\mathscr{C}(C, A)| = |\mathscr{C}(C, B)| \quad \forall C \in \Delta_G$$

where Δ_G consists of all C in $\mathscr{C}$ which admit a G-coalgebra.

Abstract Lovász' theorem, first steps

Lemma

For a comonad G on a category $\mathscr{C}$, if $\mathbf{EM}(G)$ is combinatorial then

$$F^G(A) \cong F^G(B) \iff |\mathscr{C}(C, A)| = |\mathscr{C}(C, B)| \quad \forall C \in \Delta_G$$

where Δ_G consists of all C in $\mathscr{C}$ which admit a G-coalgebra.

Proof.

$$\begin{aligned} F^G(A) \cong F^G(B) &\iff |\mathbf{EM}(G)(X, F^G(A))| = |\mathbf{EM}(G)(X, F^G(B))| \\ &\qquad \forall X \in \mathbf{EM}(G) \\ &\iff |\mathscr{C}(U^G(X), A)| = |\mathscr{C}(U^G(X), B)| \\ &\qquad \forall X \in \mathbf{EM}(G) \\ &\iff |\mathscr{C}(C, A)| = |\mathscr{C}(C, B)| \quad \forall C \in \Delta_G \end{aligned}$$

□

Theorem (Dawar–Jakl–Reggio, 2021)
If a category is locally finite, has a proper factorisation system and pushouts, then it is combinatorial.

Proof hint: Use the inclusion–exclusion principle!

Remark: [Pultr, 1973] and [Reggio, 2021] replace pushouts with stronger requirements on the factorisation systems.

Theorem (Dawar–Jakl–Reggio, 2021)
If a category is locally finite, has a proper factorisation system and pushouts, then it is combinatorial.

Proof hint: Use the inclusion–exclusion principle!

Remark: [Pultr, 1973] and [Reggio, 2021] replace pushouts with stronger requirements on the factorisation systems.

Theorem (Dawar–Jakl–Reggio, 2021)
For any comonad G over $\mathbf{Str}(\sigma)$, the category $\mathbf{EM}_{fin}(G)$ of finite coalgebras is combinatorial.

Proof hint: Use the fact that U^G (is comonadic and therefore) creates and preserves colimits and preserves epimorphisms.

Grohe's theorem for $\#\mathrm{FO}_k$ without equality

Observe that $\mathbb{E}_k$ restricts to a comonad on $\mathbf{Str}_{fin}(\sigma)$
$\Rightarrow$ the adjunction $U^{\mathbb{E}_k} \dashv F^{\mathbb{E}_k}$ restricts to $\mathbf{Str}_{fin}(\sigma) \leftrightarrows \mathbf{EM}_{fin}(\mathbb{E}_k)$

Proposition

For finite σ-structures A, B,

$$F^{\mathbb{E}_k}(A) \cong F^{\mathbb{E}_k}(B) \iff |\operatorname{hom}(C, A)| = |\operatorname{hom}(C, B)|$$

for every finite C of $\mathrm{td} \leq k$.

Grohe's theorem for $\#\mathrm{FO}_k$ without equality

Observe that $\mathbb{E}_k$ restricts to a comonad on $\mathbf{Str}_{fin}(\sigma)$
$\Rightarrow$ the adjunction $U^{\mathbb{E}_k} \dashv F^{\mathbb{E}_k}$ restricts to $\mathbf{Str}_{fin}(\sigma) \leftrightarrows \mathbf{EM}_{fin}(\mathbb{E}_k)$

Proposition

For finite σ-structures A, B,

$$F^{\mathbb{E}_k}(A) \cong F^{\mathbb{E}_k}(B) \iff |\,\mathrm{hom}(C, A)| = |\,\mathrm{hom}(C, B)|$$

for every finite C of $\mathrm{td} \leq k$.

For $A \equiv^{\#\mathrm{FO}_k} B$ we need $F^{\mathbb{E}_k}(\mathbf{t}(A)) \cong F^{\mathbb{E}_k}(\mathbf{t}(B))$!

Also, this proves equality elimination for $\#\mathrm{FO}_k$!

Adding equality

Previous proposition also holds for σ^I-structures!

$$\begin{tikzcd} \mathbf{Str}(\sigma) \arrow[r, "\mathbf{t}"] & \mathbf{Str}(\sigma^I) \arrow[loop above, "\mathbb{E}_k^I"] \arrow[r, "F^{\mathbb{E}_k^I}"] & \mathbf{EM}(\mathbb{E}_k^I) \end{tikzcd}$$

Moreover, $\mathbf{t}$ has a left adjoint:

$$\mathbf{q}\colon \mathbf{Str}(\sigma^I) \to \mathbf{Str}(\sigma), \qquad A \mapsto A/I$$

Last ingredient

Lemma (Combinatorial lemma)

The adjunction

$$\begin{tikzcd} \mathbf{Str}(\sigma^I) \arrow[d, bend right, "\mathbf{q}"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{Str}(\sigma) \arrow[u, bend right, "\mathbf{t}"'] \end{tikzcd}$$

restricts to an adjunction between finite σ-structures of $\mathrm{td} \leq k$ and finite σ^I-structures of $\mathrm{td} \leq k$.

Grohe's theorem

Theorem

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \iff |\,\mathsf{hom}(C, A)| = |\,\mathsf{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{td} \leq k$$

Theorem

For finite σ-structures A, B,

$$A \equiv^{\#\mathrm{FO}_k} B \iff |\operatorname{hom}(C, A)| = |\operatorname{hom}(C, B)| \quad \forall \textit{ finite } C \textit{ of } \mathrm{td} \leq k$$

Proof.

$|\operatorname{hom}(C, A)| = |\operatorname{hom}(C, B)|$

for every finite σ-structure C of $\mathrm{td} \leq k$

$\iff$ (from the Combinatorial lemma)

$|\operatorname{hom}(D, \mathbf{t}(A))| = |\operatorname{hom}(D, \mathbf{t}(B))|$

for every finite σ^I-structure D of $\mathrm{td} \leq k$

$\iff$

$F^{\mathbb{E}_k}(\mathbf{t}(A)) \cong F^{\mathbb{E}_k}(\mathbf{t}(B)) \iff A \equiv^{\#\mathrm{FO}_k} B$ $\square$

Adaptations

A minor adaptation needed for $\mathbb{P}_k$, it <u>does not</u> restrict to

$$\mathbf{Str}_{fin}(\sigma) \to \mathbf{Str}_{fin}(\sigma).$$

Combinatorial lemma quite involved.

$\Rightarrow$ Dvořák's theorem.

Adaptations

A minor adaptation needed for $\mathbb{P}_k$, it <u>does not</u> restrict to

$$\mathbf{Str}_{fin}(\sigma) \to \mathbf{Str}_{fin}(\sigma).$$

Combinatorial lemma quite involved.
$\Rightarrow$ Dvořák's theorem.

The abstract result yields new theorems:

- for $\mathbb{M}_k$ (no equality $\Rightarrow$ no combinatorial lemma needed),
- for the $\#\mathrm{FO}_k \cap \#\mathrm{FO}^n$ fragment,
- for the restricted conjunction fragments and path-width.

Discrete density comonads

(j.w.w. Samson Abramsky and Thomas Paine)

A <u>trivial</u> comonad for most classes

Theorem

Let $\Delta \subseteq \mathbf{Str}_{\mathit{fin}}(\sigma)$ *(resp.* $\Delta \subseteq \mathbf{Str}_{*,\mathit{fin}}(\sigma)$*) which is closed under*

- *isomorphisms,*
- *finite coproducts (*$A, B \in \Delta \implies A + B \in \Delta$*), and*
- *summands (*$A + B \in \Delta \implies A, B \in \Delta$*)*

then there is a comonad G_Δ *on* $\mathbf{Str}(\sigma)$ *(resp.* $\mathbf{Str}_*(\sigma)$*) such that*

$$A \in \Delta \iff \exists A \to G_\Delta(A)$$

A <u>trivial</u> comonad for most classes

Theorem

Let $\Delta \subseteq \mathbf{Str}_{\mathit{fin}}(\sigma)$ *(resp.* $\Delta \subseteq \mathbf{Str}_{*,\mathit{fin}}(\sigma)$*) which is closed under*

- *isomorphisms,*
- *finite coproducts (*$A, B \in \Delta \implies A + B \in \Delta$*), and*
- *summands (*$A + B \in \Delta \implies A, B \in \Delta$*)*

then there is a comonad G_Δ *on* $\mathbf{Str}(\sigma)$ *(resp.* $\mathbf{Str}_*(\sigma)$*) such that*

$$A \in \Delta \iff \exists A \to G_\Delta(A)$$

Proof idea.

Pick $\Delta' \subseteq \Delta$ representatives of each isomorphism-class, then

$$G_\Delta(A) = \coprod_{B \in \Delta'} \coprod_{f\colon B \to A} B \qquad \square$$

Examples

The theorem applies to any class closed under isomorphisms and coproducts which is either

- monotone,
- hereditary, or
- closed under minors.

Examples

The theorem applies to any class closed under isomorphisms and coproducts which is either

- monotone,
- hereditary, or
- closed under minors.

Dually, for any isomorphism closed class such that

$$A \times B \in \Delta \iff A, B \in \Delta$$

we have a monad whose algebras classify the class!

Example

Connected non-bipartite graphs

Lovász theorem for discrete density comonads

Theorem

For finite (pointed) σ-structures A, B,

$$F^{G_\Delta}(A) \cong F^{G_\Delta}(B) \quad \Longleftrightarrow \quad \forall C \in \Delta,\ \mathrm{hom}(C, A) \cong \mathrm{hom}(C, B)$$

Theorem

For finite (pointed) σ-structures A, B,

$$F^{G_\Delta}(A) \cong F^{G_\Delta}(B) \quad \Longleftrightarrow \quad \forall C \in \Delta,\ \mathsf{hom}(C, A) \cong \mathsf{hom}(C, B)$$

Example

Cospectrality from $\Delta =$ disjoint unions of cycles.

Other fragments

(due to Abramsky–Laure–Reggio)

Existential fragments

Given

$$R\colon \mathbf{Str}(\sigma) \xrightarrow{\mathbf{t}} \mathbf{Str}(\sigma^I) \xrightarrow{G^I} \mathbf{EM}(G)$$

Theorem

For $G = \mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$ capturing logic $\mathscr{L}$,

$$\begin{aligned} A \Rrightarrow^{\exists\mathscr{L}} B \quad &\iff \text{ exists a pathwise-embedding } R(A) \to R(B) \\ &\iff \text{ Duplicator wins } \exists\mathscr{G} \text{ from } R(A) \text{ to } R(B) \end{aligned}$$

Given

$$R\colon \mathbf{Str}(\sigma) \xrightarrow{\mathbf{t}} \mathbf{Str}(\sigma^I) \xrightarrow{G^I} \mathbf{EM}(G)$$

Theorem

For $G = \mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$ capturing logic $\mathscr{L}$,

$A \equiv^{+\mathscr{L}} B \iff$ *there exist open pathwise-embeddings f_1, f_2 and*

$$\begin{tikzcd} Z_1 \arrow[r, "h"] \arrow[d, "f_1"'] & Z_2 \arrow[d, "f_2"] \\ R(A) & R(B) \end{tikzcd}$$

such that h is a tree isomorphism.

$\iff$ *Duplicator wins $+\mathscr{G}$ between $R(A)$ and $R(B)$*

Overview (disregarding the special adjustments to handle equality)

Comonad G	plays in a game capturing $\mathscr{L}$
G-coalgebras	a combinatorial parameter
$\mathscr{G}^{\mathrm{PP}}$ in $\mathbf{EM}(G)$ $G(A) \to B$	$\Rrightarrow^{\mathscr{L} \cap \mathrm{PP}}$
$\mathscr{G}$ in $\mathbf{EM}(G)$ $F^G(A) \xleftarrow{ope} \cdot \xrightarrow{ope} F^G(B)$	$\equiv^{\mathscr{L}}$
$F^G(A) \cong F^G(B)$	$\equiv^{\#\mathscr{L}}$
$\exists\mathscr{G}$ resp. $F^G(A) \xrightarrow{pe} F^G(B)$	$\Rrightarrow^{\exists\mathscr{L}}$
$+\mathscr{G}$ resp. $\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ f_1\downarrow & & \downarrow f_2 \\ F^G(A) & & F^G(B) \end{array}$	$\Rrightarrow^{+\mathscr{L}}$

Overview (disregarding the special adjustments to handle equality)

Comonad G	plays in a game capturing $\mathscr{L}$
G-coalgebras	a combinatorial parameter
$\mathscr{G}^{\mathrm{PP}}$ in $\mathbf{EM}(G)$ $G(A) \to B$	$\Rrightarrow^{\mathscr{L}\cap\mathrm{PP}}$
$\mathscr{G}$ in $\mathbf{EM}(G)$ $F^G(A) \xleftarrow{ope} \cdot \xrightarrow{ope} F^G(B)$	$\equiv^{\mathscr{L}}$
$F^G(A) \cong F^G(B)$	$\equiv^{\#\mathscr{L}}$
$\exists\mathscr{G}$ resp. $F^G(A) \xrightarrow{pe} F^G(B)$	$\Rrightarrow^{\exists\mathscr{L}}$
$+\mathscr{G}$ resp. $\begin{tikzcd} Z_1 \arrow[r, "h"] \arrow[d, "f_1"'] & Z_2 \arrow[d, "f_2"] \\ F^G(A) & F^G(B) \end{tikzcd}$	$\Rrightarrow^{+\mathscr{L}}$

Still things to be checked for some comonads!

Feferman–Vaught–Mostowski theorems

(j.w.w. Dan Marsden and Nihil Shah)

Feferman–Vaught–Mostowski theorems

They have the form

$$A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 \quad \Longrightarrow \quad A_1 \times A_2 \equiv^{\mathscr{L}} B_1 \times B_2$$

Feferman–Vaught–Mostowski theorems

They have the form

$$
\begin{aligned}
A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 \quad &\Longrightarrow \quad A_1 \times A_2 \equiv^{\mathscr{L}} B_1 \times B_2 \\
A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 \quad &\Longrightarrow \quad A_1 + A_2 \equiv^{\mathscr{L}} B_1 + B_2
\end{aligned}
$$

Feferman–Vaught–Mostowski theorems

They have the form

$$\begin{aligned} A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 \quad &\Longrightarrow \quad A_1 \times A_2 \equiv^{\mathscr{L}} B_1 \times B_2 \\ A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 \quad &\Longrightarrow \quad A_1 + A_2 \equiv^{\mathscr{L}} B_1 + B_2 \\ A \equiv^{\mathscr{L}} B \quad &\Longrightarrow \quad \mathrm{fg}(A) \equiv^{\mathscr{L}} \mathrm{fg}(B) \end{aligned}$$

Feferman–Vaught–Mostowski theorems

They have the form

$$
\begin{aligned}
A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 &\implies A_1 \times A_2 \equiv^{\mathscr{L}} B_1 \times B_2 \\
A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 &\implies A_1 + A_2 \equiv^{\mathscr{L}} B_1 + B_2 \\
A \equiv^{\mathscr{L}} B &\implies \mathrm{fg}(A) \equiv^{\mathscr{L}} \mathrm{fg}(B) \\
(A, a_0) \equiv^{\mathrm{ML}} (B, b_0) &\implies (A \cup \{a_0'\}, a_0') \equiv^{\mathrm{ML}} (B \cup \{b_0'\}, b_0') \\
\ldots &\implies \ldots
\end{aligned}
$$

Feferman–Vaught–Mostowski theorems

They have the form

$$\begin{array}{rcl} A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 & \Longrightarrow & A_1 \times A_2 \equiv^{\mathscr{L}} B_1 \times B_2 \\ A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 & \Longrightarrow & A_1 + A_2 \equiv^{\mathscr{L}} B_1 + B_2 \\ A \equiv^{\mathscr{L}} B & \Longrightarrow & \mathrm{fg}(A) \equiv^{\mathscr{L}} \mathrm{fg}(B) \\ (A, a_0) \equiv^{\mathrm{ML}} (B, b_0) & \Longrightarrow & (A \cup \{a_0'\}, a_0') \equiv^{\mathrm{ML}} (B \cup \{b_0'\}, b_0') \\ \dots & \Longrightarrow & \dots \end{array}$$

The pattern

$$(\forall i \in \{1, \dots, n\} \quad A_i \equiv^{\mathscr{L}_i} B_i) \implies \Psi(A_1, \dots, A_n) \equiv^{\mathscr{K}} \Psi(B_1, \dots, B_n)$$

Feferman–Vaught–Mostowski theorems

They have the form

$$\begin{aligned}
A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 &\implies A_1 \times A_2 \equiv^{\mathscr{L}} B_1 \times B_2 \\
A_1 \equiv^{\mathscr{L}} B_1,\ A_2 \equiv^{\mathscr{L}} B_2 &\implies A_1 + A_2 \equiv^{\mathscr{L}} B_1 + B_2 \\
A \equiv^{\mathscr{L}} B &\implies \mathrm{fg}(A) \equiv^{\mathscr{L}} \mathrm{fg}(B) \\
(A, a_0) \equiv^{\mathrm{ML}} (B, b_0) &\implies (A \cup \{a_0'\}, a_0') \equiv^{\mathrm{ML}} (B \cup \{b_0'\}, b_0') \\
\dots &\implies \dots
\end{aligned}$$

The pattern

$$(\forall i \in \{1, \dots, n\} \quad A_i \equiv^{\mathscr{L}_i} B_i) \implies \Psi(A_1, \dots, A_n) \equiv^{\mathscr{K}} \Psi(B_1, \dots, B_n)$$

For simplicity we focus on the unary case

$$A \equiv^{\mathscr{L}} B \implies \Psi(A) \equiv^{\mathscr{K}} \Psi(B)$$

In terms of comonads

Assume G captures $\mathscr{L}$ and H captures $\mathscr{K}$ such that

$$\begin{tikzcd}
\mathbf{EM}(G) \arrow[d, bend right, "U^G"'] & & \mathbf{EM}(H) \arrow[d, bend right, "U^H"'] \\
\mathbf{Str}(\sigma) \arrow[u, bend right, "F^G"'] \arrow[rr, "\Psi"] \arrow[loop left, "G"] & & \mathbf{Str}(\tau) \arrow[u, bend right, "F^H"'] \arrow[loop right, "H"]
\end{tikzcd}$$

Then, the FVM theorem

$$A \equiv^{\mathscr{L}} B \implies \Psi(A) \equiv^{\mathscr{K}} \Psi(B)$$

translates as

$$F^G(A) \leftarrow Z \rightarrow F^G(B) \implies F^H(\Psi(A)) \leftarrow ?? \rightarrow F^H(\Psi(B))$$

In terms of comonads

Assume G captures $\mathscr{L}$ and H captures $\mathscr{K}$ such that

$$\begin{tikzcd}
\mathbf{EM}(G) \arrow[r, dashed, "\widehat{\Psi}"] & \mathbf{EM}(H) \\
\mathbf{Str}(\sigma) \arrow[u, "F^G"] \arrow[r, "\Psi"] \arrow[loop left, "G"] & \mathbf{Str}(\tau) \arrow[u, "F^H"'] \arrow[loop right, "H"]
\end{tikzcd}$$

Then, the FVM theorem

$$A \equiv^{\mathscr{L}} B \implies \Psi(A) \equiv^{\mathscr{K}} \Psi(B)$$

translates as

$$F^G(A) \leftarrow Z \rightarrow F^G(B) \implies \underbrace{\widehat{\Psi}(F^G(A))}_{F^H(\Psi(A))} \leftarrow \widehat{\Psi}(Z) \rightarrow \underbrace{\widehat{\Psi}(F^G(B))}_{F^H(\Psi(B))}$$

Immediate questions

1. When does $\widehat{\Psi}$ exist?
2. And when it does, will it preserve open pathwise-embeddings?
3. How does this generalise to n-ary operations?
4. How about the other fragments?
   - Counting fragments are covered by this.
   - So how about primitive positive fragments?

Recovering the n-ary

Assume

$$\Psi \colon \mathscr{A}_1 \times \ldots \mathscr{A}_n \to \mathscr{B}$$

and we have comonads G_1 on $\mathscr{A}_1$, ..., G_n on $\mathscr{A}_n$, and H on $\mathscr{B}$.

Recovering the n-ary

Assume

$$\Psi\colon \mathscr{A}_1 \times \dots \mathscr{A}_n \to \mathscr{B}$$

and we have comonads G_1 on $\mathscr{A}_1$, …, G_n on $\mathscr{A}_n$, and H on $\mathscr{B}$.

Easy to check that

- $G := G_1 \times \cdots \times G_n$ is a comonad on $\mathscr{A} := \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$
- $\mathbf{EM}(G) \cong \mathbf{EM}(G_1) \times \cdots \times \mathbf{EM}(G_n)$
- $\mathscr{A} \xrightarrow{F^G} \mathbf{EM}(G)$ is the same as $\mathscr{A} \xrightarrow{F^{G_1} \times \cdots \times F^{G_n}} \mathbf{EM}(G)$
- bisimulation relations also decompose componentwise

Recovering the n-ary

Assume

$$\Psi\colon \mathscr{A}_1 \times \dots \mathscr{A}_n \to \mathscr{B}$$

and we have comonads G_1 on $\mathscr{A}_1$, ..., G_n on $\mathscr{A}_n$, and H on $\mathscr{B}$.

Easy to check that

- $G := G_1 \times \cdots \times G_n$ is a comonad on $\mathscr{A} := \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$
- $\mathbf{EM}(G) \cong \mathbf{EM}(G_1) \times \cdots \times \mathbf{EM}(G_n)$
- $\mathscr{A} \xrightarrow{F^G} \mathbf{EM}(G)$ is the same as $\mathscr{A} \xrightarrow{F^{G_1} \times \cdots \times F^{G_n}} \mathbf{EM}(G)$
- bisimulation relations also decompose componentwise

$\Longrightarrow$ n-ary operations are special case of unary operations $\mathscr{A} \to \mathscr{B}$

Kleisli vs EM categories

$$
\begin{tikzcd}
\mathbf{K}(G) \arrow[rr, bend left, "K"] \arrow[dr, bend right, "U_G"'] & & \mathbf{EM}(G) \arrow[dl, bend right, "U^G"'] \\
& \mathscr{C} \arrow[ul, bend right, "F_G"'] \arrow[ul, phantom, "\perp"] \arrow[ur, bend right, "F^G"'] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
$$

F_G is given by

- $F_G(A) = A$
- $F_G(A \xrightarrow{f} B) = (G(A) \xrightarrow{\varepsilon_A} A \xrightarrow{f} B) : A \to_G B$

K is given by

- $K(A) = (G(A), G(A) \xrightarrow{\delta_A} G(G(A)))$
- $K(f\colon A \to_G B) = f^*$

In particular

$$\begin{tikzcd}
\mathbf{K}(G) \arrow[rr, "K"] & & \mathbf{EM}(G) \\
& \mathscr{C} \arrow[ul, "F_G"] \arrow[ur, "F^G"'] &
\end{tikzcd}$$

Lemma

K is fully faithful, i.e. $A \to_G B \overset{1-1}{\leftrightarrow} K(A) \to K(B)$

Corollary

For $\mathscr{L}$ captured by G

1. $A \Rrightarrow^{\mathscr{L}\cap\mathrm{PP}} B \iff A \to_G B \iff F^G(A) \to F^G(B)$
2. $A \equiv^{\#\mathscr{L}} B \iff A \to_G B$ $\underline{\textit{iso}}$ $\iff F^G(A) \cong F^G(B)$

Assume we have a Kleisli lift

$$\begin{tikzcd} \mathbf{K}(G) \arrow[r, dashed, "\widetilde{\Psi}"] & \mathbf{K}(H) \\ \mathbf{Str}(\sigma) \arrow[u, "F_G"] \arrow[r, "\Psi"] \arrow[loop left, "G"] & \mathbf{Str}(\tau) \arrow[u, "F_H"'] \arrow[loop right, "H"'] \end{tikzcd}$$

Theorem

Assume G and H characterise $\mathscr{L}$ and $\mathscr{K}$, respectively, and assume $\widetilde{\Psi}$ exists for Ψ then

1. $A \Rrightarrow^{\mathscr{L}\cap \mathrm{PP}} B \implies \Psi(A) \Rrightarrow^{\mathscr{K}\cap \mathrm{PP}} \Psi(B)$
2. $A \equiv^{\#\mathscr{L}} B \implies \Psi(A) \equiv^{\#\mathscr{K}} \Psi(B)$

Theorem (folklore)

$\widetilde{\Psi}$ exists iff there is a "Kleisli law", i.e. a natural transformation

$$H\Psi \overset{\kappa}{\Longrightarrow} \Psi G$$

such that, for all A,

$$\begin{tikzcd}
H\Psi(A) \arrow[rr, "\kappa_A"] \arrow[dr, "\varepsilon_\Psi"'] & & \Psi G(A) \arrow[dl, "\Psi\varepsilon"] \\
& \Psi(A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
H\Psi(A) \arrow[rr, "\kappa_A"] \arrow[d, "\delta_{\Psi(A)}"'] & & \Psi G(A) \arrow[d, "\Psi(\delta_A)"] \\
H^2\Psi(A) \arrow[r, "H(\kappa_A)"] & H\Psi G(A) \arrow[r, "\kappa_{G(A)}"] & \Psi G^2(A)
\end{tikzcd}$$

Extended Kleisli lifts

Theorem

If H preserves embeddings then $\widetilde{\Psi}$ further lifts to $\widehat{\Psi}$.

$$\begin{tikzcd}
\mathbf{EM}(G) \arrow[r, dashed, "\widehat{\Psi}"] & \mathbf{EM}(H) \\
\mathbf{K}(G) \arrow[u, "K"] \arrow[r, dashed, "\widetilde{\Psi}"] & \mathbf{K}(H) \arrow[u, "K"'] \\
\mathbf{Str}(\sigma) \arrow[u, "F_G"] \arrow[r, "\Psi"] \arrow[uu, bend left, "F^G"] & \mathbf{Str}(\tau) \arrow[u, "F_H"'] \arrow[uu, bend right, "F^H"']
\end{tikzcd}$$

The proof is a generalisation of lifting of monoidal structures to the categories of algebras, for monoidal monads.

Example

Define

$$\kappa\colon \mathbb{E}_k(A+B) \;\to\; \mathbb{E}_k A + \mathbb{E}_k B.$$

This induces a lift

$$\oplus\colon \mathbf{EM}(\mathbb{E}_k) \times \mathbf{EM}(\mathbb{E}_k) \to \mathbf{EM}(\mathbb{E}_k)$$

Preservation of open pathwise-embeddings

We say that Ψ is **smooth** if it preserves embeddings and every

$$
\begin{tikzcd}
P \arrow[rr, "f"] \arrow[d, "\Psi(A)"'] & & \Psi G(A) \arrow[d, "\Psi(\alpha)"] \\
H(P) \arrow[r, "H(f)"] & H\Psi(A) \arrow[r, "\kappa_A"] & \Psi G(A)
\end{tikzcd}
$$

factors via $\Psi(e)$ for some minimal $e\colon Q \rightarrowtail (A, \alpha)$

Theorem

If Ψ is smooth then $\widehat{\Psi}$ preserves (open) pathwise-embeddings.

Corollary

If Ψ admits a Kleisli law and is smooth then

- $A \equiv^{\mathscr{L}} B \implies \Psi(A) \equiv^{\mathscr{K}} \Psi(B)$
- $A \Rrightarrow^{\exists\mathscr{L}} B \implies \Psi(A) \Rrightarrow^{\exists\mathscr{K}} \Psi(B)$

Example

$\mathscr{L}$ = guarded $\mathcal{C}_{\infty,k}$, for structures with comeasurable relations

[Karamlou–Shah, 2023+] constructed a smooth

$$\mathbb{E}_k Q_d \xrightarrow{\kappa} Q_d \mathbb{E}_k$$

where $Q_d(A)$ = projector-valued measurements on A from **[Abramsky–Barbosa–de Silva–Zapata, 2017]**, for non-local quantum strategies.

Consequently

$$A \equiv^{\mathscr{L}} B \implies Q_d(A) \equiv^{\mathscr{L}} Q_d(B)$$

FVM theorems for free

For any functor G, we have

$$G(A \times A') \to G(A) \times G(A').$$

Theorem

If there is a comonad G identifying logic $\mathscr{L}$, then:

- $A \Rrightarrow^{\mathscr{L} \cap \mathrm{PP}} B,\ A' \Rrightarrow^{\mathscr{L} \cap \mathrm{PP}} B' \implies A \times A' \Rrightarrow^{\mathscr{L} \cap \mathrm{PP}} B \times B'$
- $A \equiv^{\# \mathscr{L}} B,\ A' \equiv^{\# \mathscr{L}} B' \implies A \times A' \equiv^{\# \mathscr{L}} B \times B'$

If, furthermore, G preserves embeddings and paths in $\mathbf{EM}(G)$ *are closed under quotients, then:*

- $A \equiv^{\mathscr{L}} B,\ A' \equiv^{\mathscr{L}} B' \implies A \times A' \equiv^{\mathscr{L}} B \times B'$

Thank you!

References

References, i

Origins of the theory:

1. Anuj Dawar, Simone Severini, Octavio Zapata. Pebble games and cospectral graphs. Electronic Notes in Discrete Mathematics 61 (2017): 323-329.
2. Samson Abramsky, Anuj Dawar, and Pengming Wang. The pebbling comonad in finite model theory. LICS 2017.

Theory basics and a survey of main achievements:

3. Samson Abramsky, Nihil Shah, Relating structure and power: Comonadic semantics for computational resources. CMCS 2018 & Journal of Logic and Computation, Vol. 31(6), 2021.
4. Samson Abramsky. Structure and Power: an emerging landscape. Fundamenta Informaticae 186(1-4), 2022.

References, ii

Papers introducing further new game comonads

5. Adam Ó Conghaile, Anuj Dawar. Game Comonads & Generalised Quantifiers, CSL 2021.
6. Samson Abramsky, Dan Marsden. Comonadic semantics for guarded fragments, LICS 2021.
7. Yoàv Montacute, Nihil Shah. The Pebble-Relation Comonad in Finite Model Theory, LICS 2022.
8. Samson Abramsky, Dan Marsden. Comonadic semantics for hybrid logic and bounded fragments, MFCS 2022.
9. Bartosz Bednarczyk, Mateusz Urbańczyk. Comonadic semantics for description logics games. Description Logics Workshop @ FLoC 2022.
10. Santiago Figueira, Gabriel Goren Roig. A Modal Logic with n-ary Relations Over Paths: Comonadic Semantics and Expressivity. arXiv:2307.09679
11. Georg Schindling: Homomorphism Indistinguishability and Game Comonads for Restricted Conjunction and Requantification. arXiv:2506.19746
12. Nihil Shah: Arboreal covers over relational structures, PhD Thesis 2024.

Categorical (F)MT theorems

13. Lovász-type homomorphism counting theorems:
    - i Anuj Dawar, Tomáš Jakl, Luca Reggio. Lovász-type theorems and game comonads, LICS 2021.
    - ii Luca Reggio. Polyadic Sets and Homomorphism Counting. Advances in Mathematics, Vol. 410, 2022.
    - iii Samson Abramsky, Tomáš Jakl, Thomas Paine. Discrete Density Comonads and Graph Parameters. CMCS 2022.
14. Tomáš Jakl, Dan Marsden, Nihil Shah. A categorical account of composition methods in logic. LICS 2023 + arXiv:2405.06664 (extended version).
15. Tomáš Jakl, Dan Marsden, Nihil Shah. A game comonadic account of Courcelle and Feferman-Vaught-Mostowski theorems. arXiv:2205.05387
16. for van Benthem-Rosen theorem, see [8]
17. Samson Abramsky, Luca Reggio. Arboreal categories and equi-resource homomorphism preservation theorems. arXiv:2211.15808
18. Luca Reggio, Colin Riba. Finitely accessible arboreal adjunctions and Hintikka formulae. arXiv:2304.12709

References, iv

Synthetic finite model theory:

19. Samson Abramsky, Luca Reggio. Arboreal categories. An axiomatic theory of resources. ICALP 2021 & LMCS 2023.
20. Samson Abramsky, Yoàv Montacute, Nihil Shah. Linear Arboreal Categories. arXiv:2301.10088
21. Also see [17] and [18] for examples of usage of this abstract setting.

No-go theorems:

22. Moritz Lichter, Benedikt Pago, Tim Seppelt. Limitations of Game Comonads for Invertible-Map Equivalence via Homomorphism Indistinguishability. CSL 2024.
23. Nihil Shah, Amin Karamlou. No Go Theorems: Directed Containers That Do Not Distribute Over Distribution Monad. LICS 2024.
24. See also [18].