# An Invitation to Game Comonads, day 4: Logical Equivalences [a]

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## Recap and motivation

On Day 2, we saw how games provide syntax-free characterisations of various logical equivalences (respectively, preorders) of the form

$$\equiv^{\mathscr{L}} \quad \text{(respectively, } \Rrightarrow^{\mathscr{L}\cap \mathrm{PP}}\text{)}$$

for an appropriate choice of the logic fragment $\mathscr{L}$.

We then defined comonads

$$\mathbb{E}_k,\ \mathbb{P}_k,\ \text{and } \mathbb{M}_k$$

based on these games (by considering the set of plays) and showed how to encode strategies in the forth-only games as $G(A) \to B$ resp. $A \to_G B$.

For our logic fragments, we obtained

$$A \Rrightarrow^{\mathscr{L}\cap \mathrm{PP}} B \quad \iff \quad A \to_G B$$

The morphisms $A \to_G B$ form the Kleisli category $\mathbf{K}(G)$.

**Question:**

> Can we capture other logical equivalences within $\mathbf{K}(G)$?
>
> For instance, what does isomorphism in $\mathbf{K}(G)$ correspond to? (We answer that on Day 5.)

In order to characterise the equivalences $\equiv^{\mathscr{L}}$, w.r.t. the full fragments

$$\mathrm{FO}_k,\ \mathrm{FO}^k \text{ and } \mathrm{ML}_k,$$

corresponding to back-and-forth games, we move to the Eilenberg–Moore category $\mathbf{EM}(G)$ of coalgebras.

# Eilenberg–Moore coalgebras as forest-ordered structures

Yesterday, we saw that the Eilenberg–Moore categories

$$\mathbf{EM}(\mathbb{E}_k),\ \mathbf{EM}(\mathbb{P}_k) \text{ and } \mathbf{EM}(\mathbb{M}_k)$$

can be identified with categories whose objects are structures equipped with an appropriate compatible forest order (and a pebbling function in the case of $\mathbb{P}_k$).

The morphisms are the homomorphisms that preserve the forest orders (and also the pebbling functions in the case of $\mathbb{P}_k$).

We use this concrete equivalent description of $\mathbf{EM}(\mathbb{E}_k)$ to characterise $\equiv^{\mathrm{FO}_k}$.

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$ and $A \in \mathrm{Ob}(\mathscr{C})$,

$$(G(A),\ G(A) \xrightarrow{\delta_A} G(G(A))\,)$$

is a $G$-coalgebra!

## Cofree coalgebras

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$ and $A \in \mathrm{Ob}(\mathscr{C})$,

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**Example**

For $G = \mathbb{E}_k$ and a $\sigma$-structure $A$, the compatible forest order $\leq$ on $\mathbb{E}_k(A)$ is

$$u \leq w \iff u \text{ is a prefix of } w$$

For $G = \mathbb{P}_k$, the forest order $\leq$ is as above and the pebble function $p\colon \mathbb{P}_k(A) \to \{1, \ldots, k\}$ is defined by

$$p([(p_1, a_1), \ldots, (p_n, a_n)]) = p_n$$

## Cofree functors

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$ there is a functor

$$F^G \colon \mathscr{C} \to \mathbf{EM}(G)$$

which sends

- an object $A \in \mathscr{C}$ to $(G(A),\ G(A) \xrightarrow{\delta_A} G(G(A)))$, and
- a morphism $f \colon A \to B$ in $\mathscr{C}$ to
$$G(f) \colon (G(A), \delta_A) \to (G(B), \delta_B).$$

**Exercise**

Verify that $F^G$ is a functor for $G = \mathbb{E}_k, \mathbb{P}_k$ and/or $\mathbb{M}_k$, from the concrete descriptions of $\mathbf{EM}(G)$.

**Main idea:**

Describe $A \equiv^{\mathscr{L}} B$ by comparing
the cofree coalgebras $F^G(A)$ and $F^G(B)$
(in the category $\mathbf{EM}(G)$).

# Path embeddings

## Paths and embeddings

An object $(A, \leq)$ of $\mathbf{EM}(\mathbb{E}_k)$ is a **path** if $\leq$ is a (finite) linear order. Paths are noted by $P, Q, R \ldots$.

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An arrow in $f : (A, \leq) \to (B, \leq)$ in $\mathbf{EM}(\mathbb{E}_k)$ is an **embedding** if it is an embedding *as a* $\sigma$-homomorphism, that is:

- $f$ is injective, and
- for any $n$-ary $R \in \sigma$,

$$(a_1, \ldots, a_n) \in R^A \iff (f(a_1), \ldots, f(a_n)) \in R^B.$$

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A **path embedding** is an embedding

$$m \colon P \rightarrowtail (B, \leq)$$

where $P$ is a path. Define $\mathbb{P}X$ as the collection of all path embeddings into $X$. (let us not worry about set-theoretic issues now...)

## Example

Consider a morphism

$$m\colon P \to F^{\mathbb{E}_k}(A)$$

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$$\forall i \in \{1, \ldots, n\}, \quad m(p_i) = [a_1, \ldots, a_i].$$

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**Example**

What if, furthermore,

$$m\colon P \rightarrowtail F^{\mathbb{E}_k}(A)$$

is a path embedding?

Also, the relations on $P$ and on $\{a_1, \ldots, a_n\}$ agree!

**Exercise**

Let

$$m\colon P \rightarrowtail F^{\mathbb{E}_k}(A) \quad \text{and} \quad n\colon P \rightarrowtail F^{\mathbb{E}_k}(B)$$

be path embeddings. Show that, if their images contain only non-repeating sequences, then $m$ and $n$ induce a partial isomorphism between $A$ and $B$.

# Partial correspondence

**Question:** What if we drop the assumption on non-repetition?

$$m\colon P \rightarrowtail F^{\mathbb{E}_k}(A) \quad \text{and} \quad n\colon P \rightarrowtail F^{\mathbb{E}_k}(B)$$

# Partial correspondence

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**Answer:** These introduce a **partial correspondence** between $A$ and $B$ i.e.

1. ~~For all~~ $\cancel{i, j \in \{1, \dots, k\},\ a_i = a_j \iff b_i = b_j.}$
2. For all $n$-ary relations $R$ and all $i_1, \dots, i_n \in \{1, \dots, k\}$,

$$(a_{i_1}, \dots, a_{i_n}) \in R^A \iff (b_{i_1}, \dots, b_{i_n}) \in R^B.$$

# Games in the categories of coalgebras

# Back-and-forth "path embedding" game

Given any two $X, Y \in \mathbf{EM}(\mathbb{E}_k)$, we define a **back-and-forth game** $\mathscr{G}$ played by Spoiler and Duplicator between $X$ and $Y$.

# Back-and-forth "path embedding" game

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Whenever $m\colon P \rightarrowtail X$ and $m'\colon Q \rightarrowtail X$ are path embeddings, we say that $m'$ **covers** $m$, written $m \prec m'$, if

- $|Q| = |P| + 1$ and
- there is an embedding $P \rightarrowtail Q$ making the following diagram commute.

$$\begin{tikzcd} P \arrow[r, tail, "m"] \arrow[d, tail, dashed] & X \\ Q \arrow[ur, tail, "m'"'] & \end{tikzcd}$$

## Back-and-forth "path embedding" game

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- Positions in the game $\mathscr{G}$ are pairs $(m, n) \in \mathbb{P}X \times \mathbb{P}Y$.

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- The winning relation

$$\mathscr{W}(X, Y) \subseteq \mathbb{P}X \times \mathbb{P}Y$$

consists of the pairs $(m \colon P \rightarrowtail X, n \colon Q \rightarrowtail Y)$ such that $P = Q$.

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  consists of the pairs $(m\colon P \rightarrowtail X,\ n\colon Q \rightarrowtail Y)$ such that $P = Q$.
- The initial position is $(\bot_X, \bot_Y)$, where

  $$\bot_X \colon \emptyset \rightarrowtail X \quad \text{and} \quad \bot_Y \colon \emptyset \rightarrowtail Y$$

  are the unique functions from the empty poset.

# Back-and-forth "path embedding" game

- At the start of each round, the position is specified by

$$(m, n) \in \mathbb{P}X \times \mathbb{P}Y$$

- Then, either Spoiler chooses some $m' \succ m$ and Duplicator must respond with some $n' \succ n$,
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- Duplicator wins the round if they are able to respond and the new position is in $\mathscr{W}(X, Y)$.
- Duplicator wins the game if they have a strategy that is winning after $j$ rounds, for all $j \geq 1$.

  ... since paths in $\mathbf{EM}(\mathbb{E}_k)$ have length $\leq k$, the game terminates after $\leq k$ rounds!

## The game $\mathscr{G}$ and its logical counterpart

Assume the game $\mathscr{G}$ is played between cofree coalgebras $X = F^{\mathbb{E}_k}(A)$ and $Y = F^{\mathbb{E}_k}(B)$. After $k$ rounds, let

$$m_1 \prec \cdots \prec m_k \in \mathbb{P}\, X \quad \text{and} \quad n_1 \prec \cdots \prec n_k \in \mathbb{P}\, Y$$

be the path embeddings that have been played. Their images yield

$$[a_1] \sqsubseteq \cdots \sqsubseteq [a_1, \ldots, a_k] \in \mathbb{E}_k A \;\; \text{and} \;\; [b_1] \sqsubseteq \cdots \sqsubseteq [b_1, \ldots, b_k] \in \mathbb{E}_k B.$$

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**Lemma**

*For any $i \in \{1, \ldots, k\}$, $(m_i, n_i) \in \mathscr{W}(X, Y)$ iff*

$$\{(a_i, b_i) \mid i = 1, \ldots, k\}$$

*is a partial correspondence between $A$ and $B$.*

# The game $\mathscr{G}$ and its logical counterpart

**Proposition**

*The following statements are equivalent for all structures $A, B$:*

1. *Duplicator has a winning strategy in the game $\mathscr{G}$ played between $F^{\mathbb{E}_k}(A)$ and $F^{\mathbb{E}_k}(B)$.*
2. $A \equiv^{\mathrm{FO}_k{}^-} B$. *I.e., for first-order sentences $\varphi$ without equality and with quantifier rank $\leq k$, $A \vDash \varphi \iff B \vDash \varphi$.*

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**Questions:**

- How to recover equivalence with respect to $\mathrm{FO}_k$?
- The previous result relies on a notion of game in the category $\mathbf{EM}(\mathbb{E}_k)$. Can we describe it in a more structural way?

## Open morphisms and bisimulations

A morphism $f\colon X \to Y$ in $\mathbf{EM}(\mathbb{E}_k)$ is **open** if it satisfies the following path-lifting property: Given any commutative square

$$\begin{tikzcd} P \arrow[r, tail] \arrow[d, tail] & Q \arrow[d, tail] \arrow[dl, dashed] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $P, Q$ paths, there is $Q \to X$ making the triangles commute.

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A morphism $f\colon X \to Y$ in $\mathbf{EM}(\mathbb{E}_k)$ is **open** if it satisfies the following path-lifting property: Given any commutative square

$$\begin{tikzcd} P \arrow[r, rightarrowtail] \arrow[d, rightarrowtail] & Q \arrow[d, rightarrowtail] \arrow[dl, dashed] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $P, Q$ paths, there is $Q \to X$ making the triangles commute.

Further, $f\colon X \to Y$ is a **pathwise embedding** if, for all path embeddings $m\colon P \rightarrowtail X$, the composite $f \circ m$ is a path embedding.

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A **bisimulation** between objects $X, Y$ of $\mathbf{EM}(\mathbb{E}_k)$ is a span of open pathwise embeddings

$$X \leftarrow Z \to Y.$$

If such a bisimulation exists, we say that $X$ and $Y$ are **bisimilar**.

# Games vs bisimulations

**Theorem (Shah–Abramsky, Jakl–Reggio)**

*The following are equivalent for all objects $X, Y$ of $\mathbf{EM}(\mathbb{E}_k)$:*

1. *Duplicator has a winning strategy in the game $\mathscr{G}$ played between $X$ and $Y$.*
2. *$X$ and $Y$ are bisimilar.*
3. *$X$ and $Y$ are behaviourally equivalent (OPEs $X \to \cdot \leftarrow Y$)*

**Sketch of proof:** See whiteboard.

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**Sketch of proof:** See whiteboard.

**Corollary**

*The following statements are equivalent for all structures $A, B$:*

1. *$F^{\mathbb{E}_k}(A)$ and $F^{\mathbb{E}_k}(B)$ are bisimilar.*
2. *$A \equiv^{\mathrm{FO}_k^-} B$.*

## The other comonads...

- Similar results hold for the comonads $\mathbb{P}_k$ and $\mathbb{M}_k$. (In the former case, we restrict to finite structures.)
- For $\mathbb{M}_k$, since there is no equality symbol in the logic, we get a characterisation of $\equiv^{\mathrm{ML}_k}$ in terms of bisimilarity in $\mathbf{EM}(\mathbb{M}_k)$.
- We shall now look at how to "add" the equality symbol in the case of $\mathbb{E}_k$ (and $\mathbb{P}_k$). That is, how to go

$$\text{from} \quad \equiv^{\mathrm{FO}_k^-} \quad \text{to} \quad \equiv^{\mathrm{FO}_k} .$$

## $I$-relations

Consider a *fresh* binary relation symbol $I$ and define the signature

$$\sigma^I := \sigma \cup \{I\}.$$

Denote the category of $\sigma^I$-structures and their homomorphisms by

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There is a functor

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that views a $\sigma$-structure $A$ as a $\sigma^I$-structure where $I^A$ is the identity relation on $A$.

Since $\mathbb{E}_k$ was defined *uniformly* for all signatures, there is an Ehrenfeucht–Fraïssé comonad $\mathbb{E}_k^I$ on $\mathbf{Str}(\sigma^I)$.

**Exercise**

Characterise pairs of path embeddings

$$m\colon P \rightarrowtail F^{\mathbb{E}_k^I}(\mathbf{t}A) \qquad n\colon P \rightarrowtail F^{\mathbb{E}_k^I}(\mathbf{t}B)$$

in $\mathbf{EM}(\mathbb{E}_k^I)$.

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in $\mathbf{EM}(\mathbb{E}_k^I)$.

They correspond to partial isomorphisms!

# Equivalence in $\mathrm{FO}_k$ with equality

$$\begin{tikzcd} \mathbf{Str}(\sigma) \arrow[r, "\mathbf{t}"] & \mathbf{Str}(\sigma^I) \arrow[loop above, "\mathbb{E}_k^I"] \arrow[r, "F^{\mathbb{E}_k^I}"] & \mathbf{EM}(\mathbb{E}_k^I) \end{tikzcd}$$

**Theorem**

*The following statements are equivalent for all $\sigma$-structures $A, B$:*

1. $F^{\mathbb{E}_k^I}(\mathbf{t}(A))$ *and* $F^{\mathbb{E}_k^I}(\mathbf{t}(B))$ *are bisimilar.*
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**Proof.**

By the characterisation of $\equiv^{\mathrm{FO}_k{}^-}$, item 1 holds iff

$$\mathbf{t}(A) \vDash \varphi \iff \mathbf{t}(B) \vDash \varphi$$

for all $\varphi \in \mathrm{FO}_k^-(\sigma^I)$. But this is equivalent to item 2. $\square$

# Generalisations

- The same strategy applies to $\mathrm{FO}^k$, the $k$-variable logic with equality
  (provided we restrict to finite structures).
- However, the argument is essentially exactly the same!
- **Question:** Is there a way to make this formal?

# One game to rule them all

## An abstract setting (Overshoot!?)

In full generality, we only need

$$R\colon \mathbf{Str}(\sigma) \to \mathscr{A}$$

together with

- a choice of objects $\mathcal{P} \subseteq \mathscr{A}$ (paths) and
- a choice of <u>mono</u>morphisms $\rightarrowtail \subseteq \mathscr{A}$ (embeddings)

which satisfy

- $\forall X \in \mathscr{A}$, there is a minimal path embedding $\perp_X\colon \mathbf{0}_X \rightarrowtail X$
- $f, g$ embeddings $\implies$ $g \circ f$ embedding
- $g \circ f$ embedding $\implies$ $f$ embedding

Then we can specify the game as before.

## Back-and-forth (abstract) “path embedding” game

Given any two $X, Y \in \mathscr{A}$, define $\mathscr{G}$ as before:

- Positions: $\mathbb{P}\,X \times \mathbb{P}\,Y$
- Winning relation: $\mathscr{W}(X, Y) \subseteq \mathbb{P}\,X \times \mathbb{P}\,Y$
- Initial position: $(\perp_X, \perp_Y)$

At position $(m, n)$

- Spoiler picks some $m' \geq m$ or $n' \geq n$, and
- Duplicator responds with $n' \geq n$ or $m' \geq m$, respectively.

Duplicator wins the round if $(m', n') \in \mathscr{W}(X, Y)$.

**Definition**

$A \sim_{\mathscr{G}} B \iff$ Duplicator wins the game $\mathscr{G}$ between $R(A)$ and $R(B)$

## Example

Extend $\sigma$ to a two-sorted signature $\sigma^+ = \sigma \cup \{I, e\}$ where $e$ goes between the two sorts.

Define

$$\mathbf{t} \colon \mathbf{Str}(\sigma) \to \mathbf{Str}(\sigma^+)$$

where $\mathbf{t}(A) = (A, \mathcal{P}(A);\ I^A, e^A)$ where $e(a, S)$ iff $a \in S$.

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where $\mathbf{t}(A) = (A, \mathcal{P}(A);\ I^A, e^A)$ where $e(a, S)$ iff $a \in S$.

Define

$$R\colon \mathbf{Str}(\sigma) \xrightarrow{\mathbf{t}} \mathbf{Str}(\sigma^+) \xrightarrow{F^{\mathbb{E}_k^+}} \mathbf{EM}(\mathbb{E}_k^+)$$

Then,

$$A \equiv^{\mathrm{MSO}_k} B \iff R(A) \sim_{\mathscr{G}} R(B).$$

Similar techniques capture Description Logic.

## Outlook

- Bisimilarity can be defined too!
- However, for it to capture $\sim_{\mathscr{G}}$, it is necessary to assume more.
- In particular, arboreal categories provide a convenient (synthetic) setting to think about games.
- Tomorrow (Day 5), we discuss how to abstractly capture also counting logics, and existential and positive fragments.
- In some cases, there are equality elimination results which, in a sense, tell us that working with $\sigma$-structures and $\sigma^I$-structures is the same.
- These will rely on homomorphism counting theorems.

# References


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# Bonus slides:
# Different presentations of comonads

## Natural transformations

Natural transformations are "morphisms of functors".

Given functors $F\colon \mathscr{C} \to \mathscr{D}$ and $F'\colon \mathscr{C} \to \mathscr{D}$, a **natural transformation**

$$\alpha\colon F \Rightarrow F' \qquad \text{or} \qquad \begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{D} \end{tikzcd}$$

is given by a collection of morphisms

$$\{F(A) \xrightarrow{\alpha_A} F'(A) \mid A \in \mathrm{Ob}(\mathscr{C})\}$$

such that, for every $h\colon A \to B$ in $\mathscr{C}$,

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(h)"'] & F'(A) \arrow[d, "F'(h)"] \\ F(B) \arrow[r, "\alpha_B"] & F'(B) \end{tikzcd} \qquad \text{(i.e. } F'(h) \circ \alpha_A = \alpha_B \circ F(h)\text{)}$$

# Example: the identity natural transformations

For any functor $F\colon \mathscr{C} \to \mathscr{D}$, the collection

$$\{\, \mathrm{id}_{F(A)}\colon F(A) \to F(A) \mid A \in \mathrm{Ob}(\mathscr{C})\,\}$$

is a natural transformation $\mathrm{id}_F\colon F \Rightarrow F$ since

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_{F(A)}} & A \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ B & \xrightarrow{\mathrm{id}_{F(B)}} & B \end{array}$$

# Example: the counit natural transformation

For a comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$,

$$\{\, \varepsilon_A \colon G(A) \to A \mid A \in \mathrm{Ob}(\mathscr{C}) \,\}$$

is a natural transformation $\varepsilon \colon G \Rightarrow \mathrm{Id}_{\mathscr{C}}$. That is, for any $f \colon A \to B$ in $\mathscr{C}$, we have

$$\begin{tikzcd}
G(A) \arrow[r, "\varepsilon_A"] \arrow[d, "G(f)"'] & A \arrow[d, "f"] \\
G(B) \arrow[r, "\varepsilon_B"] & B
\end{tikzcd}$$

Which follows by

$$\varepsilon_B \circ G(f) = \varepsilon_B \circ (f \circ \varepsilon_A)^* = f \circ \varepsilon_A$$

## Example: the comultiplication natural transformation

For every comonad $(G, \varepsilon, (\cdot)^*)$ there is a natural transformation

$$\delta \colon G \Rightarrow GG$$

The component $\delta_A$ of $\delta$ is obtained as the coextension $\mathrm{id}^*_{G(A)} \colon G(A) \to GG(A)$ of $\mathrm{id}_{G(A)} \colon G(A) \to G(A)$.

**Exercise**

Show that $\delta$ is a natural transformation.

## Two comonad presentations

For any comonad $(G, \varepsilon, (\cdot)^*)$ on $\mathscr{C}$,

- $G\colon \mathscr{C} \to \mathscr{C}$ is a functor.
- $\varepsilon\colon G \Rightarrow \mathrm{Id}_{\mathscr{C}}$ is a natural transformation.
- $\delta\colon G \Rightarrow GG$ is a natural transformation.
- These satisfy

$$
\begin{tikzcd}
 & G \arrow[dl, Rightarrow, "\mathrm{id}_G"'] \arrow[d, Rightarrow, "\delta"] \arrow[dr, Rightarrow, "\mathrm{id}_G"] & \\
G & GG \arrow[l, Rightarrow, "G(\varepsilon)"] \arrow[r, Rightarrow, "\varepsilon_G"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\delta"'] & GG \arrow[d, Rightarrow, "\delta_G"] \\
GG \arrow[r, Rightarrow, "G(\delta)"] & GGG
\end{tikzcd}
$$

**Fact:** The presentation that we use $(G, \varepsilon, (\cdot)^*)$ can be recovered from the data $(G, \varepsilon, \delta)$, by defining $(\cdot)^*$ as $f^* := G(f) \circ \delta$.