An Invitation to Game Comonads, day 2: Games and Game Comonads [a]

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The What and Why of games, i

- (Finite) model theory looks at structures up to definable properties.
- Given a logic fragment $\mathscr{L}$ (e.g. $\mathscr{L} = \mathrm{FO}^k$, FO_k, or ML_k), define the equivalence relation

$$A \equiv^{\mathscr{L}} B \quad \text{iff} \quad \forall \varphi \in \mathscr{L}.\ (A \vDash \varphi \iff B \vDash \varphi).$$

- However, reasoning about $\equiv^{\mathscr{L}}$ can get convoluted, it depends on syntactic properties of $\mathscr{L}$.
- Games provide semantic characterisations of the syntactic equivalences $\equiv^{\mathscr{L}}$.

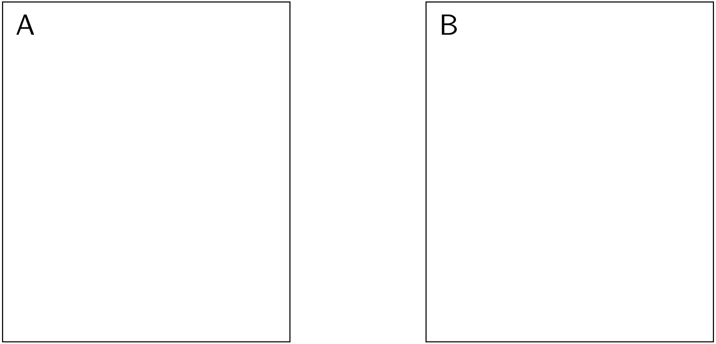
The What and Why of games, ii

- Two players:
    - **Spoiler** aims to show that $A \not\equiv^{\mathscr{L}} B$ and
    - **Duplicator** that $A \equiv^{\mathscr{L}} B$.
- Syntactic logical resources (e.g. the k in FO_k) correspond to natural semantic resource parameters in a game, e.g.:

Logical resource		Game resource
quantifier rank	$\leftrightarrow$	number of rounds
variable count	$\leftrightarrow$	number of pebbles
modal depth	$\leftrightarrow$	number of rounds
...		...

Games and bounded quantifier rank

Back-and-forth EF games intuitively



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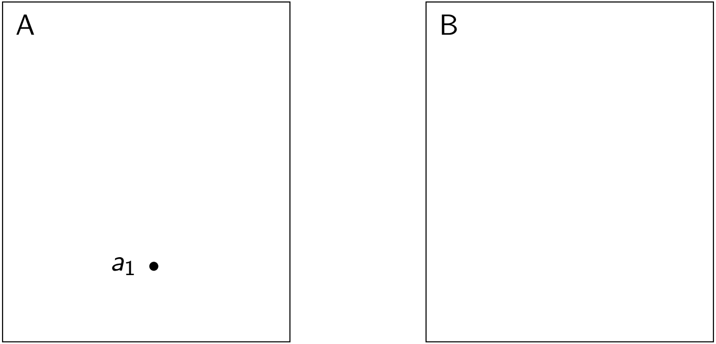

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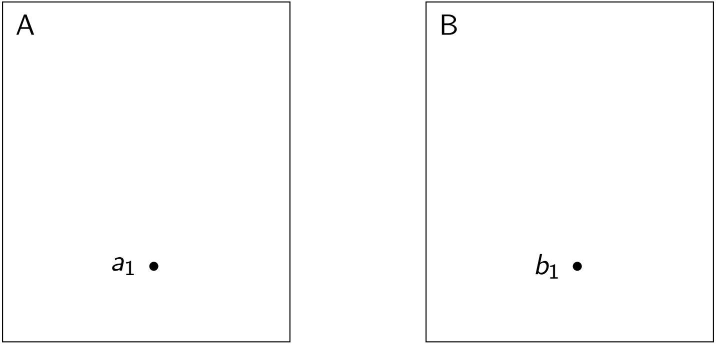

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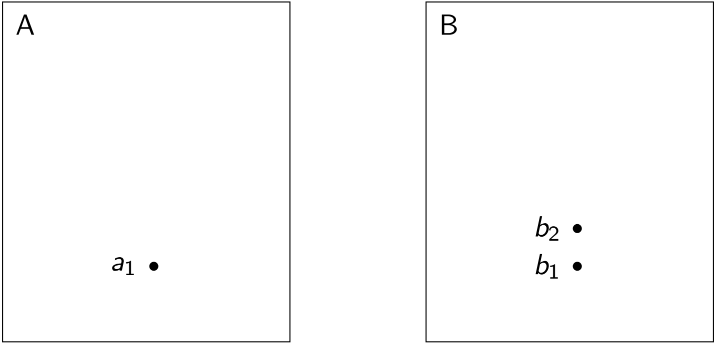

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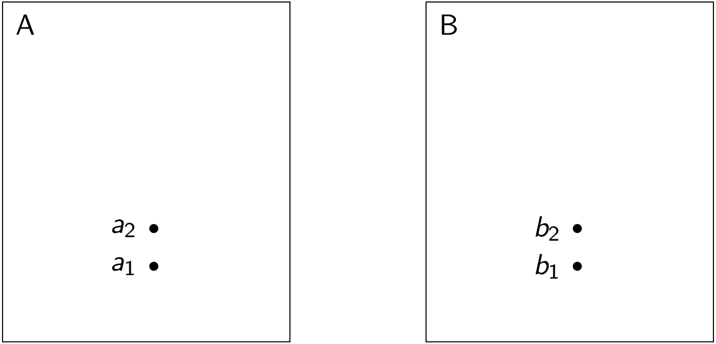

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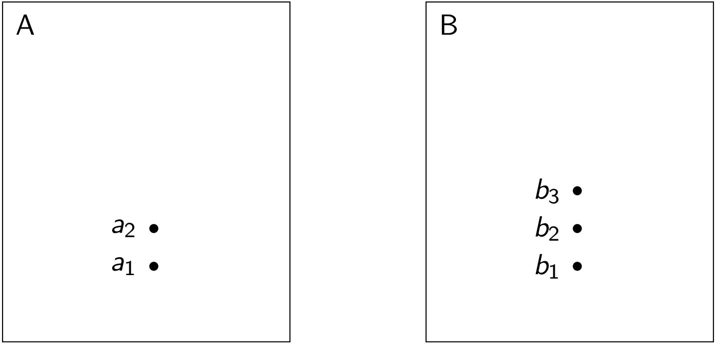

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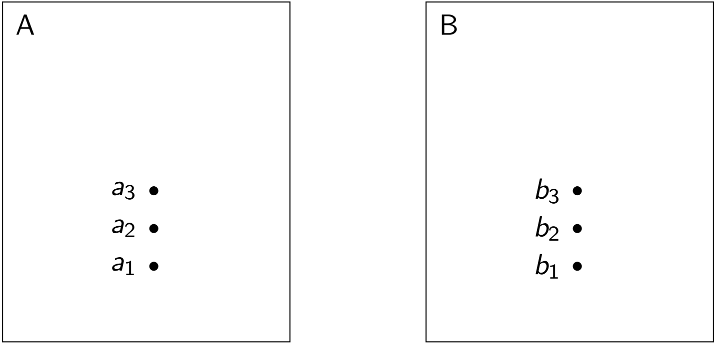

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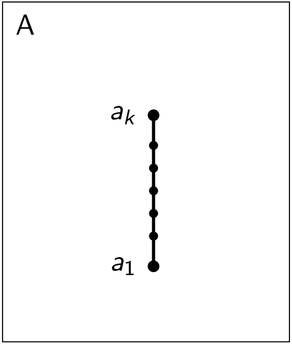

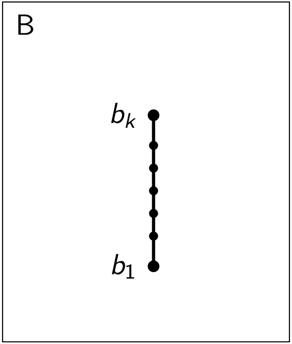

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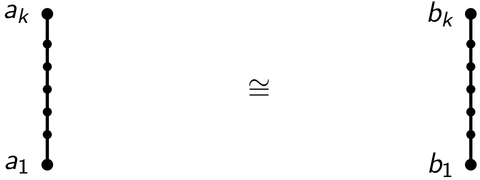

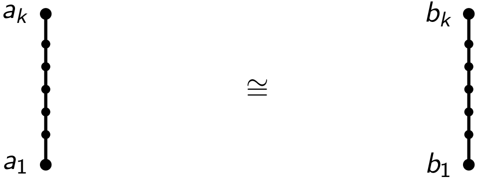

Theorem

$A \equiv^k B$ iff Duplicator wins in the k-round E–F game.

Back-and-forth EF games formally

The (back-and-forth) Ehrenfeucht–Fraïssé game between structures A and B:

- In the i^{th} round Spoiler and Duplicator pick elements a_i, b_i as follows:
  - Spoiler chooses an element $a_i \in A$ or $b_i \in B$;
  - Duplicator responds by picking an element b_i or a_i in the other structure.
- Duplicator wins after k rounds if $\{(a_i, b_i) \mid i = 1, \ldots, k\}$ is a **partial isomorphism** between A and B.

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1. For all $i, j \in \{1, \ldots, k\}$, $a_i = a_j \iff b_i = b_j$.
2. For all relation symbols R of arity n and all $i_1, \ldots, i_n \in \{1, \ldots, k\}$,

$$(a_{i_1}, \ldots, a_{i_n}) \in R^A \iff (b_{i_1}, \ldots, b_{i_n}) \in R^B.$$

Theorem (Ehrenfeucht & Fraïssé, 1954 and 1961)

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round back-and-forth Ehrenfeucht–Fraïssé game between A and B.*
2. $A \equiv^{\mathrm{FO}_k} B$. *That is, for all first-order sentences φ with quantifier rank at most k, $A \models \varphi \iff B \models \varphi$.*

Exercise

Let $A = (\mathbb{N}, <)$ and $B = (\{1, \ldots, 5\}, <)$. Does Duplicator have a winning strategy in the 2-round back-and-forth EF game?

Back-and-forth EF games and logic

Theorem (Ehrenfeucht & Fraïssé, 1954 and 1961)

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Exercise

Let $A = (\mathbb{N}, <)$ and $B = (\{1, \dots, 5\}, <)$. Does Duplicator have a winning strategy in the 2-round back-and-forth EF game?

Answer: No. Bonus exercise: Find a quantifier rank 2 formula that distinguishes A and B.

Forth-only EF games

Forth-only variant of the EF game: Spoiler plays always in A and Duplicator responds in B.

- Duplicator wins after k rounds if $\{(a_i, b_i) \mid i = 1, \ldots, k\}$ is a **partial homomorphism** from A to B.

1. For all $i, j \in \{1, \ldots, k\}$, $a_i = a_j \implies b_i = b_j$.
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$$(a_{i_1}, \ldots, a_{i_n}) \in R^A \implies (b_{i_1}, \ldots, b_{i_n}) \in R^B.$$

Note: Duplicator can win the forth-only game in both directions but still lose the back-and-forth game!

Consider e.g. $A = (\mathbb{N}, \leq)$ and $B = (\{1, \ldots, 5\}, \leq)$.

Forth-only EF games and logic

Theorem

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round forth-only Ehrenfeucht–Fraïssé game played from A to B.*
2. $A \Rrightarrow^{\mathrm{PP}_k} B$. *That is, for all primitive positive sentences φ with quantifier rank at most k, $A \models \varphi \implies B \models \varphi$.*

Exercise

Show that Spoiler has a winning strategy in the 3-round forth-only EF game from $A = (\mathbb{N}, <)$ to $B = (\{1, \ldots, 5\}, <)$.

Find a primitive positive φ with quantifier rank at most 3 such that $A \models \varphi$ and $B \not\models \varphi$.

The Ehrenfeucht–Fraïssé comonad

Intuition:

- View games as semantic constructions in the category $\mathbf{Str}(\sigma)$.
- Given a σ-structure A, define a new σ-structure $\mathbb{E}_k(A)$ on the set of all possible plays in A in the k-round EF game.
- This yields an 'operation':

$$\mathbb{E}_k \colon \mathbf{Str}(\sigma) \to \mathbf{Str}(\sigma)$$

Theorem

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round forth-only EF game played from A to B.*
2. *There exists a Kleisli morphism $\mathbb{E}_k(A) \to B$.*

The Ehrenfeucht–Fraïssé comonad formally

For every structure A, define $\mathbb{E}_k(A)$ by:

- The universe:

$$\mathbb{E}_k(A) = \{[a_1, \ldots, a_j] \mid a_1, \ldots, a_j \in A,\ 1 \leq j \leq k\}$$

$\approx$ plays in A

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- The "last move" function:

$$\varepsilon_A \colon \mathbb{E}_k(A) \to A, \qquad [a_1, \ldots, a_j] \mapsto a_j$$

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- Extend $\mathbb{E}_k(A)$ to a σ-structure: for an n-ary relation $R \in \sigma$, set

$$(s_1, \dots, s_n) \in R^{\mathbb{E}_k(A)}$$

  if and only if

  1. $s_1, \dots, s_n$ are pairwise comparable in the prefix order, and
  2. $(\varepsilon_A(s_1), \dots, \varepsilon_A(s_n)) \in R^A$.

Exercise

Visualise $\mathbb{E}_3 A$ where A is the following graph:

$A =$ 

Recall: The signature of graphs is just $\sigma = \{R(\cdot, \cdot)\}$.

Strategies as Kleisli morphisms: the case of $\mathbb{E}_k$

Theorem

The following statements are equivalent for all structures A, B:

1. *Duplicator has a winning strategy in the k-round forth-only EF game played from A to B.*
2. *There exists a homomorphism $\mathbb{E}_k(A) \to B$.*

Strategies as Kleisli morphisms: the case of $\mathbb{E}_k$

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1. *Duplicator has a winning strategy in the k-round forth-only EF game played from A to B.*
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Proof.

$1 \Rightarrow 2$. A Duplicator strategy in the k-round forth-only EF game from A to B defines a function $\mathbb{E}_k(A) \to B$. The winning condition ensures that this function is a homomorphism.

$2 \Rightarrow 1$. Fix a homomorphism $f \colon \mathbb{E}_k(A) \to B$ and suppose Spoiler plays $a_1, \ldots, a_k$. Duplicator responds with $b_i = b_j$ if $a_i = a_j$ for some $j < i$, or $b_i = f([a_1, \ldots, a_i])$ otherwise. □

Composition of logical relations

Notice that $A \Rrightarrow^{\mathrm{PP}_k} B$ and $B \Rrightarrow^{\mathrm{PP}_k} C$ imply $A \Rrightarrow^{\mathrm{PP}_k} C$.

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Composition of logical relations

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Strategies compose too: forth-only Duplicator winning strategies from A to B and from B to C yield a strategy from A to C.

But how do we compose $\mathbb{E}_k(A) \to B$ and $\mathbb{E}_k(B) \to C$?

The rest of the comonad structure

- The functions $\varepsilon_A\colon \mathbb{E}_k(A) \to A$ are homomorphisms in $\mathbf{Str}(\sigma)$.
- Reconstructing the history of Duplicator's answers:

  Each homomorphism $f\colon \mathbb{E}_k(A) \to B$ induces a homomorphism

$$f^*\colon \mathbb{E}_k(A) \to \mathbb{E}_k(B)$$
$$[a_1, \dots, a_j] \mapsto [f([a_1]), f([a_1, a_2]), \dots, f([a_1, \dots, a_j])].$$

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These data define a **comonad** on the category $\mathbf{Str}(\sigma)$, called the Ehrenfeucht–Fraïssé comonad.

Rather a family of comonads, indexed by the *resource parameter k* (number of rounds).

Comonads defined

A **comonad** (in Kleisli–Manes form) on a category $\mathscr{C}$ is given by:

- an object map $G\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$,
- a **counit** morphism $\varepsilon_A\colon G(A) \to A$ for every $A \in \mathrm{Ob}(\mathscr{C})$,
- a **coextension operation** associating with any morphism $f\colon G(A) \to B$ a morphism $f^*\colon G(A) \to G(B)$,

such that for all morphisms $f\colon G(A) \to B$ and $g\colon G(B) \to C$:

$$\varepsilon_A^* = \mathrm{id}_{G(A)}, \quad \varepsilon_B \circ f^* = f, \quad (g \circ f^*)^* = g^* \circ f^*.$$

Exercise

- Check, for $\mathbb{E}_k$, the comonad laws:
  1. $\varepsilon_A^* = \mathrm{id}_{\mathbb{E}_k(A)}$
  2. $\varepsilon_B \circ f^* = f$
  3. $(g \circ f^*)^* = g^* \circ f^*$

- Observe that $A \to B$ implies $\mathbb{E}_k(A) \to B$, but not vice versa. (Btw, what is the logical reading of this?)

Games and bounded variable count

Pebble games

(Back-and-forth) k-pebble game: Players place two sets of pebbles $\{p_1, \ldots, p_k\}$ each on one of the structures A, B.

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(Back-and-forth) k-pebble game: Players place two sets of pebbles $\{p_1, \ldots, p_k\}$ each on one of the structures A, B.

- In the i^{th} round, Spoiler places pebble p_u on an element of one of the structures.
- Duplicator places the corresponding pebble p_u on an element of the other structure.
- Duplicator wins after n rounds if the relation determined by the *current placings* of the pebbles is a partial isomorphism.
- Duplicator wins the k-pebble game if they have a strategy which is winning after n rounds, for all $n \geq 0$.

Note: Pebbles can be moved forever, this is an infinite game.

Pebble games and logic

Theorem

*The following are equivalent for all *finite* structures A, B:*

1. *Duplicator has a winning strategy in the back-and-forth k-pebble game between A and B.*
2. $A \equiv^{\mathrm{FO}^k} B$. *That is, for all first-order sentences φ with at most k variables,* $A \vDash \varphi \iff B \vDash \varphi$.

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2. $A \equiv^{\mathrm{FO}^k} B$*. That is, for all first-order sentences φ with at most k variables,* $A \models \varphi \iff B \models \varphi$.

Similarly, the following are equivalent:

3. *Duplicator has a winning strategy in the <u>forth-only</u> k-pebble game played from A to B.*
4. $A \Rrightarrow^{\mathrm{PP}^k} B$*. That is, for all <u>primitive positive</u> sentences φ with at most k variables,* $A \models \varphi \implies B \models \varphi$.

The pebble comonad

For every structure A and fixed $\boldsymbol{k} := \{p_1, \ldots, p_k\}$, let

- The universe:

$$\mathbb{P}_k(A) = \{[(p_1, a_1), \ldots, (p_j, a_j)] \mid p_i \in \boldsymbol{k}, a_i \in A\}$$

  where p_i is the **pebble index** of move (p_i, a_i).

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- $\varepsilon_A \colon \mathbb{P}_k(A) \to A$, $[(p_1, a_1), \ldots, (p_j, a_j)] \mapsto a_j$.

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- $\varepsilon_A \colon \mathbb{P}_k(A) \to A$, $[(p_1, a_1), \ldots, (p_j, a_j)] \mapsto a_j$.

- For each relation R of arity n, set $(s_1, \ldots, s_n) \in R^{\mathbb{P}_k(A)}$ iff
  1. $s_1, \ldots, s_n$ are pairwise comparable in the prefix order,
  2. $(\varepsilon_A(s_1), \ldots, \varepsilon_A(s_n)) \in R^A$,
  3. for all $i, j \in \{1, \ldots, n\}$, if s_i is a prefix of s_j, the pebble index of the last move of s_i does not appear in the suffix of s_i in s_j.

The pebble comonad

- The functions $\varepsilon_A \colon \mathbb{P}_k(A) \to A$ are homomorphisms.
- Reconstructing the history of Duplicator's answers:

  Each homomorphism $f \colon \mathbb{P}_k(A) \to B$ induces a homomorphism

$$f^* \colon \mathbb{P}_k(A) \to \mathbb{P}_k(B)$$
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  where $b_i := f([(p_1, a_1), \dots, (p_i, a_i)])$ for all $i = 1, \dots, j$.

These data define a comonad on the category $\mathbf{Str}(\sigma)$, called the pebbling comonad.

Family of comonads, indexed by the *resource parameter k* (number of pebbles)

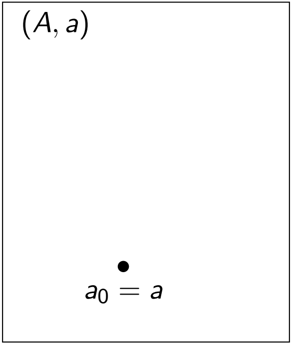
Theorem

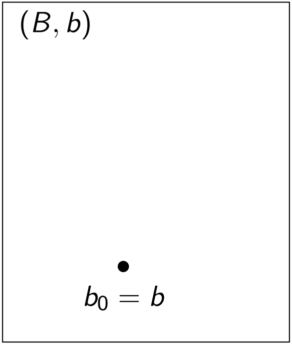
*The following are equivalent for all *finite* structures A, B:*

1. *Duplicator has a winning strategy in forth-only k-pebble game from A to B.*
2. *There exists a homomorphism $\mathbb{P}_k(A) \to B$.*

Games and bounded modal depth

k-round bisimulation game





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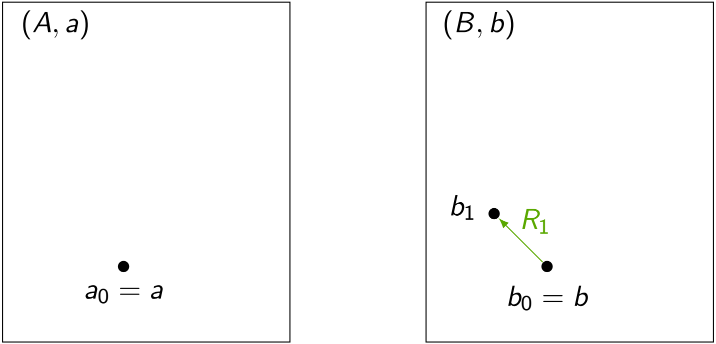

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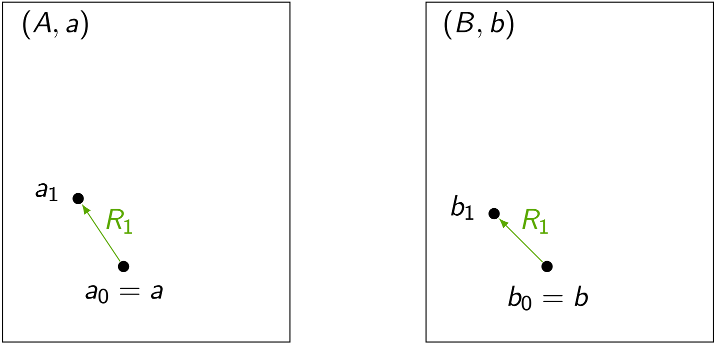

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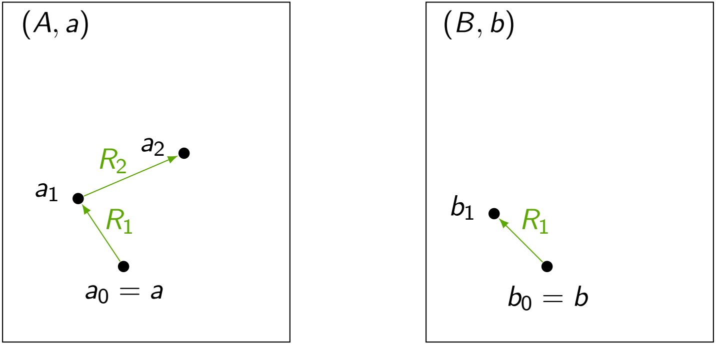

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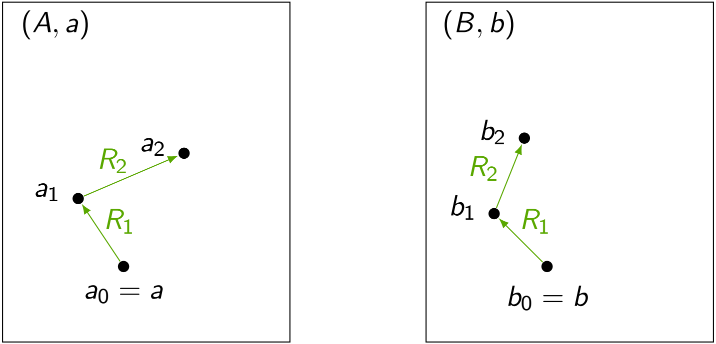

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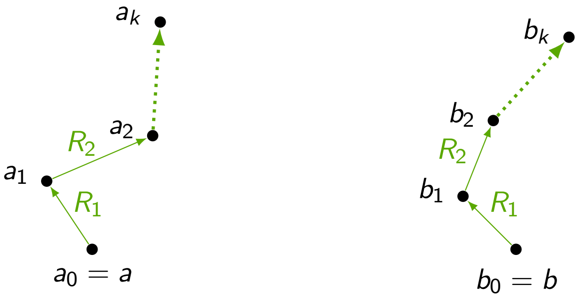
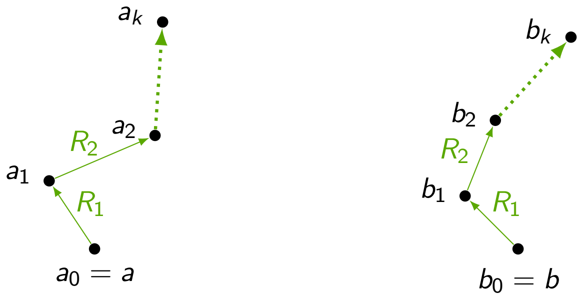


Check if $\quad a_i \in P^A \iff b_i \in P^B \qquad (\forall P, i)$



Check if $\quad a_i \in P^A \iff b_i \in P^B \qquad (\forall P, i)$

Theorem

$(A, a) \equiv^{\mathrm{ML}_k} (B, b)$ *iff Duplicator wins the k-round bisim. game.*

Bisimulation games

Bisimulation game (for modal logic) between pointed Kripke structures (A, a) and (B, b):

Bisimulation games

Bisimulation game (for modal logic) between pointed Kripke structures (A, a) and (B, b):

- The initial position is $(a_0, b_0) \coloneqq (a, b)$.
- In the i^{th} round, where the previous position was (a_{i-1}, b_{i-1}), Spoiler chooses a binary relation R, one of the two structures, say A, and $a_i \in A$ such that $(a_{i-1}, a_i) \in R^A$.
- Duplicator responds with an element of the other structure, say $b_i \in B$, such that $(b_{i-1}, b_i) \in R^B$.

Bisimulation games

Bisimulation game (for modal logic) between pointed Kripke structures (A, a) and (B, b):

- The initial position is $(a_0, b_0) := (a, b)$.

- In the i^{th} round, where the previous position was (a_{i-1}, b_{i-1}), Spoiler chooses a binary relation R, one of the two structures, say A, and $a_i \in A$ such that $(a_{i-1}, a_i) \in R^A$.
- Duplicator responds with an element of the other structure, say $b_i \in B$, such that $(b_{i-1}, b_i) \in R^B$.

- Duplicator loses if there is no such response available.
- Duplicator wins after k rounds if, for all unary predicates P, we have $a_i \in P^A \iff b_i \in P^B$ for all $i \in \{0, \dots, k\}$.

(Bi)simulation games and logic

Theorem

The following statements are equivalent for all pointed Kripke structures $(A, a), (B, b)$*:*

1. *Duplicator has a winning strategy in the* k*-round bisimulation game between* (A, a) *and* (B, b)*.*
2. $A \equiv^{\mathrm{ML}_k} B$*. That is, for all modal formulas* φ *of modal depth at most* k*,* $A, a \vDash \varphi \iff B, b \vDash \varphi$*.*

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Theorem

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Similarly, the following are equivalent:

i.e. forth-only bisimulations

3. *Duplicator has a winning strategy in the k-round simulation game played from (A, a) to (B, b).*
4. *For all primitive positive modal formulas φ of modal depth at most k, $A, a \models \varphi \implies B, b \models \varphi$.*

The modal comonad

For every pointed Kripke structure $\mathbf{A} = (A, a)$,

- $\mathbb{M}_k(\mathbf{A}) =$ the set of paths of length $\leq k$ starting from a:

$$a \xrightarrow{R_1} a_1 \xrightarrow{R_2} a_2 \rightarrow \cdots \xrightarrow{R_n} a_n$$

  where $R_1, \ldots, R_n$ are binary relations.

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- $\varepsilon_{\mathbf{A}} : \mathbb{M}_k(\mathbf{A}) \to A$ sends a path to its last element.
- For a unary relation P, set $P^{\mathbb{M}_k(\mathbf{A})} = \{s \mid \varepsilon_{\mathbf{A}}(s) \in P^A\}$.
- For a binary relation R, set $(s, t) \in R^{\mathbb{M}_k(\mathbf{A})}$ if and only if

$$t \;=\; \underbrace{a \xrightarrow{R_1} \cdots \xrightarrow{R_n} a_n}_{s} \xrightarrow{R} a_{n+1}$$

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- The distinguished element of $\mathbb{M}_k(\mathbf{A})$ is the trivial path (a).

The modal comonad

- The functions $\varepsilon_{\mathbf{A}} \colon \mathbb{M}_k(\mathbf{A}) \to \mathbf{A}$ become homomorphisms of pointed Kripke structures.
- Each homomorphism $f \colon \mathbb{M}_k(\mathbf{A}) \to \mathbf{B}$ yields a homomorphism

$$f^* \colon \mathbb{M}_k(\mathbf{A}) \to \mathbb{M}_k(\mathbf{B})$$

$$(a \xrightarrow{R_1} a_1 \xrightarrow{R_2} \cdots \xrightarrow{R_n} a_n) \mapsto (b \xrightarrow{R_1} b_1 \xrightarrow{R_2} \cdots \xrightarrow{R_n} b_n)$$

where $b_i := f(a \xrightarrow{R_1} a_1 \xrightarrow{R_2} \cdots \xrightarrow{R_i} a_i)$.

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where $b_i \coloneqq f(a \xrightarrow{R_1} a_1 \xrightarrow{R_2} \cdots \xrightarrow{R_i} a_i)$.

These data define a comonad, called modal comonad, on the category $\mathbf{Str}_*(\sigma)$ of pointed Kripke structures and their homomorphisms.

Family of comonads, indexed by the *resource parameter* k (number of rounds)

Theorem

The following statements are equivalent for all pointed Kripke structures $\mathbf{A}, \mathbf{B}$*:*

1. *Duplicator has a winning strategy in the* k*-round simulation game played from* $\mathbf{A}$ *to* $\mathbf{B}$*.*
2. *There exists a Kleisli morphism* $\mathbb{M}_k(\mathbf{A}) \to \mathbf{B}$*.*

The big picture

The Kleisli category of a comonad

Morphisms $G(A) \to B$ in $\mathscr{C}$, for a comonad G, are called **Kleisli morphism**, we also denote them by $A \to_G B$.

They induce the **Kleisli category** of G, denoted $\mathbf{K}(G)$, such that

- $\mathrm{Ob}(\mathbf{K}(G)) = \mathrm{Ob}(\mathscr{C})$
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Exercise

Check that $\mathbf{K}(G)$ is a category, from the comonad axioms for G.

Logic vs Kleisli morphisms

The typical scenario:

logic $\mathscr{L}$	$\leftrightarrow$	game	$\rightsquigarrow$	comonad G
$\Rrightarrow^{\mathscr{L}\cap\mathrm{PP}}$	$\leftrightarrow$	winning Duplicator strategy in the forth-only game	$\cong$	Kleisli morphism $\to_G$

Consequently,

$$\Rrightarrow^{\mathscr{L}\cap\mathrm{PP}} \quad = \quad \to_G$$

for appropriate choices of G and $\mathscr{L}$.

E.g., if $G = \mathbb{E}_k$ and $\mathscr{L} = \mathrm{FO}_k \cap \mathrm{PP} = \mathrm{PP}_k$ then

$$A \Rrightarrow^{\mathrm{PP}_k} B \quad \iff \quad A \to_{\mathbb{E}_k} B$$

Outlook

The Kleisli category $\mathbf{K}(G)$ arises naturally by considering winning strategies in various forth-only games.

- From a logical viewpoint $\mathbf{K}(G)$ captures preservation of primitive positive fragments.
- $\mathbf{K}(G)$ sits in a larger category of **coalgebras** for G that capture combinatorial parameters of structures.
  This is the topic of tomorrow's lecture.

References

Model comparison games:

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Game comonads:

- S. Abramsky, A. Dawar, P. Wang, *The pebbling comonad in finite model theory*, 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS, pp. 1–12, 2017.
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Bonus slides:
Comparing logics

Exercise

Given two comonads G, H on $\mathscr{C}$, if there is a collection of morphisms

$$\{\alpha_A \colon G(A) \to H(A)\}_{A \in \mathrm{Ob}(\mathscr{C})}$$

show that

$$\frac{A \to_H B}{A \to_G B}$$

Exercise

Find some examples of

$$\{\alpha_A \colon G(A) \to H(A)\}_{A \in \mathrm{Ob}(\mathbf{Str}(\sigma))}$$

for our comonads $\mathbb{E}_k, \mathbb{P}_k, \mathbb{M}_k$.