An Invitation to Game Comonads, day 1: Overview, Syntax vs Semantics [a]

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Overview

Motivation

Notoriously difficult problems:

Constraint Satisfaction Problem (CSP)

Input: finite structures A, B
Decide: is there a homomorphism $A \to B$?

Isomorphism Problem

Input: finite structures A, B
Decide: is $A \cong B$?

Difficult even with B fixed!

Approximations

Polynomial-time decidable $\rightsquigarrow$ and $\approx$ such that

$$\frac{A \to B}{A \rightsquigarrow B} \qquad \text{and} \qquad \frac{A \cong B}{A \approx B}$$

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Examples: local consistency and Weisfeiler-Leman tests

$$\frac{A \to B}{A \Rrightarrow^{\mathscr{L}} B} \qquad \text{and} \qquad \frac{A \cong B}{A \equiv^{\mathscr{I}} B}$$

where

$$A \Rrightarrow^{\mathscr{L}} B \quad \Longleftrightarrow \quad \forall \varphi \in \mathscr{L} \quad A \models \varphi \text{ implies } B \models \varphi$$

$$A \equiv^{\mathscr{I}} B \quad \Longleftrightarrow \quad \forall \varphi \in \mathscr{I} \quad A \models \varphi \text{ iff } B \models \varphi$$

First-Order Logic

In our case $\mathscr{L} \subseteq \mathrm{FO}$ or $\mathscr{L} \subseteq \mathrm{ML}$.

First-Order Logic (FO) in a relational signature $\sigma = \{R_1, \dots, R_t\}$ has

- atomic formulas: $x = y$, $R(x_1, \dots, x_n)$ (for n-ary $R \in \sigma$)
- connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$
- quantifiers: $\forall x\, \varphi$, $\exists x\, \varphi$

Models: σ-structures A, given as tuples

$$(A, R_1^A, \dots, R_t^A)$$

where, for n-ary $R \in \sigma$,

$$R^A \subseteq A^n.$$

Then, $A \vDash R(a_1, \dots, a_n)$ iff $(a_1, \dots, a_n) \in R^A$.

Modal Logic

A **(multi)modal signature** $\sigma = \{R_1, \ldots, R_n, P_1, \ldots, P_m\}$ is given by $R_1, \ldots, R_n$ binary and $P_1, \ldots, P_m$ unary relations.

Modal Logic (ML) in a modal signature σ has

- propositional letters: P (for unary $P \in \sigma$)
- connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$
- modalities: $\Box_R\, \varphi$, $\Diamond_R\, \varphi$ (for binary $R \in \sigma$)

Models: pointed σ-structures (A, a), i.e. $a \in A$

$$
\begin{aligned}
(A, a) \models P &\iff a \in P^A \\
(A, a) \models \Box_R\, \varphi &\iff \forall (a, b) \in R^A \;\; (A, b) \models \varphi \\
(A, a) \models \Diamond_R\, \varphi &\iff \exists (a, b) \in R^A \;\; (A, b) \models \varphi
\end{aligned}
$$

For certain $\mathscr{L} \subseteq \mathrm{FO}$ there exists a (turn-based) game $\mathscr{G}$ of two players

- **Spoiler** wants to show $A \ncong B$
- **Duplicator** wants to show $A \cong B$

and

$$A \equiv^{\mathscr{L}} B \quad \overset{\text{(Thm)}}{\Longleftrightarrow} \quad \text{Duplicator has a winning strategy in } \mathscr{G}$$

Model comparison games

For certain $\mathscr{L} \subseteq \mathrm{FO}$ there exists a (turn-based) game $\mathscr{G}$ of two players

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Typically, $\mathscr{L}$ and $\mathscr{G}$ parametrised by a resource parameter k, e.g.

$$\begin{array}{rcl} \text{quantifier rank} & \leftrightarrow & \text{number of rounds} \\ \text{variable count} & \leftrightarrow & \text{number of pebbles} \end{array}$$

Semantic encoding of Duplicator winning strategies (days 2 & 4)

$G(A)$ encoding Spoiler's possible moves on A such that

$$A \Rrightarrow^{\mathscr{L}} B \quad \overset{\text{(Thm)}}{\Longleftrightarrow} \quad G(A) \to B$$

$$A \equiv^{\mathscr{I}} B \quad \overset{\text{(Thm)}}{\Longleftrightarrow} \quad G(A) \approx G(B)$$

Giving approximations

$$\frac{A \to B}{G(A) \to B} \qquad \text{and} \qquad \frac{A \cong B}{G(A) \approx G(B)}$$

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Giving approximations

$$\frac{A \to B}{G(A) \to B} \qquad \text{and} \qquad \frac{A \cong B}{G(A) \approx G(B)}$$

$G(\cdot)$ is a **comonad**

$\Rightarrow$ new shiny tools from category theory!

Syntactic vs Combinatorial Parameters

Coalgebras for comonads reveal a structural connection:

Comonad	Combinatorial property	logic fragment
$\mathbb{E}_k$	tree-depth	qauntifier rank
$\mathbb{P}_k$	tree-width	variable count
$\mathbb{M}_k$	sync. tree depth	modal depth
$\mathbb{PR}_k$	path-width	restricted conjunction & variable count
$\mathbb{He}_k$	k-ary generalised t.-w.	generalised quantifier k-variable extension
$\mathbb{G}_k$	guarded tree decomp	quantifier-guarded
$\mathbb{LG}_k$	hypertree-width	k-conjunct guarded
$\mathbb{Hy}_k$	generated tree-depth	hybrid modal depth
$\mathbb{B}_k$	generated tree-depth	bounded quantifiers
...	...	...

Categorified statements

Uniform proofs of

- Lovász-type homomorphism-counting theorems
- van Benthem-type theorems
- Feferman–Vaught–Mostowski theorems
- Homomorphism Preservation Theorems
- Hudges' word construction
- Gaifman & Hanf locality

A synthetic framework for generic results

- arboreal categories

Category Theory 101

Origins

Common patterns in mathematics

Objects of study	their structure-preserving mappings
sets	functions
vector spaces	linear maps
monoids	monoid homomorphisms
posets	monotone maps
topological spaces	continuous maps

Many properties and constructions of these structures are characterised by *universal properties* of their mappings.

Category theory studies properties of mappings abstractly.
$\Rightarrow$ Generic results that apply to many scenarios.

The main definition

A **category** $\mathscr{C}$ consists of

- a class of objects $\mathrm{Ob}(\mathscr{C})$
- for $A, B \in \mathrm{Ob}(\mathscr{C})$, a set of morphisms $\mathscr{C}(A, B)$, which we designate by

$$f\colon A \to B$$

- for $A \in \mathrm{Ob}(\mathscr{C})$, identity morphism $\mathrm{id}_A\colon A \to A$
- for $A, B, C \in \mathrm{Ob}(\mathscr{C})$, a composition operation

$$\circ : \mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

Such that, whenever the compositions are defined:

$$f \circ \mathrm{id}_A = f$$

$$\mathrm{id}_A \circ f = f$$

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Examples of categories

Set

- objects: sets
- morphisms: functions
- identity morphisms: identity functions
- composition operation: function composition

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- objects: sets
- morphisms: functions
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Set$_*$

- objects: pointed sets (X, x), with $x \in X$,
- morphisms: $(X, x) \to (Y, y)$ are functions $f : X \to Y$
  such that $f(x) = y$.

Categories of relational structures

Str(σ)

- objects: σ-structures A
- morphisms: homomorphisms of σ-structures $f\colon A \to B$

$$(a_1, \ldots, a_n) \in R^A \quad \Longrightarrow \quad (f(a_1), \ldots, f(a_n)) \in R^B$$

for an n-ary $R \in \sigma$

$\mathbf{Str}_{\mathit{fin}}(\sigma) =$ restriction of **Str**(σ) to <u>finite</u> σ-structures

Categories of relational structures

$\mathbf{Str}(\sigma)$

- objects: σ-structures A
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for an n-ary $R \in \sigma$

$\mathbf{Str}_{fin}(\sigma)$ = restriction of $\mathbf{Str}(\sigma)$ to <u>finite</u> σ-structures

$\mathbf{Str}_*(\sigma)$

- objects: pointed σ-structures (A, a) i.e. $a \in A$
- morphisms: $(A, a) \to (B, b)$ are σ-structure homomorphisms
  $f\colon A \to B$ such that $f(a) = b$.

Functors = "homomorphisms of categories"

For categories $\mathscr{C}, \mathscr{D}$, a functor $F\colon \mathscr{C} \to \mathscr{D}$ is given by

- a mapping on objects $F\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{D})$
- a mapping on morphisms, for every $A, B \in \mathscr{C}$,

$$F\colon \mathscr{C}(A, B) \to \mathscr{D}(F(A), F(B))$$

  I.e. $f\colon A \to B$ is mapped to $F(f)\colon F(A) \to F(B)$.

These must preserve the rest of the category structure:

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$$
$$F(f \circ g) = F(f) \circ F(g)$$

Examples of functors

For a category $\mathscr{C}$, the **identity functor** $\mathrm{Id}_{\mathscr{C}} \colon \mathscr{C} \to \mathscr{C}$ is given by

- the identity mapping on objects $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$
- the identity mapping on morphisms $\mathscr{C}(A, B) \to \mathscr{C}(A, B)$

Forgetful functors:

(1) $\mathbf{Set}_* \to \mathbf{Set}$

- on objects $(A, a) \mapsto A$
- on morphisms $f \mapsto f$

(2) $\mathbf{Str}(\sigma) \to \mathbf{Set}$

- on objects $(A, R_1^A, \ldots, R_t^A) \mapsto A$
- on morphisms $f \mapsto f$

Exercise: Show that, for every relational signature σ, we have a functor $\mathbf{Set} \to \mathbf{Str}(\sigma)$ which maps a set A to $(A, R_1^A, \ldots, R_t^A)$ where $R_i^A = A^n$, for an n-ary $R_i \in \sigma$.

Syntax vs Semantics

Semantics to logic

For any fragment $\mathscr{L}$

$$\frac{A \cong B}{A \equiv^{\mathscr{L}} B}$$

But when do we get

$$\frac{A \to B}{A \Rrightarrow^{\mathscr{L}} B} \quad ?$$

Recall

$$A \Rrightarrow^{\mathscr{L}} B \quad \Longleftrightarrow \quad \forall \varphi \in \mathscr{L} \quad A \vDash \varphi \text{ implies } B \vDash \varphi$$

Primitive positive fragment

Primitive positive sentences $\mathrm{PP} \subseteq \mathrm{FO}$ are formed by

- atomic formulas: $\mathbf{t}$, $R(x_1, \ldots, x_n)$ (for n-ary $R \in \sigma$)
- conjunctions: $\varphi \wedge \psi$
- existential quantifiers: $\exists x\, \varphi$

I.e. we do not allow equality $x = y$, disjunctions $\varphi \vee \psi$, negations $\neg\varphi$, universal quantifications $\forall x\, \varphi$.

We have added the always true sentence $\mathbf{t}$, which holds $A \vDash \mathbf{t}$ in every σ-structure A.

Examples of PP sentences

(1) A well-formed PP sentence

$$\exists xyz\,(R(x,y) \wedge P(y) \wedge S(y,z))$$

in signature $\sigma = \{R(\cdot,\cdot),\ P(\cdot),\ S(\cdot,\cdot),\ T(\cdot,\cdot,\cdot)\}$.

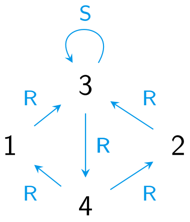
(2) Despite equivalence of

$$(\exists x.\ R(x,x)) \vee (\exists x.\ R(x,x)) \qquad \text{and} \qquad \exists x.\ R(x,x)$$

The former is not PP!

Exercise

Given a σ-structure A:



in signature $\sigma = \{R(\cdot,\cdot), S(\cdot,\cdot)\}$, and a PP sentence φ:

$$\exists x \Big(\exists y \, (R(x,y) \wedge \exists z (R(y,z) \wedge R(z,x))) \\ \wedge \exists z \, (S(z,z) \wedge R(x,z)) \Big)$$

Decide if $A \vDash \varphi$.

Evaluating PP sentences, I

Step 1: Variable renaming in φ

$$\exists x_1 \, (\exists x_2 \, (R(x_1, x_2) \wedge \exists x_3 (R(x_2, x_3) \wedge R(x_3, x_1))) \\ \wedge \exists x_4 \, (S(x_4, x_4) \wedge R(x_1, x_4)))$$

Observation

If x does not occur freely in ψ then

$$\exists x \, (\varphi \wedge \psi) \quad \text{and} \quad (\exists x \, \varphi) \wedge \psi$$

are equivalent, in first-order logic.

Step 2: Rewrite φ into the prenex normal form

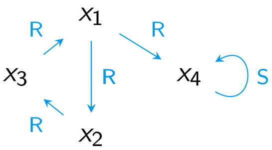
$$\exists x_1, x_2, x_3, x_4 \, (R(x_1, x_2) \wedge R(x_2, x_3) \wedge R(x_3, x_1) \\ \wedge S(x_4, x_4) \wedge R(x_1, x_4))$$

Evaluating PP sentences, II

Step 3: From the prenex normal form $\exists x_1, x_2, x_3, x_4\ \varphi_0$ where

$$\begin{aligned}\varphi_0(x_1, x_2, x_3, x_4) \quad &= \quad R(x_1, x_2) \land R(x_2, x_3) \land R(x_3, x_1) \\ &\quad\quad \land S(x_4, x_4) \land R(x_1, x_4)\end{aligned}$$

we build a σ-structure $\mathbf{M}(\varphi)$ on universe $\{x_1, x_2, x_3, x_4\}$



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we build a σ-structure $\mathbf{M}(\varphi)$ on universe $\{x_1, x_2, x_3, x_4\}$

$$\begin{tikzcd}
& x_1 \arrow[dd, "R"] \arrow[dr, "R"] & \\
x_3 \arrow[ur, "R"] & & x_4 \arrow[loop right, "S"] \\
& x_2 \arrow[ul, "R"] &
\end{tikzcd}$$

Observation: There is a bijection

homomorphisms $\mathbf{M}(\varphi) \to A \quad \overset{1-1}{\longleftrightarrow} \quad$ assignments $v\colon x_i \mapsto a_i$ such that $A \vDash \varphi_0(v(x_1), v(x_2), v(x_3), v(x_4))$

Approximating the homomorphism order

Theorem

For any $\varphi \in \mathrm{PP}$ *there is an* $\mathbf{M}(\varphi) \in \mathbf{Str}_{fin}(\sigma)$ *such that*

$$\mathbf{M}(\varphi) \to A \quad \iff \quad A \models \varphi$$

for any σ*-structure* A*.*

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Corollary

For σ*-structures* A, B*,*

$$\frac{A \to B}{A \Rrightarrow^{\mathrm{PP}} B}$$

Proof.

For a $\varphi \in \mathrm{PP}$, if $A \vDash \varphi$ then $\mathbf{M}(\varphi) \to A \to B$.

Therefore, $B \vDash \varphi$. $\square$

From finite structures to sentences

Conversely, for a finite $A \in \mathbf{Str}_{fin}(\sigma)$, we construct a $\Psi(A) \in \mathrm{PP}$ by listing everything true in A in a prenex normal form.

Example

Take A to be as follows

$$A \quad = \quad \begin{tikzcd} a_1 \arrow[r, "S"] \arrow[d, "R"'] & a_4 \arrow[d, "R"] \\ a_2 \arrow[r, "S"] \arrow[d, "R"'] & a_5 \arrow[d, "R"] \\ a_3 \arrow[r, "S"] & a_6 \end{tikzcd}$$

Set $\Psi(A)$ to be

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Set $\Psi(A)$ to be

$$\exists x_1, \ldots, x_6 \; (\bigwedge_{i \in \{1,2,4,5\}} R(x_i, x_{i+1}) \wedge \bigwedge_{i \in \{1,2,3\}} S(x_i, x_{i+3}))$$

Approximating $\Rrightarrow^{\mathrm{PP}}$

Theorem

For any finite $A \in \mathbf{Str}(\sigma)$ *there is a* $\Psi(A) \in \mathrm{PP}$ *such that*

$$A \to B \quad \Longleftrightarrow \quad B \models \Psi(A)$$

for any σ*-structure* B*.*

Approximating $\Rrightarrow^{\mathrm{PP}}$

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For any finite $A \in \mathbf{Str}(\sigma)$ *there is a* $\Psi(A) \in \mathrm{PP}$ *such that*

$$A \to B \quad \Longleftrightarrow \quad B \models \Psi(A)$$

for any σ*-structure* B.

Corollary

For σ*-structures* A, B *with* A *finite,*

$$\frac{A \Rrightarrow^{\mathrm{PP}} B}{A \to B}$$

Proof.

From $A \models \Psi(A)$ and $A \Rrightarrow^{\mathrm{PP}} B$ we get $B \models \Psi(A)$.
Therefore, $A \to B$. $\square$

The Chandra–Merlin Correspondence [1977]

For finite A and B arbitrary,

$$A \to B \qquad \iff \qquad A \Rrightarrow^{\mathrm{PP}} B$$

And we have

$$\begin{tikzcd} \mathbf{Str}_{fin}(\sigma) \arrow[r, bend left, "\mathbf{\Psi}"] & \mathrm{PP} \arrow[l, bend left, "\mathbf{M}"] \end{tikzcd}$$

such that

$$\mathbf{M}(\varphi) \to A \quad \iff \quad A \models \varphi \quad \overset{(Thm)}{\iff} \quad \Psi(A) \vdash \varphi$$

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such that

$$\mathbf{M}(\varphi) \to A \quad \Longleftrightarrow \quad A \models \varphi \quad \overset{(Thm)}{\Longleftrightarrow} \quad \Psi(A) \vdash \varphi$$

In fact

$$\mathbf{Th}_{\mathrm{PP}}(A) = \{\varphi \in \mathrm{PP} \mid A \models \varphi\} = \{\varphi \in \mathrm{PP} \mid \Psi(A) \vdash \varphi\}$$

Logic fragments

Logic restriction: quantifier rank

For a natural number k, define

$$\mathrm{FO}_k \subseteq \mathrm{FO}$$

as the restriction to sentences φ of **quantifier rank** at most k, that is, $\texttt{qrank}(\varphi) \leq k$.

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Quantifier rank is defined inductively

$$\begin{aligned}
\mathtt{qrank}(A) &= 0 && \text{(for an atomic } A\text{)} \\
\mathtt{qrank}(\neg\varphi) &= \mathtt{qrank}(\varphi) \\
\mathtt{qrank}(\varphi \wedge \psi) &= \mathtt{qrank}(\varphi \vee \psi) = \max(\mathtt{qrank}(\varphi), \mathtt{qrank}(\psi)) \\
\mathtt{qrank}(\exists x\, \varphi) &= \mathtt{qrank}(\forall x\, \varphi) = \mathtt{qrank}(\varphi) + 1
\end{aligned}$$

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\mathtt{qrank}(\exists x\, \varphi) &= \mathtt{qrank}(\forall x\, \varphi) = \mathtt{qrank}(\varphi) + 1
\end{aligned}$$

$\boxed{\text{Define } \mathrm{PP}_k = \mathrm{FO}_k \cap \mathrm{PP}.}$

Exercise

What is the quantifier rank of

$$\exists xy\,(R(x,y) \wedge \exists z\, S(z,z,x) \wedge \exists z\, S(x,y,z)) \quad ?$$

Bounded quantifier rank approximations

For every natural number k:

$$\frac{A \to B}{A \Rrightarrow^{\mathrm{PP}_k} B} \qquad \text{and} \qquad \frac{A \cong B}{A \equiv^{\mathrm{FO}_k} B}$$

Both are polynomial-time decidable.

Logic restriction: number of variables

For a natural number k, define

$$\mathrm{FO}^k \subseteq \mathrm{FO}$$

as the restriction to sentences φ which only use variables from $x_1, \ldots, x_k$.

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$\boxed{\text{Define } \mathrm{PP}^k = \mathrm{FO}^k \cap \mathrm{PP}}$

Bounded variable count approximations

For every natural number k:

$$\frac{A \to B}{A \Rrightarrow^{\mathrm{PP}^k} B} \qquad \text{and} \qquad \frac{A \cong B}{A \equiv^{\mathrm{FO}^k} B}$$

Again, both are polynomial-time decidable.

Exercise

Is any of these true?

$$\frac{A \Rrightarrow^{\mathrm{PP}_k} B}{A \Rrightarrow^{\mathrm{PP}^k} B} \qquad\qquad \frac{A \Rrightarrow^{\mathrm{PP}^k} B}{A \Rrightarrow^{\mathrm{PP}_k} B}$$

Restrictions of ML: Modal depth

Define ML_k as the restriction of ML to formulas of **modal depth** at most k, written as $\mathtt{modep}(\varphi) \leq k$.

Modal depth is defined inductively

$$
\begin{aligned}
\mathtt{modep}(p) &= 0 \qquad\qquad\qquad \text{(for a propositional letter } p\text{)} \\
\mathtt{modep}(\neg\varphi) &= \mathtt{modep}(\varphi) \\
\mathtt{modep}(\varphi \wedge \psi) &= \mathtt{modep}(\varphi \vee \psi) = \max(\mathtt{modep}(\varphi), \mathtt{modep}(\psi)) \\
\mathtt{modep}(\Box_R\, \varphi) &= \mathtt{modep}(\Diamond_R\, \varphi) = \mathtt{modep}(\varphi) + 1
\end{aligned}
$$

Restrictions of ML: Primitive positive

Primitive positive modal formulas are formed by

- propositional letters, true statement **t**, conjunctions $\wedge$, and modalities $\Diamond_R$

$\Longrightarrow$ We also consider restrictions of ML to primitive positive formulas of modal depth at most k.

We saw that

$$A \to B \iff A \Rrightarrow^{\mathrm{PP}} B$$

However, for approximations we prefer

$$A \Rrightarrow^{\mathrm{PP}_k} B \quad \text{and} \quad A \Rrightarrow^{\mathrm{PP}^k} B$$

Question: Can we express these relations as homomorphisms of some sort?

Yes, we'll see tomorrow!

Bonus slides:
Restricting Chandra–Merlin

For

$$\mathcal{F}_k = \mathbf{M}[\mathrm{PP}_k] \quad \text{and} \quad \mathcal{F}^k = \mathbf{M}[\mathrm{PP}^k]$$

by the Chandra–Merlin correspondence we have

$$A \Rrightarrow^{\mathrm{PP}_k} B \quad \iff \quad \forall C \in \mathcal{F}_k \;\; C \to A \text{ implies } C \to B$$

$$A \Rrightarrow^{\mathrm{PP}^k} B \quad \iff \quad \forall C \in \mathcal{F}^k \;\; C \to A \text{ implies } C \to B$$

In fact, the structures in $\mathcal{F}_k$ and $\mathcal{F}^k$ have nice characterisations.

Characterising structures in $\mathcal{F}_k$

Theorem

A finite σ-structure A is in $\mathcal{F}_k$ iff there exists a binary relation $\leq$ on the universe of A such that

1. *$\leq$ is a partial order*
2. *Every set $\downarrow a = \{x \in A \mid x \leq a\}$ has cardinality $\leq k$, and is linearly ordered by $\leq$.*
3. $(a_1, \ldots, a_n) \in R^A$ *implies* $a_i \leq a_j$ *or* $a_j \leq a_i$ $\quad (\forall i, j)$.

Characterising structures in $\mathcal{F}^k$

Theorem

A finite σ-structure A is in $\mathcal{F}^k$ iff there exists a binary relation $\leq$ on the universe of A and a function $p\colon A \to \{1, \dots, k\}$ such that

1. $\leq$ *is a partial order*
2. *Every set* $\downarrow a = \{x \in A \mid x \leq a\}$ *is finite and linearly ordered by* $\leq$.
3. $(a_1, \dots, a_n) \in R^A$ *implies*
   - $a_i \leq a_j$ *or* $a_j \leq a_i \quad (\forall i, j)$.
   - $\forall z \quad a_i < z \leq a_j \implies p(a_i) \neq p(z)$

Logic extensions: Existential Positive fragment

Existential positive sentences $\mathrm{EP} \subseteq \mathrm{FO}$ are formed by

- atomic formulas: $\mathbf{t}$, $x = y$, $R(x_1, \ldots, x_n)$ (for n-ary $R \in \sigma$)
- logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$
- existential quantifiers: $\exists x\, \varphi(x)$

Theorem (Łoś–Tarski–Lyndon, 1955 & 1959)

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Consequently, since $\mathrm{PP} \subseteq \mathrm{EP}$,

$$A \to B \qquad \Longleftrightarrow \qquad A \Rrightarrow^{\mathrm{EP}} B$$

(for a finite A)

Lemma

Every EP *sentence* φ *is equivalent to*

$$\varphi_1 \vee \cdots \vee \varphi_n$$

for some PP *sentences* $\varphi_1, \ldots, \varphi_n$ *(possibly with equalities).*

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Follows from $\quad A \models \exists x\,(\psi \vee \psi') \leftrightarrow (\exists x\,\psi) \vee (\exists x\,\psi')$. $\square$

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Through Chandra–Merlin lenses

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$\boxed{\text{Define } \mathrm{EP}_k \text{ and } \mathrm{EP}^k \text{ as earlier.}}$