Additive cellular automata graded-monadically

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Comonads meetup, 16 Nov. 2022

What is this about?

- **Cellular automata** (such as Conway's Game of Life) are an archetypical **comonadic** notion of computation—

  computation happens in the coKleisli category of a comonad.

- Cellular automata are also **graded comonadic**.

- **Additive cellular automata** are a special class of cellular automata.

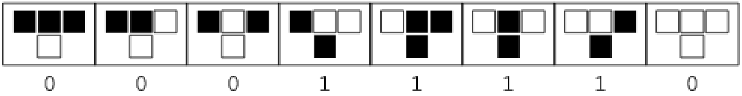
- Additive cellular automata are both graded comonadic and **graded monadic**.

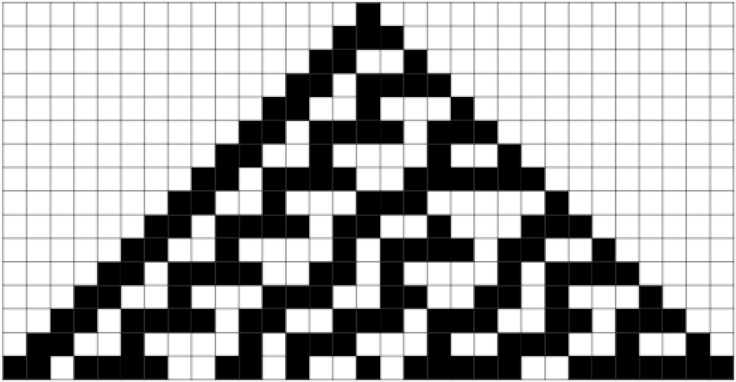
- That is explainable by a theorem about adjoint (graded) comonad-monad pairs.

Outline

- Cellular automata
- Cellular automata as a comonadic notion of computation
- Graded comonads, locally graded (l.g.) categories,
  the coKleisli, coEM l.g. categories of a graded comonad
- Cellular automata as graded comonadic
- Additive cellular automata
- Adjoint comonad-monad pairs, graded version
- Additive cellular automata as graded monadic

Wolfram's Rule 30





Cellular automata

- A *cellular automaton* is given by
  - a monoid $G = (G, 1_G, \cdot)$ (the *grid*, not necessarily a group),
  - sets X and Y (the *source and target alphabets*, not necessarily finite, not necessarily the same),
  - a *local rule*.

- A *local rule* is
  - a function $k : X^G \to Y$

  such that
  - there is a finite $M \subseteq G$ (a *neighborhood*) such that $k\, c$ is for all $c : X^G$ determined by the X^M part of c.

- The local rule determines the output configuration letter $k\, c : Y$ at node 1_G for a given input configuration $c : X^G$.

- Rule 30 has $(G, 1_G, \cdot) = (\mathbb{Z}, 0, +)$, $X = Y = \mathrm{Bool}$, $k\, c = c\,(-1)\,\mathrm{xor}\,(c\, 0 \vee c\, 1)$.

Cellular automata

- A *global rule* is
  - a function $f : X^G \to Y^G$

  such that
  - $f\,(c \rhd_X h) = f\,c \rhd_Y h$ for all $c : X^G$, $h : G$
    where $\rhd_X : X^G \times G \to X^G$ (*translation*) is defined by
    $c \rhd_X h = \lambda g : G.\, c\,(h \cdot g)$,
  - there is a finite $M \subseteq G$ such that $f\,c\,h : Y$ is for all $c : X^G$ and $h : G$
    determined by the $X^{\{h\} \cdot M}$ part of c.

- The global rule determines the whole output configuration $f\,c : Y^G$ for a given input configuration $c : X^G$.

- Rule 30 has $f\,c\,h = c\,(h-1)\,\mathrm{xor}\,(c\,h \vee c\,(h+1))$.

- Local and global rules are in bijection (Curtis, Hedlund).
- Given k, the corresponding f is defined by $f\,c\,h = k\,(c \rhd_X h)$.
- Given f, the corresponding k is defined by $k\,c = f\,c\,1_G$.

Cellular automata as comonadic

(Capobianco, U., 2010)

- Define a comonad $D = (D, \varepsilon, \delta)$ on Set by
  - $DX = X^G$,
    $D(f : X \to Y)(c : X^G) = f \circ c : Y^G$,
  - $\varepsilon_X(c : X^G) = c\,1_G : X$,
  - $\delta_X(c : X^G) = \lambda h : G.\, c \rhd_X h = \lambda h : G.\, \lambda g : G.\, c\,(h \cdot g) : (X^G)^G$

  (the *cowriter comonad* for G).

- Ignoring the requirements of uniform continuity,

  local rules $k : X^G \to Y$ are exactly coKleisli maps of D,
  global rules $f : X^G \to Y^G$ are exactly cofree coalgebra maps of D,
  with the identities and composition of the coKleisli and coE-M categories.

- The Curtis-Hedlund theorem is an instance of the isomorphism of the coKleisli category of any comonad D to the full subcategory of the coE-M category of D given by the cofree coalgebras.

Cellular automata as comonadic

- To incorporate uniform continuity, one can switch from Set to the category Unif of uniform spaces, consider the cowriter comonad for G on Unif.
- But one can remain in Set, viewing $M \subseteq G$ as part of the data of a cellular automaton and replacing the comonad with a graded comonad.
- This is the approach of this talk.

Graded comonads

(dual of graded monads of Smirnov 2008, Melliès 2012, Katsumata 2014)

- Suppose given a pomonoid $\mathcal{M} = (|\mathcal{M}|, \leq, 1, \cdot)$.
- A *$\mathcal{M}$-graded comonad* is
  - a family of functors $D_M : \mathcal{C} \to \mathcal{C}$ functorial in M
    via a family natural transformations $D_{M \leq M'} : D_M \to D_{M'}$,
  - a natural transformation $\varepsilon : D_1 \to \mathrm{Id}$ (the *counit*),
  - a family of natural transformations $\delta_{N,M} : D_{N \cdot M} \to D_N \cdot D_M$ natural in N, M (the *comultiplication*)

  such that

$$
\begin{tikzcd}
D_M \arrow[r, "\delta_{1,M}"] \arrow[d, "\delta_{M,1}"'] \arrow[dr, equal] & D_1 \cdot D_M \arrow[d, "\varepsilon \cdot D_M"] \\
D_M \cdot D_1 \arrow[r, "D_M \cdot \varepsilon"] & D_M
\end{tikzcd}
\qquad
\begin{tikzcd}
D_{P \cdot N \cdot M} \arrow[r, "\delta_{P \cdot N, M}"] \arrow[d, "\delta_{P, N \cdot M}"'] & D_{P \cdot N} \cdot D_M \arrow[d, "\delta_{P,N} \cdot D_M"] \\
D_P \cdot D_{N \cdot M} \arrow[r, "D_P \cdot \delta_{N,M}"] & D_P \cdot D_N \cdot D_M
\end{tikzcd}
$$

- A $\mathcal{M}$-graded comonad is the same as an oplax monoidal functor from $\mathcal{M}$ (as a thin strict monoidal category) to $[\mathcal{C}, \mathcal{C}]$.

Cellular automata as graded comonadic

- Let $\mathcal{M} = (|\mathcal{M}|, \leq, 1, \cdot)$ be the pomonoid defined by $|\mathcal{M}| = \mathcal{P}_{\mathrm{f}}(G)$, $\leq\, = \,\supseteq$, $1 = \{1_G\}$, $N \cdot M = \{n \cdot m \mid n \in N, m \in M\}$.

- We can define a $\mathcal{M}$-graded comonad $D = (D, \varepsilon, \delta)$ on Set by
  - $D_M X = X^M$,
    $D_M(f : X \to Y)(c : X^M) = f \circ c : Y^M$,
  - $D_{M \leq M', X}(c : X^M) = c|_{M'} = \lambda m : M'.\, c\, m : X^{M'}$
    (note that $M' \subseteq M$),
  - $\varepsilon_X(c : X^1) = c\, 1_G : X$,
  - $\delta_{N,M,X}(c : X^{N \cdot M}) =$
    $\lambda n : N.\, c \rhd_{N,M,X} n = \lambda n : N.\, \lambda m : M.\, c\,(n \cdot m) : (X^M)^N$

  (an $\mathcal{M}$-graded version of the cowriter comonad for G).

- ($\rhd_{N,M,X} : X^{N \cdot M} \times N \to X^M$ is defined by $c \rhd_{N,M,X} n = \lambda m : M.\, c\,(n \cdot m)$.)

- What are coKleisli and cofree coalgebra maps of D like?

- We do not even get maps, but graded maps...

Locally graded categories

(Wood 1976)

- Suppose given a pomonoid $\mathcal{M} = (|\mathcal{M}|, \leq, 1, \cdot)$.
- A *locally $\mathcal{M}$-graded category* is given by:
  - a set $|\mathcal{C}|$ of *objects*,
  - for any $X, Y \in |\mathcal{C}|$,
    a family of sets $\mathcal{C}_M(X, Y)$ of *maps of grade M*
    (we write $f : X \to_M Y$ for $f \in \mathcal{C}_M(X, Y)$),
  - if $M \leq M'$, then, for any map $f : X \to_M Y$,
    a map $(M \leq M')^* f : X \to_{M'} Y$ (the *coercion*)
  - for any $X \in |\mathcal{C}|$,
    a map $\mathrm{id}_X : X \to_1 X$ (the *identity*);
  - for any maps $f : X \to_M Y$, $g : Y \to_N Z$,
    a map $g \circ f : X \to_{M \cdot N} Z$ (the *composition*)

  such that
  - $(M \leq M)^* f = f$, $(M \leq M'')^* f = (M' \leq M'')^*((M \leq M')^* f)$,
  - $f \circ \mathrm{id} = f = \mathrm{id} \circ f$, $h \circ (g \circ f) = (h \circ g) \circ f$,
  - $(N \leq N')^* g \circ (M \leq M')^* f = (M \cdot N \leq M' \cdot N')^*(g \circ f)$.

Locally graded functors, natural transformations

- A *functor* between two locally $\mathcal{M}$-graded categories $\mathcal{C}$ and $\mathcal{D}$ is
  - a mapping $F : |\mathcal{C}| \to |\mathcal{D}|$,
  - for any $X, Y \in |\mathcal{C}|$,
    a family of mappings $F : \mathcal{C}_M(X, Y) \to \mathcal{D}_M(FX, FY)$

  such that
  - $F\mathrm{id}_X = \mathrm{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$.

- A *natural transformation* between functors F, G between locally $\mathcal{M}$-graded categories $\mathcal{C}$, $\mathcal{D}$ is
  - for any $X \in |\mathcal{C}|$,
    a map $\tau_X : FX \to_1 GX$ of $\mathcal{D}$

  such that
  - for any map $f : X \to_M Y$ of $\mathcal{C}$, one has
    $Gf \circ \tau_X = \tau_Y \circ Ff$.

CoKleisli

(McDermott 2022)

- The *coKleisli locally* $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$*-graded category* $\mathsf{CoKl}(D)$ of a $\mathcal{M}$-graded comonad $D = (D, \varepsilon, \delta)$ on $\mathcal{C}$ has as
  - objects: objects of $\mathcal{C}$,
  - maps of grade M from X to Y: maps $D_M X \to Y$ of $\mathcal{C}$,
  - the coercion $X \to_{M'} Y$ of $k : X \to_M Y$ along $M \geq M'$:

    $$D_{M'} X \xrightarrow{D_{M' \leq M}} D_M X \xrightarrow{k} Y ,$$
  - the identity $X \to_1 X$ on X:

    $$D_1 X \xrightarrow{\varepsilon_X} X ,$$
  - the composition $X \to_{M \cdot^{\mathrm{rev}} N} Z$ of $k : X \to_M Y$ and $\ell : Y \to_N Z$:

    $$\begin{tikzcd} D_{N \cdot M} X \arrow[r, "\delta_{N,M,X}"] \arrow[rr, bend right, "k^\dagger_N"'] & D_N D_M X \arrow[r, "D_N k"] & D_N Y \arrow[r, "\ell"] & Z \end{tikzcd} .$$

- Here $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}} = (|\mathcal{M}|, \geq, 1, \cdot^{\mathrm{rev}})$ where $M \geq M'$ iff $M' \leq M$ and $M \cdot^{\mathrm{rev}} N = N \cdot M$.

CoEilenberg-Moore

- A *coalgebra* of D is
  - a functor X from $\mathcal{M}$ to $\mathcal{C}$ with
  - a family of maps $\xi_{N,M} : X_{N \cdot M} \to D_N X_M$ of $\mathcal{C}$ natural in N, M

  such that

$$
\begin{tikzcd}
X_M \arrow[r, "\xi_{1,M}"] \arrow[dr, equal] & D_1 X_M \arrow[d, "\varepsilon_{X_M}"] \\
 & X_M
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{P \cdot N \cdot M} \arrow[r, "\xi_{P \cdot N, M}"] \arrow[d, "\xi_{P, N \cdot M}"'] & D_{P \cdot N} X_M \arrow[d, "\delta_{P,N,X_M}"] \\
D_P X_{N \cdot M} \arrow[r, "D_P \xi_{N,M}"] & D_P D_N X_M
\end{tikzcd}
$$

- A *map* of grade M *between coalgebras* (X, ξ) and (Y, χ) is
  - a family of maps $f_N : X_{N \cdot M} \to Y_N$ of $\mathcal{C}$ natural in N

  such that

$$
\begin{tikzcd}
X_{P \cdot N \cdot M} \arrow[r, "f_{P \cdot N}"] \arrow[d, "\xi_{P, N \cdot M}"'] & Y_{P \cdot N} \arrow[d, "\chi_{P,N}"] \\
D_P X_{N \cdot M} \arrow[r, "D_P f_N"] & D_P Y_N
\end{tikzcd}
$$

CoEilenberg-Moore

- The *coEilenberg-Moore locally* $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$*-graded category* $\mathrm{CoEM}(D)$ of a $\mathcal{M}$-graded comonad D on $\mathcal{C}$ has as
  - objects: coalgebras of D,
  - maps of grade M: coalgebra maps of grade M,
  - the N-component of the coercion $(X, \xi) \to_{M'} (Y, \chi)$ of $f : (X, \xi) \to_M (Y, \chi)$ along $M \geq M'$:

    $$X_{N \cdot M'} \xrightarrow{X_{N \cdot M' \leq N \cdot M}} X_{N \cdot M} \xrightarrow{f_N} Y_N \ ,$$

  - the M-component of identity $(X, \xi) \to_1 (X, \xi)$ on (X, ξ):

    $$X_M \xrightarrow{\mathrm{id}_{X_M}} X_M \ ,$$

  - the P-component of the composition $(X, \xi) \to_{N \cdot^{\mathrm{rev}} M} (Z, \zeta)$ of $f : (X, \xi) \to_M (Y, \chi)$ and $g : (Y, \chi) \to_N (Z, \zeta)$:

    $$X_{P \cdot N \cdot M} \xrightarrow{f_{P \cdot N}} Y_{P \cdot N} \xrightarrow{g_P} Z_P \ .$$

The comparison functor

- The coKleisli and coEilenberg-Moore locally graded categories define the initial and final resolutions of the graded comonad.

- The unique map between these resolutions is the locally graded functor $E : \mathrm{CoKl}(D) \to \mathrm{CoEM}(D)$ defined by
  - $EX = (D_{-}X, \delta_{-,-,X})$ (the cofree coalgebra on X),
  - $E(k : X \to_M Y) = k^\dagger : (D_{-}X, \delta_{-,-,X}) \to_M (D_{-}Y, \delta_{-,-,Y})$ (note that $k : D_M X \to Y$, so $k^\dagger_N : D_{N \cdot M} X \to D_N Y$).

- This functor (the *comparison functor*) is fully-faithful.

Cellular automata as graded comonadic

- Recall the pomonoid $\mathcal{M} = (|\mathcal{M}|, \leq, 1, \cdot)$ is defined by $|\mathcal{M}| = \mathcal{P}_{\mathrm{f}}(G)$, $\leq = \supseteq$, $1 = \{1_G\}$, $N \cdot M = \{n \cdot m \mid n \in N, m \in M\}$

- and the $\mathcal{M}$-graded comonad $D = (D, \varepsilon, \delta)$ on Set is defined by
  - $D_M X = X^M$,
    $D_M(f : X \to Y)(c : X^M) = f \circ c : Y^M$,
  - $D_{M \leq M', X}(c : X^M) = c|_{M'} = \lambda m : M'.\, c\, m : X^{M'}$
    (note that $M' \subseteq M$).
  - $\varepsilon_X(c : X^1) = c\, 1_G : X$,
  - $\delta_{N,M,X}(c : X^{N \cdot M}) =$
    $\lambda n : N.\, c \triangleright_{N,M,X} n = \lambda n : N.\, \lambda m : M.\, c\,(n \cdot m) : (X^M)^N$.

- CoKleisli maps of grade M are functions $k : X^M \to Y$.

- Cofree coalgebra maps of grade M are families of functions $f_N : X^{N \cdot M} \to Y^N$ such that
  - if $N' \subseteq N$, then $(f_N\, c)|_{N'} = f_{N'}\,(c|_{N' \cdot M}) : Y^{N'}$ for all $c : X^{N \cdot M}$,
  - $f_{P \cdot N}\, c \triangleright_{P,N,Y} p = f_N\,(c \triangleright_{P,N \cdot M,X} p)$ for all $c : X^{P \cdot N \cdot M}$, $p : P$.

- These are local and global rules made resource-aware!

Additive cellular automata

- An additive CA has commutative monoids $(X, 0_X, +_X)$, $(Y, 0_Y, +_Y)$ instead of just sets X, Y as input and output alphabets.

- An additive CA local rule is a CA local rule $k : X^G \to Y$ for the underlying sets X and Y that is additive (a commutative monoid homomorphism), i.e., $k\,0_{X^G} = 0_Y$ and $k\,(c +_{X^G} c') = k\,c +_Y k\,c'$.
- Ditto for additive CA global rules: they are additive global functions for the underlying sets.

- Evidently, local and global rules of additive CA are precisely coKleisli maps and cofree coalgebra maps of the cowriter comonad for G on UnifCommMon.
- Also, resource-sensitive versions thereof are precisely coKleisli maps and cofree coalgebra maps of the $\mathcal{M}$-graded cowriter comonad for G on CommMon.

- But there is more!

An observation

- Suppose given a finite $M \subseteq G$.
- An additive function $k : X^M \to Y$ is fully determined by what it does on relevant *point configurations*, i.e., configurations of the form $[m \mapsto x]_M : X^M$ defined by
$[m \mapsto x]_M = \lambda m' : M.\, \text{if } m' = m \text{ then } x \text{ else } 0_X$ (for $m : M$, $x : X$).
- Indeed, if k is additive, then, for any $c : X^M$, one has

$$k\, c = \bigoplus_{m:M} k\, [m \mapsto c\, m]$$

(As M is finite and addition is commutative, this sum is well-defined.)

- It follows readily that additive functions $k : X^M \to Y$ are in bijection with additive functions $\ell : X \to Y^M$.
- Given k, the corresp. ℓ is defined by $\ell\, x = \lambda m : M.\, k\, [m \mapsto x]_M$.
- Given ℓ, the corresp. k is defined by $k\, c = \bigoplus_{m:M} \ell\, (c\, m)\, m$.

An observation

- Consider $(G, 1_G, \cdot) = (\mathbb{Z}, 0, +)$, $(X, 0_X, +_X) = (Y, 0_Y, +_Y) = (\mathbb{Q}, 0, +)$.
- Let $M = \{-1, 0\}$, $k\, c = \frac{1}{3} * c\,(-1) + \frac{2}{3} * c\, 0$.
- Here is an evolution:

$$\begin{array}{ccccccccc}
\dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\
\hline
\dots & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & \dots \\
\hline
\dots & 0 & 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} & 0 & 0 & \dots \\
\hline
\dots & 0 & 0 & \frac{8}{27} & \frac{12}{27} & \frac{6}{27} & \frac{1}{27} & 0 & \dots \\
\hline
\dots & 0 & 0 & \frac{16}{81} & \frac{32}{81} & \frac{24}{81} & \frac{8}{81} & \frac{1}{81} & \dots \\
\hline
 & & & & \vdots & & & &
\end{array}$$

- $\ell\, x\,(-1) = \frac{1}{3} * x$, $\ell\, x\, 0 = \frac{2}{3} * x$.

Additive CA as graded monadic?

- Define a family of endofunctors T_M on CommMon by
  - $T_M X = X^M$ ($= \; D_M X$).
- We have just seen that $D_M \dashv T_M$.
- (So far there was no good reason for the new name T_M. But wait.)
- Define a family of natural transformations $T_{M' \geq M} : T_{M'} \to T_M$ by
  - $T_{M' \geq M}(s : Y^{M'}) = s|^M = \lambda m : M.\, \text{if } m \in M' \text{ then } s\, m \text{ else } 0_Y : Y^M$ (note that $M' \subseteq M$).
- This makes T into a functor from $\mathcal{M}^{\mathrm{op}} = (|\mathcal{M}|, \geq)$ while D was a functor from $\mathcal{M} = (|\mathcal{M}|, \leq)$.
- Moreover, it turns out that $D_{M \leq M'} \dashv T_{M' \geq M}$.
- We can apply a graded version of a folklore result.

Adjoint natural transformations

- Suppose given functors $L, L' : \mathcal{C} \to \mathcal{D}$ and $R, R' : \mathcal{D} \to \mathcal{C}$ such that $L \dashv R$ and $L' \dashv R'$.

- Then a nat. transf. $\tau : L' \to L$ is said to be left adjoint to a nat. transf. $\theta : R \to R'$ if
  - for any $f : LX \to Y$, $(f \circ \tau_X)^* = \theta_Y \circ f^* : X \to R'Y$.

Adjoint comonad-monad pairs

(folklore?, Kleiner 1990)

- Suppose given two endofunctors D and T on a category $\mathcal{C}$ such that $D \dashv T$.
- If D carries a comonad structure (ε, δ), then T carries a monad structure (η, μ).
- If T carries a monad structure (η, μ), then D carries a comonad structure (ε, δ).
- In both cases, $\varepsilon \dashv \eta$ and $\delta \dashv \mu$.
- The two constructions form a bijection between adjoint comonad structures on D and monad structures on T.
- For corresponding comonad structures on D and monad structures on T, the categories $\mathrm{CoKl}(D)$ and $\mathrm{Kl}(T)$ are isomorphic.
- On objects, the isomorphism is identity. On maps, it is the bijection between maps $DX \to Y$ and $X \to TY$ provided by the transposes.

A version for graded comonad-monad pairs

- Suppose given a pomonoid $\mathcal{M} = (|\mathcal{M}|, \leq, 1, \cdot)$ and two functors $D : \mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ and $T : \mathcal{M}^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$ such that $D_M \dashv T_M$ and $D_{M' \leq M} \dashv T_{M \geq M'}$ (notice that $D_{M' \leq M} : D_{M'} \to D_M$ and $T_{M \geq M'} : T_M \to T_{M'}$).
- If D carries an $\mathcal{M}$-graded comonad structure, then T carries a right adjoint $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$-graded monad structure.
- If T carries an $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$-graded monad structure, then D carries a left adjoint $\mathcal{M}$-graded comonad structure.
- The two constructions form a bijection between adjoint $\mathcal{M}$-graded comonad and $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$-graded monad structures on D and T.
- For corresponding comonad and monad structures on D and T, the locally $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$-graded categories $\mathsf{CoKl}(D)$ and $\mathsf{Kl}(T)$ are isomorphic.

An aside: adjoint monad-comonad pairs

(Eilenberg, Moore 1964)

- Suppose $T \dashv D$ instead.
- Then, similarly, there is a bijection of adjoint monad structures on T and comonad structures on D.
- But in this case, it is $\mathrm{EM}(T)$ and $\mathrm{CoEM}(D)$ (not $\mathrm{Kl}(T)$ and $\mathrm{CoKl}(D)$!) that are isomorphic for the corresponding monad and comonad structures on T and D.

- This theorem, too, admits a graded version.

Additive CA as graded monadic

- Recall that $D_M \dashv T_M$ and $D_{M \leq M'} \dashv T_{M' \geq M}$.
- By the theorem, T carries a $(\mathcal{M}^{\mathrm{op}})^{\mathrm{rev}}$-graded monad structure η, μ.
- Explicitly, it is defined by
  - $\eta_X(x : X) = \lambda\_.\, x : X^1$,
  - $\mu_{N,M,X}(s : (X^M)^N) = \lambda p : M \cdot N.\ \bigoplus_{m:N, n:N, p=m\cdot n} s\, n\, m : X^{N \cdot^{\mathrm{rev}} M}$.

- Kleisli maps of grade M of T are maps $\ell : X \to Y^M$ that we saw to be in bijection with maps $k : X^M \to Y$, ie. coKleisli maps of grade M of D.
- Free algebra maps of grade M are families of maps $h : X^N \to Y^{M \cdot N}$ such that
  - if $N' \subseteq N$, then $(h_{N'}\, s)|^{M \cdot N} = h_N(s|^N) : Y^{M \cdot N}$ for $s : X^{N'}$,
  - $\lambda r : M \cdot N \cdot P.\ \bigoplus_{o:M\cdot N, p:P, r=o\cdot p} h\,(s\,p)\,o : Y^{M\cdot N\cdot P} =$
    $h_{N\cdot P}\,(\lambda q : N \cdot P.\ \bigoplus_{n:N, p:P, q=n\cdot p} s\, p\, n)$ for $s : (X^N)^P$.

- It is useful to think of $s : X^M$ as formal polynomials (assignments of coefficients from X to exponents from M).
- T is a graded version of the polynomial monad.

Takeaway

- CA are a nice example of a comonadic notion of computation, but they exemplify more!
- It is natural to consider grading in this example.
- The CA example helps with intuitions for the complications present in the locally graded coKleisli and coEM constructions.
- Additive CA (but also linear CA) are among the rare examples of notions of computation that are both (graded) comonadic and monadic.