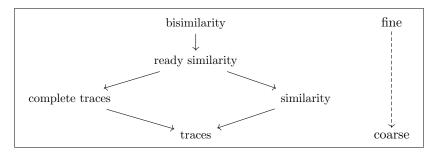
Graded Monads and Behavioural Equivalence Games

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Where are we?



Linear-time-Branching-time spectrum for LTS (Van Glabbeek, 1990)

Graded semantics: framework for spectra of behavioural semantics

coalgebra [system-type] + graded monads [granularity]

Milius/Pattinson/Schröder (CALCO 15) Dorsch/Milius/Schröder (CONCUR 19)

Ford/Milius/Schröder (LICS 21)



- Graded monads and their algebras
- Graded coalgebraic semantics
- Generic determinization of coalgebras under graded semantics
- Game characterizations of graded semantics, for free

Graded monads

• In this talk, a graded monad is a lax monoidal action

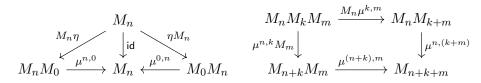
$$\mathscr{M} \times \mathsf{Set} \to \mathsf{Set}$$

where \mathcal{M} is the discrete category induced by $(\mathbb{N}, +, 0)$.

• Graded monad $\mathbb{M} = (M, \eta, \mu)$ on Set $(n, k \in \mathbb{N})$:

$M_n \colon Set \to Set$	$\eta \colon Id \to M_0$	$\mu^{n,k} \colon M_n M_k \to M_{n+k}$
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subject to *unit* and *multiplication laws*:



Examples

Functor iteration

Given a functor $G: \mathsf{Set} \to \mathsf{Set}$, define \mathbb{M}_G by

$$M_n := G^n \qquad \eta := \operatorname{Id} \xrightarrow{\operatorname{id}} G^0 \qquad \mu^{n,k} := G^n G^k \xrightarrow{\operatorname{id}} G^{n+k}$$

Kleisli distributive laws

Each distributive law $\lambda \colon FT \to TF$ with

 (T, η, μ) a monad $F: \mathsf{Set} \to \mathsf{Set}$ a functor

yields a graded monad with $M_n := TF^n$, unit η , and multiplication

$$\mu^{n,k} := TF^n TF^k \xrightarrow{T\lambda^n F^k} TTF^n F^k \xrightarrow{\mu F^{n+k}} TF^{n+k}$$

where $\lambda^n \colon F^n T \to TF^n$.

e.g. for a set \mathcal{A} , taking $M_n = \mathscr{P}_f(\mathcal{A}^n \times -)$ yields a graded monad

Graded algebras

- Graded monads admit a notion of graded algebra [FKM16, MPS15] ...generalizing the EM category of ordinary monads
- For $n \in \mathbb{N}$, the category $\mathsf{Alg}_n(\mathbb{M})$ has
 - \triangleright objects: families of sets $(A_k)_{k \leq n}$ with structure maps

$$a^{m,k} \colon M_m A_k \to A_{m+k} \qquad (m+k \le n)$$

compatible with \mathbb{M} :



▷ morphisms: graded algebra homorphisms

Examples

- M_0 -algebras: EM algebras for $(M_0, \eta, \mu^{0,0})$
- M_1 -algebras: a pair of EM algebras

$$a^{0,0} \colon M_0 A_0 \to A_0 \qquad a^{0,1} \colon M_0 A_1 \to A_1$$

equipped with a main structure map

$$a^{1,0}\colon M_1A_0\to A_1$$

 $a^{1,0}\colon (M_1A_0,\mu_A^{0,1}) \to (A_1,a^{0,1})$ an M_0 -algebra homom. and

$$M_1 M_0 A_0 \xrightarrow[M_1 a^{0,0}]{} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

Proposition [MPS15]

The free M_n -algebra on X has carrier $(M_k X)_{k \le n}$ and multiplication $\mu^{n,k} \colon M_n M_k X \to M_{n+k} X$ as structure.

Canonical algebras

- The **0-part** of M_1 -algebra (A, a) is the M_0 -algebra $(A_0, a^{0,0})$
- Taking 0-parts defines a forgetful functor

 $(-)_0: \operatorname{Alg}_1(\mathbb{M}) \to \operatorname{Alg}_0(\mathbb{M}), \qquad A \mapsto (A_0, a^{0,0})$

An M_1 -algebra A is canonical if it is free over its 0-part w.r.t. $(-)_0$.

Proposition [DMS19]

An M_1 -algebra A is canonical iff

$$M_1 M_0 A_0 \xrightarrow[M_1 a^{0,0}]{\mu^{1,0}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

is a coequalizer diagram in $\mathsf{Alg}_0(\mathbb{M})$.

e.g. (M_0X, M_1X) is canonical...sometimes(!)

Finitary graded monads admit presentations by graded theories:

- graded signature Σ : algebraic signature + depth on operations
- terms of uniform-depth n with variables in X, denoted T_{Σ,n}(X):
 ▷ each variable is a term of uniform depth 0;
 - $\rhd \text{ given } m\text{-ary } \sigma \in \Sigma \text{ and } t_1, \dots, t_m \in T_{\Sigma,k}(X), \\ \text{ then } \sigma(t_1, \dots, t_m) \in T_{\Sigma,d(\sigma)+k}(X).$
- uniform-depth equations: pairs of terms of the same depth
- graded theory: pairs $\mathbb{T} = (\Sigma, \mathcal{E})$, where \mathcal{E} is a set of u.d. equations

Theorem

Every graded monad is the free-algebra graded monad of a graded equational theory. In particular, $M_n X$ has the form $T_{\Sigma,n}(X)/=_{\mathcal{E}}$ for some (Σ, \mathcal{E}) , and $\mathsf{Alg}(\mathbb{M}) \cong \mathsf{Alg}(\mathbb{T})$.

Graded theory of trace equivalence

Graded theory of \mathcal{A} -traces

- Depth-0: operations/equations of join semilattices
- Depth-1: unary actions a(-) satisfying a(x + y) = a(x) + a(y)
- The theory above captures the graded monad with $M_n X = \mathscr{P}_{\omega}(\mathcal{A}^n \times X)$
- join semilattices \rightsquigarrow convex algebras: theory of prob. traces

Depth-1 graded monads and theories

- A graded theory is depth-1 if its ops/eqns have depth at most 1.
- M is depth-1 if it is presentable by a depth-1 graded theory
 - \triangleright i.e. $\mathsf{Alg}(\mathbb{M}) \cong \mathsf{Alg}(\mathbb{T})$ for some depth-1 graded theory \mathbb{T}
 - \triangleright almost expressible in terms of a coequalizer [MPS15]
- Depth-1 graded monads have 'nice' canonical algebras:

Proposition

Let $k \in \mathbb{N}$ and let \mathbb{M} be depth-1. Then $(M_k X, M_{k+1} X)$ is canonical.

e.g. the graded theory of \mathcal{A} -traces is depth-1 hence also $M_n = \mathscr{P}_f(\mathcal{A}^n \times -)$

Graded semantics

Graded semantics: framework for spectra of behavioural semantics

coalgebra [system-type] + graded monads [granularity]

Graded semantics on G-coalgebras

A pair (α, \mathbb{M}) with \mathbb{M} a graded monad and $G \xrightarrow{\alpha} M_1$ a natural transformation.

Given $X \xrightarrow{\gamma} GX$, define $\gamma^{(n)} \colon X \to M_n 1$:

$$\gamma^{(0)} := X \xrightarrow{\eta} M_0 X \xrightarrow{M_0!} M_0 1$$

$$\gamma^{(n+1)} := X \xrightarrow{\alpha \cdot \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu^{1,n}} M_{1+n} 1$$

$$\boxed{x \sim_{(\alpha,\mathbb{M})} y :\iff \gamma^{(n)}(x) = \gamma^{(n)}(y) \text{ for all } n \in \mathbb{N}}$$

Examples of graded semantics

Coalgebraic behavioural equivalence

Recall that \mathbb{M}_G has $M_n = G^n$. Then for $(\mathsf{Id}, \mathbb{M}_G)$ we see:

• $\gamma^{(n)}: X \to M_n 1$ form the canonical cone into the final chain:

$$\gamma^{(0)} = X \xrightarrow{!} 1 \qquad \gamma^{(n+1)} = X \xrightarrow{\gamma} GX \xrightarrow{G\gamma^{(n)}} G^{n+1}1$$

• G finitary implies $\sim_{(\mathsf{Id},\mathbb{M}_G)}$ is coalgebraic behavioural equivalence

Trace equivalence on LTS

Let $\gamma: X \to \mathscr{P}_f(\mathcal{A} \times X)$ be an LTS.

 \triangleright Trace equivalence is the relation defined for all $x, y \in X$ by

$$x \sim_{\mathsf{Tr}} y :\iff \mathsf{Tr}_n(x) = \mathsf{Tr}_n(y) \text{ for all } n \in \omega$$

 \triangleright Trace equivalence captured by $M_n X = \mathscr{P}_f(\mathcal{A}^n \times X)$ and $\alpha = \mathsf{id}$.

(Pre-)determinization

Assumption: (α, \mathbb{M}) a depth-1 graded semantics on G-coalgebras

• Each M_0 -algebra $(A_0, a^{0,0})$ extends to a canonical algebra EA:

$$M_1 M_0 A_0 \xrightarrow[M_1 a^{0,0}]{\mu^{1,0}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

- This assignment is part of a functor $E\colon \mathsf{Alg}_0(\mathbb{M})\to\mathsf{Alg}_1(\mathbb{M})$
- Define

$$\left| \overline{M_1} := \mathsf{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathsf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathsf{Alg}_0(\mathbb{M}) \right|$$

• For instance,

$$\overline{M_1}(M_0X,\mu^{0,0}) = (M_1X,\mu^{0,1})$$

Thus, where F ⊢ U: Alg₀(M) → Set: M
₁(M₀X, μ^{0,0}) = (M₁X, μ^{0,1}) ⇒ UM
₁F = M₁
Given γ: X → M₁X = UM
₁FX, we obtain γ[#]: FX → M
₁FX

via adjoint transposition. Explicitly:

$$\gamma^{\#} = M_0 X \xrightarrow{M_0 \alpha \cdot \gamma} M_0 M_1 X \xrightarrow{\mu^{0,1}} M_1 X$$

Definition

The predeterminization of $\gamma: X \to GX$ is the $\overline{M_1}$ -coalgebra $\gamma^{\#}$.

Theorem

Suppose that $M_0 = 1$. Then $x \sim_{(\alpha, \mathbb{M})} y$ iff $\eta(x)$ and $\eta(y)$ are finite-depth $\overline{M_1}$ -behaviourally equivalent.

 \mathcal{G}_n captures (α, \mathbb{M}) -equivalence at depth n on

$$\gamma \colon X \to GX \qquad \rightsquigarrow \qquad \bar{\gamma} = (X \xrightarrow{\gamma} GX \xrightarrow{\alpha} M_1X)$$

...starting from $\eta(x) = \eta(y)$ for target states x, y:

Position	Player	Admissible Moves	
$(s,t) \in (M_0 X)^2$	D	$\{Z \subseteq (M_0 X)^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$	
$Z \subseteq (M_0 X)^2$	S	$Z = \{(s,t) \in (M_0 X)^2 \mid (s,t) \in Z\}$	
Play of \mathcal{G}_n : (s,t) Z_1 (s_1,t_1) Z_n (s_n,t_n)			

Slogan: equivalence games play out equational proofs in graded theories

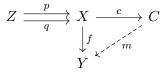
Theorem

Suppose (α, \mathbb{M}) is depth-1 such that $\overline{M_1}$ preserves monos. Then:

 $x \sim_{(\alpha, \mathbb{M})} y \iff D$ wins the *n*-round *S*-game for all $n \in \omega$

Currently, restricted to graded semantics in Set:

- We use that the EM category of a monad on Set is regular...
- ...ensuring that for the kernel pair $p, q: Z \to X$ of a map $f: X \to Y$ we have m monic below



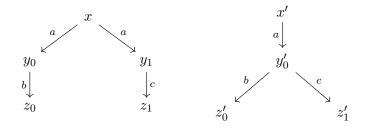
Bisimilarity game

Position	Player	Admissible Moves
$(s,t) \in (M_0 X)^2$	D	$\{Z \subseteq (M_0 X)^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$
$Z \subseteq (M_0 X)^2$	S	$Z = \{(s,t) \in (M_0 X)^2 \mid (s,t) \in Z\}$

Our bisimilarity game is somewhat non-standard:

- $(id, \mathbb{M}_{\mathscr{P}_{f}(\mathcal{A}\times -)})$ captures bisimilarity on f.b. LTS
- Positions for D are state pairs in a LTS $\gamma: X \to GX$ since $M_0 = \mathsf{Id}$
- Z is admissible for D at (x, y) if it is a *local bisimulation* at (x, y)
- Given Z, S picks the next state pair to continue the game
- D wins every full play because the $M_0 1 = 1$

Trace equivalence game



- At (x, x'), D plays $Z_1 := \{y_0 + y_1 = y'_0\}$ admissible: $Z_1 \vdash_1 a(y_0) + a(y_1) = a(y'_0)$?
- At position Z_1 , S must play $(y_0 + y_1 = y'_0) \in Z_1$
- At $(y_0 + y_1, y)$, D plays $Z_2 := \{z_0 = z'_0, z_1 = z'_1\}$ admissibile: $Z_2 \vdash_1 b(z_0) + c(z_1) = b(z'_0) + c(z'_1)$?
- S plays a challenge from Z_2 inducing $(x, x') Z_1 (y_0 + y_1, y'_0) Z_2 (z_i, z'_i)$ D wins because * = * is valid in JSL

Concluding remarks

See here for our arXiv preprint:



- Graded semantics: framework for capturing spectra of behavioural semantics based on coalgebra and graded monads
- In this talk:
 - \triangleright a generic determinization construction under graded semantics
 - $\,\vartriangleright\,$ game characterizations of graded behavioural equivalences for free
- Many interesting problems for future work:
 - \triangleright extensions beyond Set (e.g. games for preorders, metrics, etc.)
 - ▷ minimization under graded semantics (...learning algorithms)

- Ulrich Dorsch, Stefan Milius, and Lutz Schröder. 2019. Graded Monads and Graded Logics for the Linear Time-Branching Time Spectrum. In Concurrency Theory, CONCUR 2019 (LIPIcs).
- Soichiro Fujii, Shin-ya Katsumata, and Paul-André Melliès. 2016. *Towards a Formal Theory of Graded Monads*. In Foundations of Software Science and Computation Structures, FOSSACS 2016 (LNCS).
- Chase Ford, Stefan Milius, and Lutz Schröder. 2021. *Behavioural Preorders via Graded Monads*. In Logic in Computer Science, LICS 2021.
- Stefan Milius, Dirk Pattinson, and Lutz Schröder. 2015. *Generic Trace Semantics and Graded Monads*. In Algebra and Coalgebra in Computer Science, CALCO 2015 (LIPIcs).