Graded Monads and Behavioural Equivalence Games

Chase Ford

joint work with S. Milius, L. Schröder, H. Beohar, & B. König

Comonads Seminar, 01 February 2023

$$
\begin{tikzcd}
 & \text{bisimilarity} \arrow[d] & & \text{fine} \arrow[ddd, dashed] \\
 & \text{ready similarity} \arrow[dl] \arrow[dr] & & \\
\text{complete traces} \arrow[dr] & & \text{similarity} \arrow[dl] & \\
 & \text{traces} & & \text{coarse}
\end{tikzcd}
$$

Linear-time–Branching-time spectrum for LTS (Van Glabbeek, 1990)

`Graded semantics`: framework for spectra of behavioural semantics

$\boxed{\text{coalgebra [system-type] + graded monads [granularity]}}$

Milius/Pattinson/Schröder (CALCO 15) Dorsch/Milius/Schröder (CONCUR 19)

Ford/Milius/Schröder (LICS 21)

- Graded monads and their algebras
- Graded coalgebraic semantics
- Generic determinization of coalgebras under graded semantics
- Game characterizations of graded semantics, for free

- In this talk, a graded monad is a lax monoidal action

$$\boxed{\mathscr{M} \times \mathsf{Set} \to \mathsf{Set}}$$

where $\mathscr{M}$ is the discrete category induced by $(\mathbb{N}, +, 0)$.

- Graded monad $\mathbb{M} = (M, \eta, \mu)$ on Set $(n, k \in \mathbb{N})$:

$$\boxed{M_n \colon \mathsf{Set} \to \mathsf{Set} \qquad \eta \colon \mathsf{Id} \to M_0 \qquad \mu^{n,k} \colon M_n M_k \to M_{n+k}}$$

subject to *unit* and *multiplication laws*:

$$\begin{tikzcd}
 & M_n \arrow[dl, "M_n\eta"'] \arrow[d, "\mathsf{id}"] \arrow[dr, "\eta M_n"] & \\
M_n M_0 \arrow[r, "\mu^{n,0}"] & M_n & M_0 M_n \arrow[l, "\mu^{0,n}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M_n M_k M_m \arrow[r, "M_n \mu^{k,m}"] \arrow[d, "\mu^{n,k} M_m"'] & M_n M_{k+m} \arrow[d, "\mu^{n,(k+m)}"] \\
M_{n+k} M_m \arrow[r, "\mu^{(n+k),m}"] & M_{n+k+m}
\end{tikzcd}$$

Functor iteration

Given a functor $G\colon \mathsf{Set} \to \mathsf{Set}$, define $\mathbb{M}_G$ by

$$M_n := G^n \qquad \eta := \mathsf{Id} \xrightarrow{\mathsf{id}} G^0 \qquad \mu^{n,k} := G^n G^k \xrightarrow{\mathsf{id}} G^{n+k}$$

Kleisli distributive laws

Each distributive law $\lambda\colon FT \to TF$ with

$$(T, \eta, \mu) \text{ a monad} \qquad F\colon \mathsf{Set} \to \mathsf{Set} \text{ a functor}$$

yields a graded monad with $M_n := TF^n$, unit η, and multiplication

$$\mu^{n,k} := TF^n TF^k \xrightarrow{T\lambda^n F^k} TTF^n F^k \xrightarrow{\mu F^{n+k}} TF^{n+k}$$

where $\lambda^n\colon F^n T \to TF^n$.

e.g. for a set $\mathcal{A}$, taking $M_n = \mathscr{P}_f(\mathcal{A}^n \times -)$ yields a graded monad

- Graded monads admit a notion of *graded algebra* [FKM16, MPS15]

  ...generalizing the EM category of ordinary monads

- For $n \in \mathbb{N}$, the category $\mathsf{Alg}_n(\mathbb{M})$ has
  - ▷ **objects**: families of sets $(A_k)_{k \leq n}$ with structure maps

$$a^{m,k} \colon M_m A_k \to A_{m+k} \qquad (m + k \leq n)$$

compatible with $\mathbb{M}$:

$$\begin{tikzcd} A_k \arrow[r, "\mathsf{id}"] \arrow[d, "\eta"'] & A_k \\ M_0 A_k \arrow[ur, "a^{0,k}"'] & \end{tikzcd} \qquad \begin{tikzcd} M_\ell M_m A_k \arrow[r, "M_\ell a^{m,k}"] \arrow[d, "\mu^{\ell,m}"'] & M_\ell A_{m+k} \arrow[d, "a^{\ell,m+k}"] \\ M_{\ell+m} A_k \arrow[r, "a^{\ell+m,k}"] & A_{\ell+m+k} \end{tikzcd}$$

  - ▷ **morphisms**: *graded algebra homorphisms*

- M_0-algebras: EM algebras for $(M_0, \eta, \mu^{0,0})$
- M_1-algebras: a pair of EM algebras

$$a^{0,0} \colon M_0 A_0 \to A_0 \qquad\qquad a^{0,1} \colon M_0 A_1 \to A_1$$

equipped with a main structure map

$$\boxed{a^{1,0} \colon M_1 A_0 \to A_1}$$

$a^{1,0} \colon (M_1 A_0, \mu^{0,1}_A) \to (A_1, a^{0,1})$ an M_0-algebra homom. and

$$M_1 M_0 A_0 \underset{M_1 a^{0,0}}{\overset{\mu^{1,0}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

Proposition [MPS15]

The free M_n-algebra on X has carrier $(M_k X)_{k \leq n}$ and multiplication $\mu^{n,k} \colon M_n M_k X \to M_{n+k} X$ as structure.

- The **0-part** of M_1-algebra (A, a) is the M_0-algebra $(A_0, a^{0,0})$
- Taking 0-parts defines a forgetful functor

$$(-)_0 \colon \mathsf{Alg}_1(\mathbb{M}) \to \mathsf{Alg}_0(\mathbb{M}), \qquad A \mapsto (A_0, a^{0,0})$$

An M_1-algebra A is canonical if it is free over its 0-part w.r.t. $(-)_0$.

Proposition [DMS19]

An M_1-algebra A is canonical iff

$$\begin{tikzcd} M_1 M_0 A_0 \arrow[r, shift left, "\mu^{1,0}"] \arrow[r, shift right, "M_1 a^{0,0}"'] & M_1 A_0 \arrow[r, "a^{1,0}"] & A_1 \end{tikzcd}$$

is a coequalizer diagram in $\mathsf{Alg}_0(\mathbb{M})$.

$\boxed{\text{e.g. } (M_0 X, M_1 X) \text{ is canonical...}\textbf{sometimes}(!)}$

Finitary graded monads admit presentations by graded theories:

- *graded signature* Σ: algebraic signature + *depth* on operations
- *terms of uniform-depth* n with variables in X, denoted $T_{\Sigma,n}(X)$:
  - ▷ each variable is a term of uniform depth 0;
  - ▷ given m-ary $\sigma \in \Sigma$ and $t_1, \ldots, t_m \in T_{\Sigma,k}(X)$, then $\sigma(t_1, \ldots, t_m) \in T_{\Sigma,d(\sigma)+k}(X)$.
- *uniform-depth equations*: pairs of terms of the same depth
- *graded theory*: pairs $\mathbb{T} = (\Sigma, \mathcal{E})$, where $\mathcal{E}$ is a set of u.d. equations

Theorem

Every graded monad is the free-algebra graded monad of a graded equational theory. In particular, $M_n X$ has the form $T_{\Sigma,n}(X)/{=_{\mathcal{E}}}$ for some $(\Sigma, \mathcal{E})$, and $\mathsf{Alg}(\mathbb{M}) \cong \mathsf{Alg}(\mathbb{T})$.

Graded theory of $\mathcal{A}$-traces

- Depth-0: operations/equations of join semilattices
- Depth-1: unary *actions* $a(-)$ satisfying $\boxed{a(x+y) = a(x) + a(y)}$

- The theory above captures the graded monad with $M_n X = \mathscr{P}_\omega(\mathcal{A}^n \times X)$
- join semilattices $\rightsquigarrow$ convex algebras: theory of prob. traces

- A graded theory is depth-1 if its ops/eqns have depth at most 1.
- $\mathbb{M}$ is depth-1 if it is presentable by a depth-1 graded theory
  - ▷ i.e. $\mathsf{Alg}(\mathbb{M}) \cong \mathsf{Alg}(\mathbb{T})$ for some depth-1 graded theory $\mathbb{T}$
  - ▷ *almost* expressible in terms of a coequalizer [MPS15]
- Depth-1 graded monads have 'nice' canonical algebras:

Proposition

Let $k \in \mathbb{N}$ and let $\mathbb{M}$ be depth-1. Then $(M_k X, M_{k+1} X)$ is canonical.

e.g. the graded theory of $\mathcal{A}$-traces is depth-1 hence also $M_n = \mathscr{P}_f(\mathcal{A}^n \times -)$

`Graded semantics`: framework for spectra of behavioural semantics

$$\boxed{\text{coalgebra}\ _{\text{[system-type]}} + \text{graded monads}\ _{\text{[granularity]}}}$$

Graded semantics on G-coalgebras

A pair $(\alpha, \mathbb{M})$ with $\mathbb{M}$ a graded monad and $G \xrightarrow{\alpha} M_1$ a natural transformation.

Given $X \xrightarrow{\gamma} GX$, define $\gamma^{(n)} \colon X \to M_n 1$:

$$\gamma^{(0)} := X \xrightarrow{\eta} M_0 X \xrightarrow{M_0 !} M_0 1$$

$$\gamma^{(n+1)} := X \xrightarrow{\alpha \cdot \gamma} M_1 X \xrightarrow{M_1 \gamma^{(n)}} M_1 M_n 1 \xrightarrow{\mu^{1,n}} M_{1+n} 1$$

$$\boxed{x \sim_{(\alpha, \mathbb{M})} y \ :\Longleftrightarrow\ \gamma^{(n)}(x) = \gamma^{(n)}(y) \text{ for all } n \in \mathbb{N}}$$

Coalgebraic behavioural equivalence

Recall that $\mathbb{M}_G$ has $M_n = G^n$. Then for $(\mathsf{Id}, \mathbb{M}_G)$ we see:

- $\gamma^{(n)} \colon X \to M_n 1$ form the canonical cone into the final chain:

$$\gamma^{(0)} = X \xrightarrow{!} 1 \qquad \gamma^{(n+1)} = X \xrightarrow{\gamma} GX \xrightarrow{G\gamma^{(n)}} G^{n+1}1$$

- G finitary implies $\sim_{(\mathsf{Id}, \mathbb{M}_G)}$ is coalgebraic behavioural equivalence

Trace equivalence on LTS

Let $\gamma \colon X \to \mathscr{P}_f(\mathcal{A} \times X)$ be an LTS.

▷ Trace equivalence is the relation defined for all $x, y \in X$ by

$$x \sim_{\mathsf{Tr}} y :\Longleftrightarrow \mathsf{Tr}_n(x) = \mathsf{Tr}_n(y) \text{ for all } n \in \omega$$

▷ Trace equivalence captured by $M_n X = \mathscr{P}_f(\mathcal{A}^n \times X)$ and $\alpha = \mathsf{id}$.

Assumption: $(\alpha, \mathbb{M})$ a depth-1 graded semantics on G-coalgebras

- Each M_0-algebra $(A_0, a^{0,0})$ extends to a canonical algebra EA:

$$M_1 M_0 A_0 \underset{M_1 a^{0,0}}{\overset{\mu^{1,0}}{\rightrightarrows}} M_1 A_0 \xrightarrow{a^{1,0}} A_1$$

- This assignment is part of a functor $E\colon \mathsf{Alg}_0(\mathbb{M}) \to \mathsf{Alg}_1(\mathbb{M})$
- Define

$$\boxed{\overline{M_1} := \mathsf{Alg}_0(\mathbb{M}) \xrightarrow{E} \mathsf{Alg}_1(\mathbb{M}) \xrightarrow{(-)_1} \mathsf{Alg}_0(\mathbb{M})}$$

- For instance,

$$\overline{M_1}(M_0 X, \mu^{0,0}) = (M_1 X, \mu^{0,1})$$

- Thus, where $F \dashv U \colon \mathsf{Alg}_0(\mathbb{M}) \to \mathsf{Set}$:

$$\overline{M_1}(M_0 X, \mu^{0,0}) = (M_1 X, \mu^{0,1}) \quad \Longrightarrow \quad U\overline{M_1}F = M_1$$

- Given $\gamma \colon X \to M_1 X = U\overline{M_1}FX$, we obtain

$$\gamma^{\#} \colon FX \to \overline{M_1}FX$$

via adjoint transposition. Explicitly:

$$\gamma^{\#} = M_0 X \xrightarrow{M_0\alpha \cdot \gamma} M_0 M_1 X \xrightarrow{\mu^{0,1}} M_1 X$$

Definition

The predeterminization of $\gamma \colon X \to GX$ is the $\overline{M_1}$-coalgebra $\gamma^{\#}$.

Theorem

Suppose that $M_0 1 = 1$. Then $x \sim_{(\alpha,\mathbb{M})} y$ iff $\eta(x)$ and $\eta(y)$ are finite-depth $\overline{M_1}$-behaviourally equivalent.

The n-round equivalence game $\mathcal{G}_n$

$\mathcal{G}_n$ captures $(\alpha, \mathbb{M})$-equivalence at depth n on

$$\gamma\colon X \to GX \qquad \rightsquigarrow \qquad \bar{\gamma} = (X \xrightarrow{\gamma} GX \xrightarrow{\alpha} M_1 X)$$

...starting from $\eta(x) = \eta(y)$ for target states x, y:

Position	Player	Admissible Moves
$(s,t) \in (M_0X)^2$	D	$\{Z \subseteq (M_0X)^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$
$Z \subseteq (M_0X)^2$	S	$Z = \{(s,t) \in (M_0X)^2 \mid (s,t) \in Z\}$

$$\text{Play of } \mathcal{G}_n : \quad (s,t)\ Z_1\ (s_1,t_1)\ \dots\ Z_n\ \boxed{(s_n,t_n)}$$

Slogan: equivalence games play out equational proofs in graded theories

Main theorem

Theorem

Suppose $(\alpha, \mathbb{M})$ is depth-1 such that $\overline{M_1}$ preserves monos. Then:

$$x \sim_{(\alpha,\mathbb{M})} y \quad \iff \quad \text{D wins the } n\text{-round } \mathcal{S}\text{-game for all } n \in \omega$$

Currently, restricted to graded semantics in Set:

- We use that the EM category of a monad on Set is regular...
- ...ensuring that for the kernel pair $p, q\colon Z \to X$ of a map $f\colon X \to Y$ we have m monic below

$$\begin{tikzcd} Z \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & X \arrow[r, "c"] \arrow[d, "f"] & C \arrow[dl, dashed, "m"] \\ & Y & \end{tikzcd}$$

Position	Player	Admissible Moves
$(s,t) \in (M_0X)^2$	D	$\{Z \subseteq (M_0X)^2 \mid Z \vdash_1 s\bar{\gamma} = t\bar{\gamma}\}$
$Z \subseteq (M_0X)^2$	S	$Z = \{(s,t) \in (M_0X)^2 \mid (s,t) \in Z\}$

Our bisimilarity game is somewhat non-standard:

- $(\mathsf{id}, \mathbb{M}_{\mathscr{P}_f(\mathcal{A}\times -)})$ captures bisimilarity on f.b. LTS
- Positions for D are state pairs in a LTS $\gamma\colon X \to GX$ since $M_0 = \mathsf{Id}$
- Z is admissible for D at (x,y) if it is a *local bisimulation* at (x,y)
- Given Z, S picks the next state pair to continue the game
- D wins every full play because the $M_0 1 = 1$

$$\begin{tikzcd}
& x \arrow[dl, "a"'] \arrow[dr, "a"] & \\
y_0 \arrow[d, "b"'] & & y_1 \arrow[d, "c"] \\
z_0 & & z_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& x' \arrow[d, "a"'] & \\
& y'_0 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
z'_0 & & z'_1
\end{tikzcd}$$

- At (x, x'), D plays $Z_1 := \{y_0 + y_1 = y'_0\}$

  admissible: $Z_1 \vdash_1 a(y_0) + a(y_1) = a(y'_0)$?
- At position Z_1, S must play $(y_0 + y_1 = y'_0) \in Z_1$
- At $(y_0 + y_1, y)$, D plays $Z_2 := \{z_0 = z'_0, z_1 = z'_1\}$

  admissibile: $Z_2 \vdash_1 b(z_0) + c(z_1) = b(z'_0) + c(z'_1)$?
- S plays a challenge from Z_2 inducing $\boxed{(x, x')\ Z_1\ (y_0 + y_1, y'_0)\ Z_2\ (z_i, z'_i)}$

  D wins because $* = *$ is valid in JSL

`See here for our arXiv preprint:`

- Graded semantics: framework for capturing spectra of behavioural semantics based on coalgebra and graded monads
- In this talk:
  - ▷ a generic determinization construction under graded semantics
  - ▷ game characterizations of graded behavioural equivalences *for free*
- Many interesting problems for future work:
  - ▷ extensions beyond Set (e.g. games for preorders, metrics, etc.)
  - ▷ minimization under graded semantics (...learning algorithms)

References

- Ulrich Dorsch, Stefan Milius, and Lutz Schröder. 2019. *Graded Monads and Graded Logics for the Linear Time-Branching Time Spectrum*. In Concurrency Theory, CONCUR 2019 (LIPIcs).
- Soichiro Fujii, Shin-ya Katsumata, and Paul-André Melliès. 2016. *Towards a Formal Theory of Graded Monads*. In Foundations of Software Science and Computation Structures, FOSSACS 2016 (LNCS).
- Chase Ford, Stefan Milius, and Lutz Schröder. 2021. *Behavioural Preorders via Graded Monads*. In Logic in Computer Science, LICS 2021.
- Stefan Milius, Dirk Pattinson, and Lutz Schröder. 2015. *Generic Trace Semantics and Graded Monads*. In Algebra and Coalgebra in Computer Science, CALCO 2015 (LIPIcs).