Classification of homogeneous structures and EPPA

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Recall

Definition (Age of structure L-structure $\mathbf{A}$)

By $\mathrm{Age}(\mathbf{A})$ we denote the class of all L-structures isomorphic to some (induced) substructure of $\mathbf{A}$.

Definition (Amalgamation class)

Class $\mathcal{C}$ is an amalgamation class if it is

1. closed under isomorphism,
2. closed under taking induced substructures,
3. contains only countably many non-isomorphic structures, and,
4. has the amalgamation property.

Definition (Homogeneous structure)

Structure is homogeneous if every isomorphism of its two finite substructures extends to an automorphism.

Theorem (Fraïssé 1953)

(a) *A class $\mathcal{C}$ of finite (relational) structures is the age of a countable homogeneous structure $\mathbf{H}$ if and only if $\mathcal{C}$ is an amalgamation class.*

(b) *If conditions of (a) are satisfied then the structure $\mathbf{H}$ is unique up to isomorphism.*

Classification of homogeneous graphs

Theorem (Lachlan–Woodrow 1980)

Let G be an infinite homogeneous graph. Then either G or its complement is isomorphic to one of:

1. *Rado graph (universal homogeneous graph),*
2. *universal homogeneous graph omitting complete graphs of size n,*
3. *a disjoint union of complete graphs, all of same size.*

Classification of homogeneous partial orders

Theorem (Schmerl 1979)

Every homogeneous partial order is isomorphic to one of the following:

1. *A (possibly infinite) antichain*
2. *A (possibly infinite) antichain of chains*
3. *chain of antichains*
4. *The generic partial order*

Classification of homogeneous tournaments

Theorem (Lachlan 1984)

Every homogeneous tournament

1. *Finite cases: one-point tournament, oriented cycle of length 3.*
2. $(\mathbb{Q}, \leq)$
3. *Dense local order*
4. *Generic tournament*

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Classification of homogeneous digraphs was completed by Cherlin in 1998

EPPA

Definition (Extension property for partial automorphisms)

A class $\mathcal{C}$ of finite L-structures has extension property for partial automorphisms (EPPA or Hrushovski property) iff for every $\mathbf{A} \in \mathcal{C}$ there exists $\mathbf{B} \in \mathcal{C}$ containing $\mathbf{A}$ such that every partial automorphism of $\mathbf{A}$ extends to automorphism of $\mathbf{B}$

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Joint embedding + EPPA $\Longrightarrow$ amalgamation property

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The class of all finite graphs has EPPA.

- Represent vertices by sets of edges they belong to
- Make representation symmetric: extend sets freely so their sizes are the same

Construction of EPPA-witness B

1. Vertices: all k-element subsets of $\{1, 2, \ldots, n\}$.
2. Edges: Two sets are joined by an edge if they intersect.

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Proof extends naturally to oriented graphs and graphs with multiple types of edges.
Hypergraphs or relatoinal structures are bit more difficult.
It does not extend naturally to triangle-free graphs.

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- Represent vertex $x \in A$ by function $\psi_x \colon A \setminus \{x\} \to \{0,1\}$ that corresponds to given row of the asymmetric adjacency matrix of $\mathbf{A}$.

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- **B** is an EPPA witness of **A**.

New construction of EPPA for graphs

Construction of EPPA-witness B of A

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Embeding $\psi : \mathbf{A} \to \mathbf{B}$ is $\psi(x) = \psi_x$ s.t. $\psi_x(y) = I_{x,y}$ (row of assym. adjacency matrix).

Theorem (H., Konečný, Nešetřil, 2018)

Every partial automorphism φ of $\psi(\mathbf{A})$ extends to automorphism $\mathbf{B}$.

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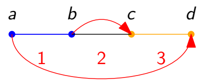
Proof.

- φ induces partial permutation of A which extends to full permutation $\hat{\varphi}(A)$.

□



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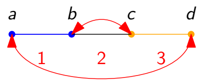

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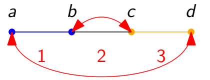

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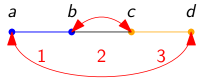
- Automorphism extending φ is $\theta(\chi_x) = \chi'$ where

$$\chi'(\hat{\varphi}(y)) = \begin{cases} \chi_x(y) & \text{if } \{x,y\} \notin F \\ 1 - \chi_x(y) & \text{if } \{x,y\} \in F. \end{cases}$$

For all $y \in A \setminus \hat{\varphi}(x)$. □



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Generalization for relational structures

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Construction of EPPA-witness $\mathbf{B}$ for $\mathbf{A}$

1. Vertices: All x-valuation functions.
2. Relations: $(\chi_{x_1}, \ldots, \chi_{x_a}) \in R_{\mathbf{B}}$ if and only if the tuple is transversal and the following sum is odd:

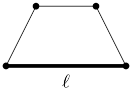
$$\sum_{\chi \in \{\chi_{x_1}, \ldots, \chi_{x_a}\}} \chi(R)((x_1, x_2, \ldots, x_n)).$$

New construction of EPPA for metric spaces

We consider metric spaces to be complete edge-labeled graphs
(or relational structures).

Lemma

Let **A** *be a metric space seen (complete) edge-labeled graph.*
If there is EPPA-witness **B** *(possibly with no distance defined for some pairs of vertices) which contains no induced non-metric cycles, then* **B** *can be completed to a metric space* **C** *which is EPPA-witeness of* **A**.



Non-metric cycle is edge-labeled cycle with one long edge ℓ longer than sum of the lengths of others.

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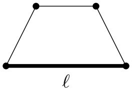

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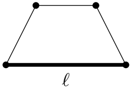
Proof.

C has same vertex set as **B** and distance of x, y in **C** is the length of shortest path connecting x and y in **B**. □

New proof of EPPA for metric spaces

Main idea: Repeat the valuation trick to unwind cycles to "Möbius strips"



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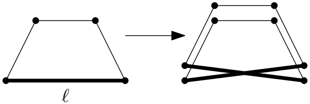

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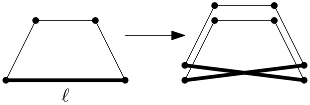

- Let **A** be metric space, $\mathbf{B}_0$ an EPPA-witness, C_0 smallest non-metric cycle of $\mathbf{B}_0$.
- For $x \in B_0$ let $U(x)$ be the set of all induced non-metric cycles containing x.
- For $x \in B_0$ let x-valuation function be any $\chi_x : U(x) \to \{0, 1\}$.

Construction of EPPA-witness **B**

1. Vertices: all pairs (x, χ_x) where $x \in B$ and χ_x is an x-valuation function.
2. Distances: $d_{\mathbf{B}}((x, \chi_x), (y, \chi_y)) = n$ iff $d_{\mathbf{B}_0}(x, y) = n$ and

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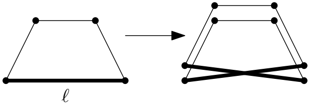

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Proof.

- Induced non-metric C cycle of $\mathbf{B}_0$ intersects $\mathbf{A}$ by at most 2 vertices x, y.
  Put $\chi_C(x) = 0, \chi_C(y) = 1$ if edge x, y is long and $\chi_C(x), \chi_C(y) = 0$ otherwise.
- Embedding $\psi : \mathbf{A} \to \mathbf{B}$ maps $x \in A$ to (x, χ_x) where $\chi_x(C) = \chi_C(x)$.
- Automorphism of $\mathbf{B}_0$ induce automorphisms of $\mathbf{B}$ with "flips".

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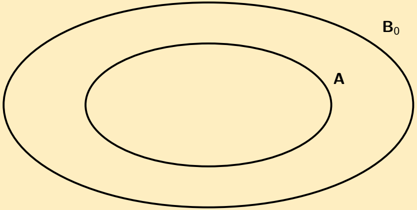

New proof of EPPA for metric spaces

Lemma

$\mathbf{B}$ is an EPPA-witness of $\mathbf{A}$; non-metric cycles of $\mathbf{B}$ has more edges than C_0 edges.

Proof.

- Induced non-metric C cycle of $\mathbf{B}_0$ intersects $\mathbf{A}$ by at most 2 vertices x, y.
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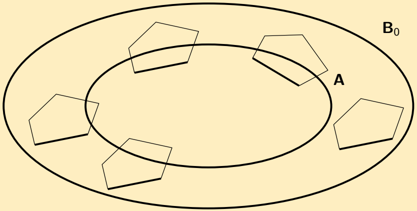

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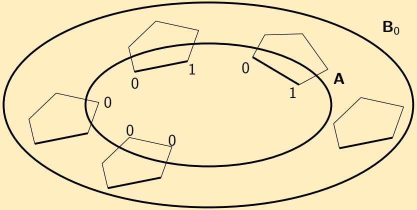

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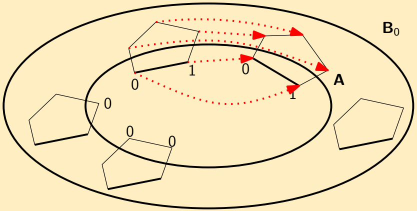

- Given automorphism $\hat{\varphi} : \mathbf{B}_0 \to \mathbf{B}_0$, F is set of "flipped" cycles. Define automorphism $\theta : \mathbf{B} \to \mathbf{B}$ by $\theta((x, \chi_x)) = (\hat{\varphi}(x), \chi')$ where $\chi'(\hat{\varphi}(C)) = \chi_x(C) \iff C \notin F$.

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Lemma

B *is an EPPA-witness of* **A***; non-metric cycles of* **B** *has more edges than* C_0 *edges.*

Theorem (Solecki 2005, Vershik 2008; new proof by H., Konečný, Nešetřil 2018)

The class of all finite metric spaces has EPPA.

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The class of all finite metric spaces has EPPA.

Proof.

- Given metric space $\mathbf{A}$ construct EPPA-witness $\mathbf{B}_0$ (using construction for relational structures)
- Repeat Lemma N times to obtain $\mathbf{B}'$.

$$N = \left\lceil \frac{\max\{d_{\mathbf{A}}(x,y); x \neq y \in A\}}{\min\{d_{\mathbf{A}}(x,y); x \neq y \in A\}} \right\rceil$$

- Complete $\mathbf{B}'$ to $\mathbf{B}$ by the shortest path completion.

□