Universal structures II

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Apr 29 2020

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Definition (Amalgamation class)

Class $\mathcal{C}$ is an amalgamation class if it is

1. closed under isomorphism,
2. closed under taking induced substructures,
3. contains only countably many non-isomorphic structures, and,
4. has the amalgamation property.

Definition (Homogeneous structure)

Structure is homogeneous if every isomorphism of its two finite substructures extends to an automorphism.

Theorem (Fraïssé 1953)

(a) *A class $\mathcal{C}$ of finite (relational) structures is the age of a countable homogeneous structure $\mathbf{H}$ if and only if $\mathcal{C}$ is an amalgamation class.*

(b) *If conditions of (a) are satisfied then the structure $\mathbf{H}$ is unique up to isomorphism.*

Definition

Let $\mathcal{K}$ be a class of structures. We say that $\mathbf{U}$ is $\mathcal{K}$-universal if for every $\mathbf{A} \in \mathcal{K}$ there is an embedding $\mathbf{A} \to \mathbf{U}$.

Example

classes containing universal structure:

1. Class of all countable graphs, partial orders, rational metric spaces...
2. For every $k \geq 3$ the class of all graphs not containing clique of size k.
3. For every $k \geq 3$ the class of all graphs not containing independent of size k.
4. Class of all countable graphs with maximal degree 2.

Definition

Given a finite family of structure $\mathcal{F}$, denote by $\text{Forb}_e(\mathcal{F})$ / $\text{Forb}_m(\mathcal{F})$ / $\text{Forb}_h(\mathcal{F})$ the class of all finite or countable structures $\mathbf{A}$ such that for every $\mathbf{F} \in \mathcal{F}$ there is no embedding / monomorphism / homomorphism $\mathbf{A} \to \mathcal{F}$ (respectively).

Question

Does every class $\text{Forb}_e(\mathcal{F})$, for $\mathcal{F}$ finite contain an universal structure?

Theorem (Cherlin, Shelah, Shi, 1999; Covington 1990)

Let $\mathcal{F}$ be a finite family of connected structures. Then $\text{Forb}_h(\mathcal{F})$ contains an universal structure.

H.N. 2005: In fact this can be generalized to so called regular families $\mathcal{F}$.

Definition (Minimal separating cut)

A vertex cut R of graph $\mathbf{G}$ is minimal separating if there exists components $\mathbf{C}_1$ and $\mathbf{C}_2$ of $\mathbf{G} \setminus R$ such that $C = N_{\mathbf{G}}(\mathbf{C}_1) \cap N_{\mathbf{G}}(\mathbf{C}_2)$.

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Definition (Piece of a graph, Hubička, Nešetřil 2005)

Let $\mathbf{A}$ be a connected graph and R a minimal separating cut for a component $\mathbf{A}_1$ in $\mathbf{A}$.
A piece of $\mathbf{A}$ is then a rooted graph $\mathfrak{P} = (\mathbf{P}, \vec{R})$, where the tuple $\vec{R}$ consists of the vertices of the cut R in a (fixed) linear order and $\mathcal{P}$ is a structure induced by $\mathbf{A}$ on $A_1 \cup R$.

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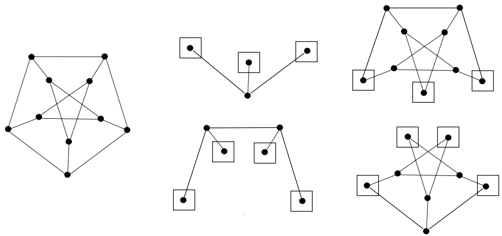

Basic idea of the proof

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Definition (Language $L^+_{\mathcal{F}}$)

Given finite family $\mathcal{F}$ of finite connected graphs. Enumerate by $\mathfrak{P}_1, \mathfrak{P}_1, \ldots, \mathfrak{P}_n$ all pieces of structures in $\mathcal{F}$. Then language $L^+_{\mathcal{F}}$ extends language of graphs by relations $R_1, R_2, \ldots R_n$ where arity of R_i is the width of $\mathfrak{P}_i$.

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Definition (Canonical $\mathcal{F}$-lift)

Given graph $\mathbf{G} \in \mathrm{Forb}_h(\mathcal{F})$ we denote by $L(\mathbf{G})$ the canonical lift of **G** with is an $L^+_{\mathcal{F}}$-structure on same vertex set and with same edges. We put $(v_1, v_2, \dots, v_w) \in R_i$ if and only if there exists homomorphism from $\mathfrak{P}_i$ to **G** sending roots of $\mathfrak{P}_i$ to $v_1, v_2, \dots, v_w$.

The proof

Definition (Maximal $\mathcal{F}$-lift)

The canonical $\mathcal{F}$-lift $L_{\mathcal{F}}(\mathbf{A})$ is maximal on $B \subseteq A$ if for every $\mathbf{C} \in \text{Forb}_h(\mathcal{F})$ such that $\mathbf{C}$ contains $\mathbf{A}$ as substructure, the $\mathcal{F}$-lift induced on $\mathbf{B}$ by $L_F(A)$ is the same as the $\mathcal{F}$-lift induced on $\mathbf{B}$ by $L_F(\mathbf{C})$.

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$\mathcal{F}$-lift $\mathbf{X}$ is maximal if there exists $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$ such that $\mathbf{X}$ is induced on $\mathbf{X}$ by $L_F(\mathbf{A})$ and the canonical $\mathcal{F}$-lift $L_{\mathcal{F}}(\mathbf{A})$ of $\mathbf{A}$ is maximal on X.

Theorem (Hubička, Nešetřil 2005)

For every finite family $\mathcal{F}$ of connected structures, the class of finite maximal $\mathcal{F}$-lifts is an amalgamation class.

The proof

Homomorphism universality

Definition

Finite relational structure $\mathbf{D}$ is *dual* of finite relational structure $\mathbf{F}$ iff for every relational structure $\mathbf{A}$ holds:

$$\mathbf{F} \not\to \mathbf{A} \iff \mathbf{A} \to \mathbf{D}.$$

In other words, $\mathbf{D}$ is homomorphism-universal for the class $\mathrm{Forb}_h(\{\mathbf{F}\})$: For every $\mathbf{A} \in \mathrm{Forb}_h(\{\mathbf{F}\})$ there exists a homomorphism to $\mathbf{D}$.

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Theorem (Nešetřil, Tardif (2000))

Directed graph core $\mathbf{G}$ *has dual if and only if* $\mathbf{G}$ *is an oriented tree.*

Homomorphism universality

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Finite relational structure **D** is *dual* of finite relational structure **F** iff for every relational structure **A** holds:

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In other words, **D** is homomorphism-universal for the class $\mathrm{Forb_h}(\{\mathbf{F}\})$: For every $\mathbf{A} \in \mathrm{Forb_h}(\{\mathbf{F}\})$ there exists a homomorphism to **D**.

Theorem (Nešetřil, Tardif (2000))

Directed graph core **G** *has dual if and only if* **G** *is an oriented tree.*
Finite family $\mathcal{F}$ *of finite cores has dual if and only if* $\mathcal{F}$ *is a family of trees.*

Infinite families

Theorem (P. L. Erdős, Pálvölgyi, Tardif, Tardos, 2017)

Family $\mathcal{F}$ of trees has dual if and only if $\mathcal{F}$ is regular.

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Theorem (Myhil-Nerode)

Let L be a set of words over alphabet Σ. Let X and Y be strings. We put $X \sim_L Y$ if and only if for every string Z

$$XZ \in L \iff YZ \in L.$$

L is regular (recognizable by a finite automaton) if and only if $\sim_L$ has finitely many equivalence classes.

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$$\mathfrak{P}_1 \oplus \mathfrak{T} \in \mathrm{Forb_h}(\mathcal{F}) \iff \mathfrak{P}_2 \oplus \mathfrak{T} \in \mathrm{Forb_h}(\mathcal{F}).$$

$\mathcal{F}$ is regular if and only if $\sim_{\mathcal{F}}$ has finitely many equivalence classes.

Regular families of structures

Let $\mathcal{F}$ be a family of structures. A piece $\mathfrak{P}$ is incompatible with rooted structure $\mathfrak{A}$ of $\mathfrak{P} \oplus \mathfrak{A}$ is isomorphic to some structure in $\mathcal{F}$.

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Denote by $I_{\mathcal{P}}$ the set of all incompatible structures of $\mathfrak{P}$ and put $\mathfrak{P}_1 \sim_{\mathcal{F}} \mathfrak{P}_2$ if $I_{\mathfrak{P}_1} = I_{\mathfrak{P}_2}$.

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Family of structures $\mathcal{F}$ is regular iff for every $n \geq 1$ $\sim_{\mathcal{F}}$ has only finitely many equivalence classes of pieces of $\mathcal{F}$ of width n.

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Family of structures $\mathcal{F}$ is regular iff for every $n \geq 1$ $\sim_{\mathcal{F}}$ has only finitely many equivalence classes of pieces of $\mathcal{F}$ of width n.

Family $\mathcal{F}$ is upwards closed if for every $\mathbf{F} \in \mathcal{F}$ we also have $\mathbf{F}' \in \mathcal{F}$ provided that $\mathcal{F}$ is connected and there is a homomorphism $\mathbf{F} \to \mathbf{F}'$.

Theorem (H., Nešetřil 2005)

Let L be a finite relational language. Let $\mathcal{F}$ be a upwards closed family of finite connected L-structures. Then the following conditions are equivalent:

1. *$\mathcal{F}$ is a regular family of connected structures.*
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Bounding arities

Theorem (H., Nešetřil 2005)

Let $\mathcal{F}$ be a finite set of cores such that there is no $\mathbf{F}_1 \neq \mathbf{F}_2 \in \mathcal{F}$ with a homomorphism $\mathbf{F}_1 \to \mathbf{F}_2$. Then every homogenization of $\mathrm{Forb}_h(\mathcal{F})$ in finite language uses relations of arity n where n is the maximal number of roots of a piece of structure in $\mathcal{F}$.