Universal structures

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2. closed under taking induced substructures,
3. contains only countably many non-isomorphic structures, and,
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Definition (Homogeneous structure)

Structure is homogeneous if every isomorphism of its two finite substructures extends to an automorphism.

Theorem (Fraïssé 1953)

(a) *A class $\mathcal{C}$ of finite (relational) structures is the age of a countable homogeneous structure $\mathbf{H}$ if and only if $\mathcal{C}$ is an amalgamation class.*

(b) *If conditions of (a) are satisfied then the structure $\mathbf{H}$ is unique up to isomorphism.*

Definition

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Definition

Given a finite family of structure $\mathcal{F}$, denote by $\mathrm{Forb_e}(\mathcal{F})$ / $\mathrm{Forb_m}(\mathcal{F})$ / $\mathrm{Forb_h}(\mathcal{F})$ the class of all finite or countable structures $\mathbf{A}$ such that for every $\mathbf{F} \in \mathcal{F}$ there is no embedding / monomorphism / homomorphism $\mathbf{A} \to \mathcal{F}$ (respectively).

Question

Does every class $\mathrm{Forb_e}(\mathcal{F})$, for $\mathcal{F}$ finite contain an universal structure?

Non-existence of universal graphs

Theorem (Füredi, Komjáth, 1997)

If $\mathbf{G}$ *is a 2-connected graph that is not complete, then* $\mathrm{Forb}_{\mathrm{m}}(\{\mathbf{G}\})$ *contains no universal graph.*

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Lemma

For every k *and* g *there exist a* $t = t(k, g)$ *and a countable* k*-uniform hypergraph* $\mathbf{H}_g^k$ *of girth* g*, vertex set* ω *and edges* $E_1, E_2, \ldots$ *such that* $(i + t) \in E_i \subseteq \{i - t, \ldots, i + t\}$*.*

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If copies of graphs $\mathbf{G}(\epsilon)$ *and* $\mathbf{G}(\epsilon')$ *overlap by vertices* $1, 2, \ldots t$ *then they overlap completely.*

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Universal graphs omitting cycles

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Theorem (Cherlin, 1996)

Let $\mathcal{C}$ be family of cycles, $\mathrm{Forb_m}(\mathcal{C})$ contains an universal graph if an only if $\mathcal{C}$ is a set of odd cycles of length $1, 3, 5, \ldots k$ for some k.

Theorem (Cherlin, Shelah, Shi, 1999; Covington 1990)

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Definition (Minimal separating cut)

A vertex cut R of graph $\mathbf{G}$ is minimal separating if there exists components $\mathbf{C}_1$ and $\mathbf{C}_2$ of $\mathbf{G} \setminus R$ such that $C = N_{\mathbf{G}}(\mathbf{C}_1) \cap N_{\mathbf{G}}(\mathbf{C}_2)$.

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Let $\mathbf{A}$ be a connected graph and R a minimal separating cut for a component $\mathbf{A}_1$ in $\mathbf{A}$.
A piece of $\mathbf{A}$ is then a rooted graph $\mathfrak{P} = (\mathbf{P}, \vec{R})$, where the tuple $\vec{R}$ consists of the vertices of the cut R in a (fixed) linear order and $\mathcal{P}$ is a structure induced by $\mathbf{A}$ on $A_1 \cup R$.

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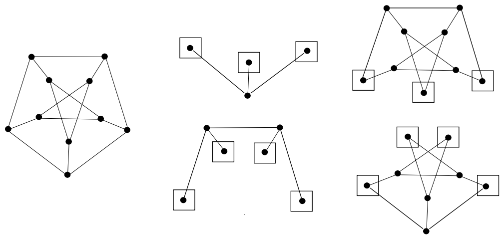

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Definition (Language $L^+_{\mathcal{F}}$)

Given finite family $\mathcal{F}$ of finite connected graphs. Enumerate by $\mathfrak{P}_1, \mathfrak{P}_1, \ldots, \mathfrak{P}_n$ all pieces of structures in $\mathcal{F}$. Then language $L^+_{\mathcal{F}}$ extends language of graphs by relations $R_1, R_2, \ldots R_n$ where arity of R_i is the width of $\mathfrak{P}_i$.

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Definition (Canonical $\mathcal{F}$-lift)

Given graph $\mathbf{G} \in \mathrm{Forb_h}(\mathcal{F})$ we denote by $L(\mathbf{G})$ the canonical lift of $\mathbf{G}$ with is an $L^+_\mathcal{F}$-structure on same vertex set and with same edges. We put $(v_1, v_2, \ldots, v_w) \in R_i$ if and only if there exists homomorphism from $\mathfrak{P}_i$ to $\mathbf{G}$ sending roots of $\mathfrak{P}_i$ to $v_1, v_2, \ldots, v_w$.

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Definition (Maximal $\mathcal{F}$-lift)

The canonical $\mathcal{F}$-lift $L_{\mathcal{F}}(\mathbf{A})$ is maximal on $B \subseteq A$ if for every $\mathbf{C} \in \mathrm{Forb_h}(\mathcal{F})$ such that $\mathbf{C}$ contains $\mathbf{A}$ as substructure, the $\mathcal{F}$-lift induced on $\mathbf{B}$ by $L_F(A)$ is the same as the $\mathcal{F}$-lift induced on $\mathbf{B}$ by $L_F(\mathbf{C})$.

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$\mathcal{F}$-lift $\mathbf{X}$ is maximal if there exists $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$ such that $\mathbf{X}$ is induced on $\mathbf{X}$ by $L_F(\mathbf{A})$ and the canonical $\mathcal{F}$-lift $L_{\mathcal{F}}(\mathbf{A})$ of $\mathbf{A}$ is maximal on X.

Theorem (Hubička, Nešetřil 2005)

For every finite family $\mathcal{F}$ of connected structures, the class of finite maximal $\mathcal{F}$-lifts is an amalgamation class.

The proof

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Finite relational structure **D** is *dual* of finite relational structure **F** iff for every relational structure **A** holds:

$$\mathbf{F} \not\to \mathbf{A} \iff \mathbf{A} \to \mathbf{D}.$$

Theorem (Nešetřil, Tardif (2000))

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Theorem (P. L. Erdős, Pálvölgyi, Tardif, Tardos, 2017)

Family $\mathcal{F}$ of trees has dual if and only if $\mathcal{F}$ is regular.