Fraïssé limits

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Lemma

Any two countable graphs with the extension property are isomorphic.

Constructions

Example (Rado)

The following graph is isomorphic to R:

- Vertices are all integers.
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Let M be a countable modes of set theory (which exists by Skolem paradox).
Then the following graph is also isomorphic to R:

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Universality

Theorem

R is universal for the class of all countable graphs: R contains every countable graph as an induced subgraph.

Homogeneity

Definition

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Can we find more objects like R?

(Model-theoretic) structures

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Definition (L-struture)

L-structure $\mathbf{A}$ has vertex set (or domain) A equipped with:

1. Relation $R_{\mathbf{A}} \subseteq A^{\text{arity}(R)}$ for every relational symbol $R \in L$;
2. Function $F_{\mathbf{A}}^{\text{arity}(F)} \to A$ for every function symbol $F \in L$.

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2. n-uniform hypergraphs

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Definition (Homomorphism of L-structures)

A homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is a mapping $f\colon A \to B$ such that:

1. For every relational symbol $R \in L$: $(x_1, x_2, \ldots, x_{\mathrm{arity}(R)}) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{\mathrm{arity}(R)})) \in R_{\mathbf{B}}$,
2. For every function symbol $F \in L$: $f(F_{\mathbf{A}}(x_1, x_2, \ldots, x_{\mathrm{arity}(F)})) = F_{\mathbf{B}}(f(x_1), f(x_2), \ldots, f(x_{\mathrm{arity}(F)}))$.

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Monomorphism, embedding and isomorphism can be defined analogously as on graphs.
Substructures are always induced.

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By $\mathrm{Age}(\mathbf{A})$ we denote the class of all L-structures isomorphic to some (induced) substructure of $\mathbf{A}$.

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Definition (Amalgamation class)

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3. contains only countably many non-isomorphic structures, and,
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Definition (Recall: Homogeneous structure)

Structure is homogeneous if every isomorphism of its two finite substructures can be extended to an automorphism.

Theorem (Recall: Fraïssé 1953)

(a) *A class $\mathcal{C}$ of finite (relational) structures is the age of a countable homogeneous structure $\mathbf{H}$ if and only if $\mathcal{C}$ is an amalgamation class.*

(b) *If conditions of (a) are satisfied then the structure $\mathbf{H}$ is unique up to isomorphism.*

Amalgamation property

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A class $\mathcal{C}$ of finite structures has the amalgamation property if, given $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$ and embeddings $f_1 : \mathbf{A} \to \mathbf{B}_1$ and $f_2 : \mathbf{A} \to \mathbf{B}_2$, there exists $\mathbf{C} \in \mathcal{C}$ and embeddings $g_1 : \mathbf{B}_1 \to C$ and $g_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $f_1 g_1 = f_2 g$.

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It is easy to see that $\mathrm{Age}(\mathbf{H})$ is an amalgamation class.
Lets prove the other direction.

Rich sequences

Definition (Rich sequence)

Suppose $\mathcal{C}$ is a class of finite L-structures. An increasing chain $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \mathbf{A}_3 \subseteq \cdots$ of structures in $\mathcal{C}$ is rich sequence if:

1. for all $\mathbf{A} \in \mathcal{C}$ there is some $i < \omega$ and a embedding $e : \mathbf{A} \to \mathbf{A}_i$;
2. for all $\mathbf{B} \in \mathcal{C}$, $f : \mathbf{A}_j \to \mathbf{B}$ there is $k \geq j$ and a $g : \mathbf{B} \to \mathbf{A}_k$ such that $g(f(a)) = a$ for all $a \in \mathbf{A}_j$.

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Proof of the Fraïssé theorem:

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Proof of the Fraïssé theorem:

1. Every amalgamation class $\mathcal{C}$ contains a rich sequence.
2. A Fraïssé limit of $\mathcal{C}$ is an L-structure which is the union of a rich sequence of substructures.
3. The Fraïssé limit has the extension property: for every $\mathbf{A} \subseteq \mathbf{B} \in \mathcal{C}$ and embedding $e : \mathbf{A} \to \mathbf{M}$ there exists embedding $f : \mathbf{B} \to \mathbf{M}$ extending e.

More homogeneous structures

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3. All homogeneous digraphs was classified by Gregory Cherlin.