Algorithms and datastructures II

Lecture 9: Introduction to the complexity theory

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Nov 30 2020

Decision problems

Definition

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4. Overall time is $O(a^\ell + a^{k\ell})$

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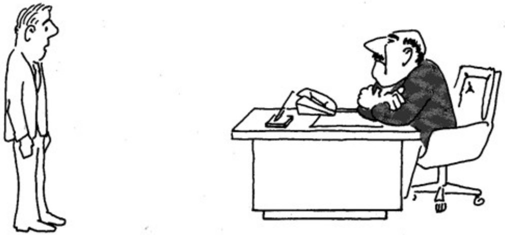

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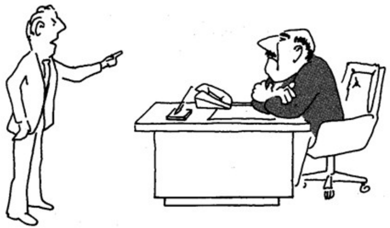

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Definition (Conjuctive normal form)

Formula φ in conjunctive normal form (CNF) consists of

1. clauses separated by $\vee$ (“and”),
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Same as SAT but every clause has at most 3 literals

Observation

3-SAT $\longrightarrow$ SAT

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1. Assume that there is some clause with $k > 3$ literals. Write it as $(\alpha \vee \beta)$ where α has 2 literals and β has k literals.

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4. Apply $3 - SAT$ to solve expanded formula. Original formula is satisfiable $\iff$ expanded formula is.

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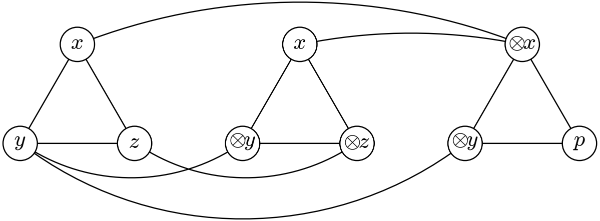

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Clique

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2. 1 if and only if G contains an clique of size k.

IndSet $\longrightarrow$ Clique
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3,3-SAT

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if variable x appears more than 3 times, replace it by multiple variables $x_1, x_2, \ldots x_k$ and add clauses requiring them to have same values $(x_1 \implies x_2)$, $(x_2 \implies x_3)$, $\ldots$, $(x_{k-1} \implies x_k)$, $(x_k \implies x_1)$.

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$P = NP$ is open since 1970's.

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Problem *L* is *NP*-hard if every problem from *NP* can be reduced to *L*.

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3. Numerical problems: Finding subset of a given sum, Knapsack, $Ax = 1$, . . .