Algorithms and datastructures II

Lecture 4: network flows 2

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Oct 26 2020

Network flow

Definition (Network)

Network is an 4-tuple $N = (V, E, s, t, c)$ where

1. (V, E) is a directed graph,
2. $s \in V$ is a source vertex,
3. $t \in V$ is a sink vertex,
4. $c : E \to \mathbb{R}_0^+$ is a function assigning every edge a capacity.

- $f^+(v) = \sum_{u,(u,v)\in E} f(u, v)$ (flow into a vertex)
- $f^-(v) = \sum_{u,(v,u)\in E} f(v, u)$ (flow out of a vertex)
- $f^\Delta(v) = f^+(v) - f^-(v)$ (surplus)

Here $f : E \to \mathbb{R}_0^+$

Definition (Flow)

Function $f : E \to \mathbb{R}_0^+$ is flow if it satisfies

1. Capacity constraint: $(\forall_{e\in E}) : f(e) \leq c(e)$
2. Conservation of flows (Kirchoff's law): $(\forall_{v\in V\setminus\{s,t\}}) : f^\Delta(v) = 0$

Value of the flow: $|f| = f^\Delta(t)$. Problem: Find a flow with maximum value.

Ford–Fulkerson algorithm, 1956

Definition (Residual capacity)

$r(u, v) = c(u, v) - f(u, v) + f(v, u)$

Definition (Augumenting path)

A path in (V, E) is augmenting if every edge has non-zero residual capacity.

FordFulkerson(V, E, s, t, c)

1. $f \leftarrow$ zero flow (or flow of your choice).
2. While there exists augmenting path P from s to t:
3. $\quad \epsilon \leftarrow \min_{e \in P} r(e)$.
4. $\quad$ For every $\{u, v\} \in P$:
5. $\qquad \delta \leftarrow \min(f(v, u), \epsilon)$.
6. $\qquad f(v, u) \leftarrow f(v, u) - \delta$.
7. $\qquad f(u, v) \leftarrow f(u, v) + \epsilon - \delta$.
8. Return f (maximum flow).

Invariant: f is a flow.

Definition (elementary cut)

(X, Y) is an (elementary) cut of graph (V, E) if:

1. $X, Y \subseteq V$,
2. $X \cup Y = V$,
3. $X \cap Y = \emptyset$,
4. $s \in X$,
5. $t \in Y$.

$$E(X, Y) = E \cap \{X \times Y\}$$

$$f(X, Y) = \sum_{e \in E(X,Y)} f(e)$$

$$f^{\Delta}(X, Y) = f(X, Y) - f(Y, X)$$

Observation C

If f is a flow and (X, Y) a cut, $|f| = c(X, Y)$ then f is a maximum flow and $c(X, Y)$ is minimum capacity.

Dinic algorithm 1970

Dinic (V, E, s, t, c)

1. $f \leftarrow$ zero flow.
2. Repeat:
3. Build residual network R and remove all edges e with $r(e) = 0$.
4. $\ell \leftarrow$length of the shortest oriented path from s to t in R.
5. If there is no such path return f.
6. $L \leftarrow$LayeredNetwork (R).
7. $g \leftarrow$BlockingFlow (L).
8. Improve flow f using g.

Theorem

Dinic algorithm will terminate in time $O(|V|^2|E|)$ and will return maximum flow.

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Today: An easier algorithm with same time complexity.

Goldberg algorithm (Golberg, Tarjan 1988): Push-relabel, Preflow-push

Definition (Wave or pre-flow)

A wave (or pre-flow) in a network $N = (V, E, s, t, c)$ is function $f : E \to \mathbb{R}^+$ satisfying:

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1. $\delta \leftarrow \min(f(v, u), \epsilon)$.
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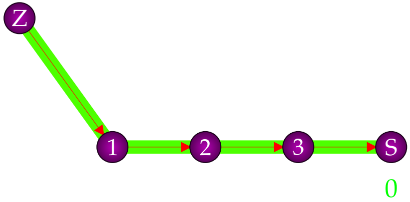

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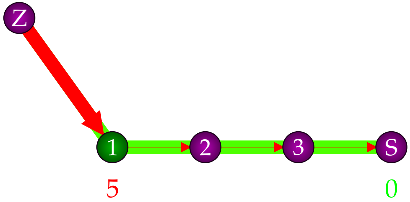

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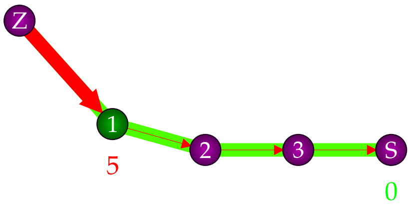

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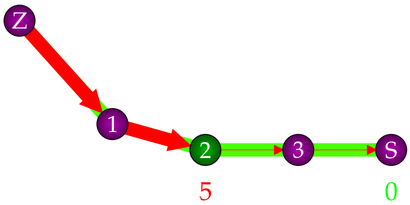

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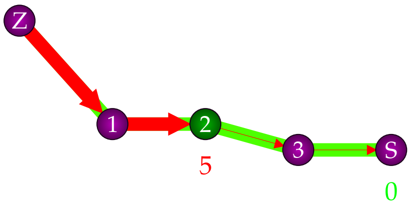

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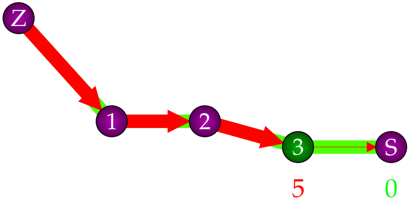

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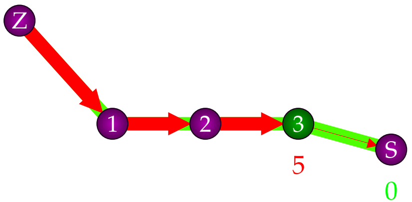

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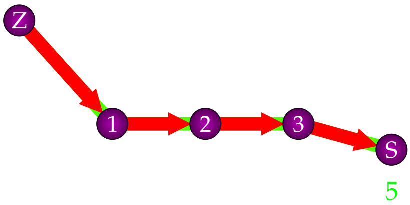

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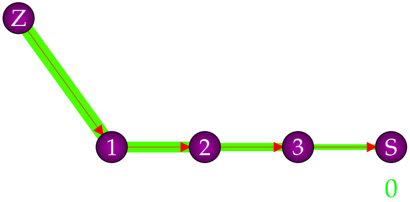

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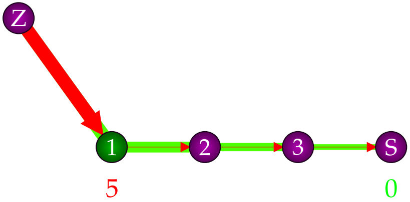

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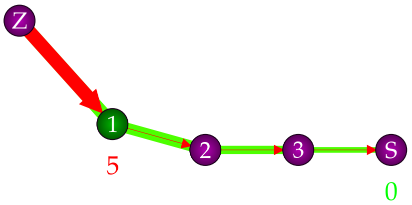

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4. $h(s) \leftarrow 0$ for all $v \in V \setminus \{s\}$
5. While there exists $u \in V \setminus \{s,t\}$, $f^{\Delta}(u) > 0$:
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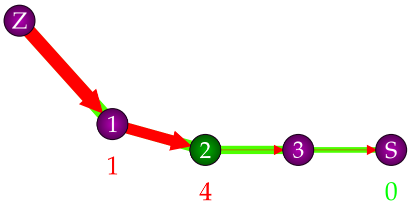

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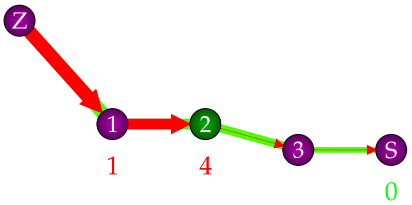

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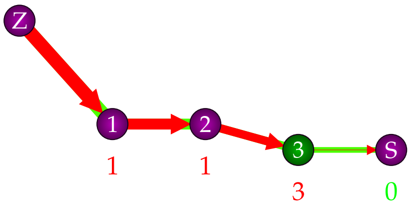

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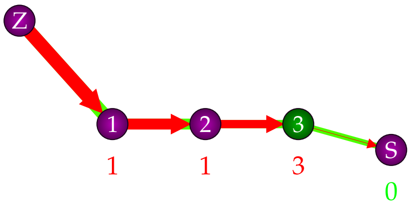

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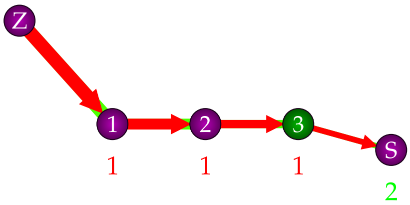

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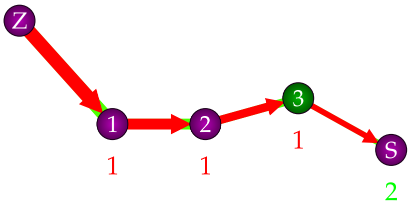

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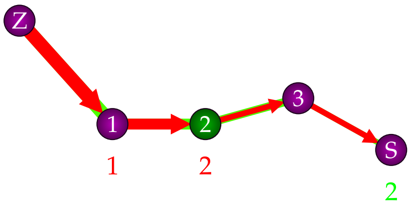

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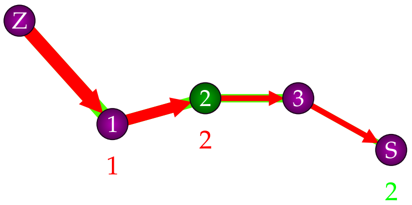

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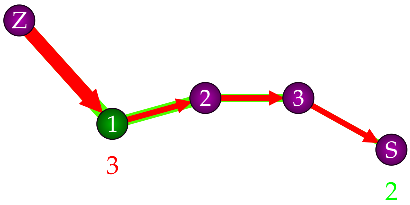

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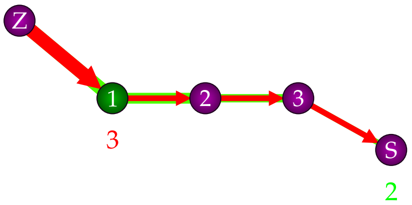

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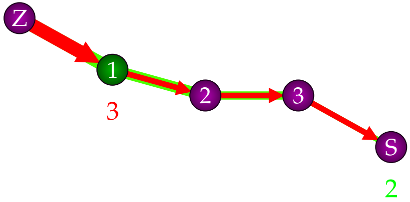

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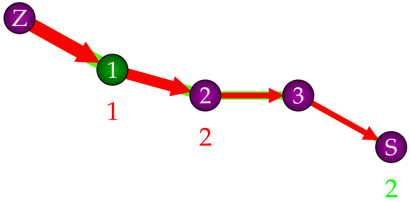

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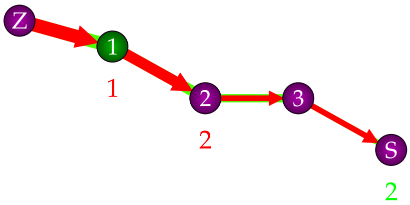

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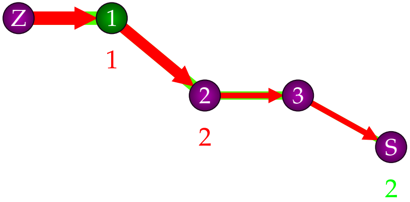

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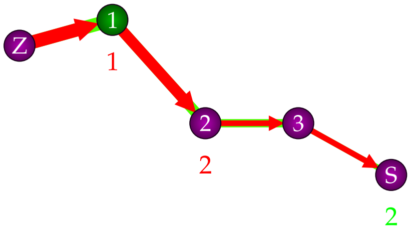

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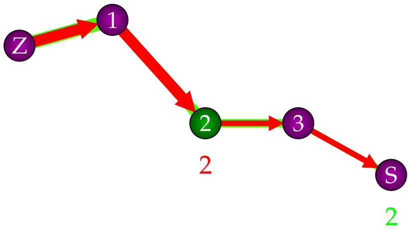

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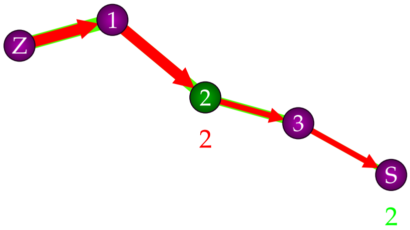

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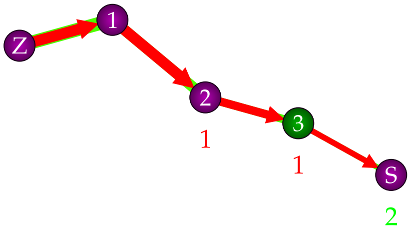

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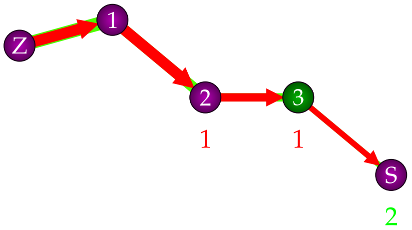

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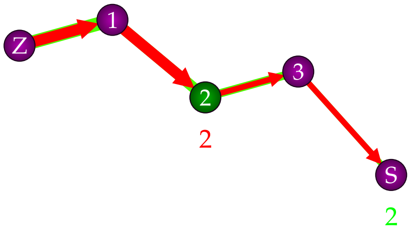

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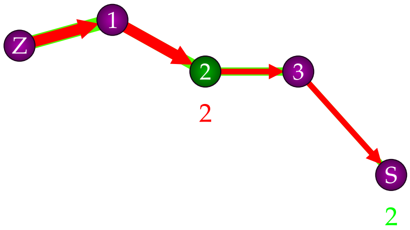

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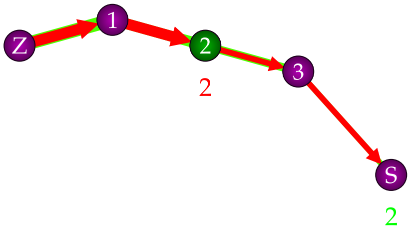

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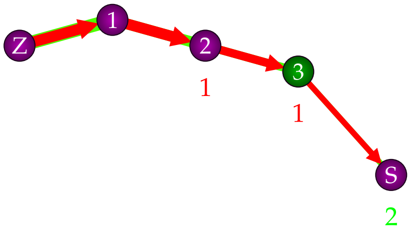

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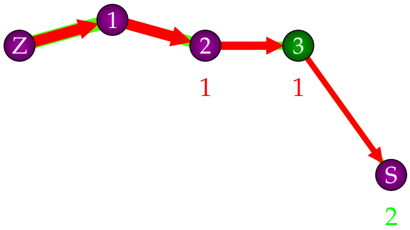

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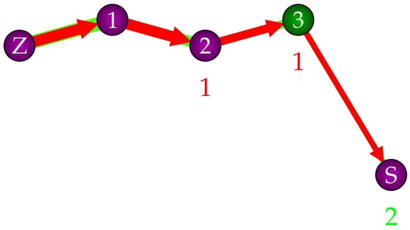

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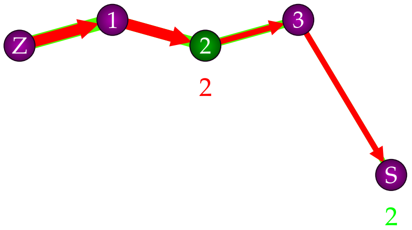

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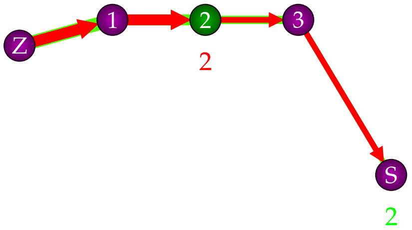

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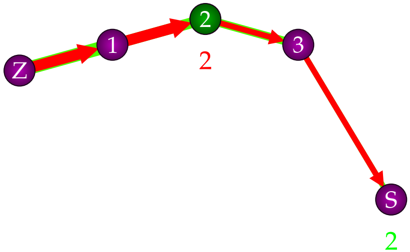

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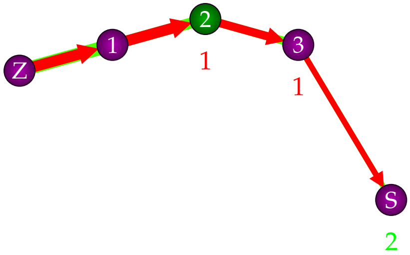
Goldberg Meets Bottleneck



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2. $f(s, v) \leftarrow c(s, v)$ for every v such that $(s, v) \in E$.
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4. $h(s) \leftarrow 0$ for all $v \in V \setminus \{s\}$
5. While there exists $u \in V \setminus \{s, t\}$, $f^{\Delta}(u) > 0$:
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7. else: $h(u) \leftarrow h(u) + 1$.

Goldberg algorithm (Golberg, Tarjan 1988): Push-relabel, Preflow-push

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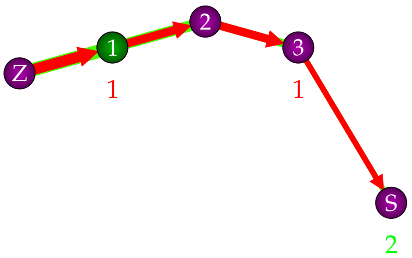

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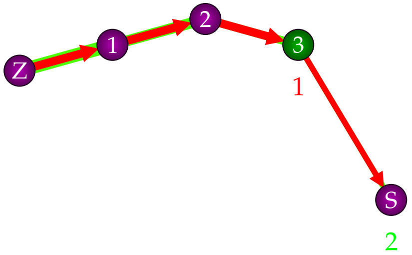

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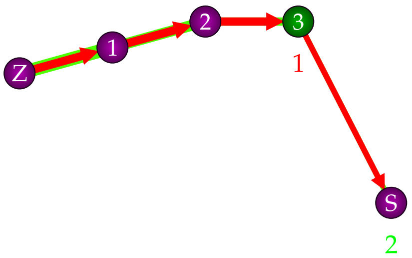

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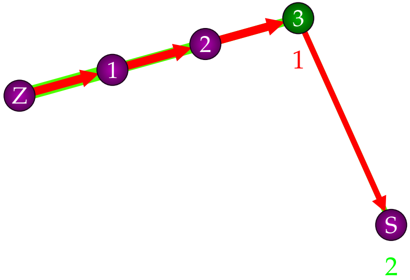

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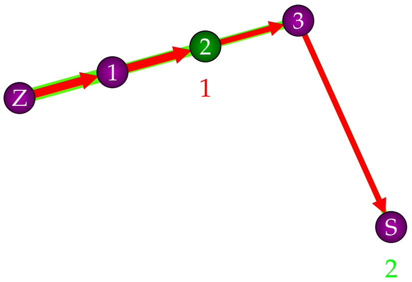

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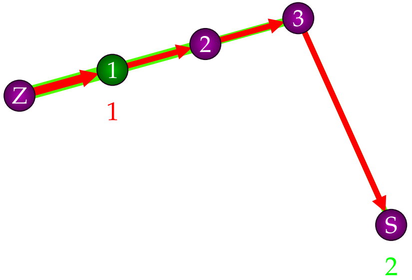

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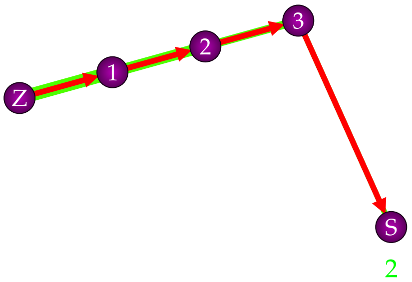

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5. Sum of all surpluses is 0.

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For every edge (u, v) with $r(u, v) > 0$ it holds that $h(u) - h(v) \leq 1$.

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We proceed by induction on run of the algorithm. After initialization the invariant holds. It can be invalidated by:

1. Increasing height of u while there exists an edge $r(u, v) > 0$.
2. Increasing $r(u, v)$ for some edge with $h(u) - h(v) > 1$.

These cases does not happen.

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Lemma (On correctness)

If PushRelabel terminates, it returns a maximum flow.

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Proof.

To see that f is a wave that is also a flow we only need to verify the preservation of flows. This is satisfied because algorithm terminated.

To see that f is maximum flow assume, to the contrary, that there is an augmenting path P. By invariant A we know that $h(s) = |V|$ and $h(t) = 0$. P thus goes down by $|V|$ but has at most $|V| - 1$ edges. So one of edges must have gradient 2 that contradicts invariant G. □

Definition (Push (u,v))

Put $\epsilon = \min(f^{\Delta}(v), r(u,v))$ and send ϵ from u to v:

1. $\delta \leftarrow \min(f(v,u), \epsilon)$.
2. $f(v,u) \leftarrow f(v,u) - \delta$.
3. $f(u,v) \leftarrow f(u,v) + \epsilon - \delta$.

PushRelabel($N = (V, E, s, t, c)$)

1. $f \leftarrow$ zero wave.
2. $f(s,v) \leftarrow c(s,v)$ for every v such that $(s,v) \in E$.
3. $h(s) \leftarrow |V|$ ($h(s)$ is an height array)
4. $h(s) \leftarrow 0$ for all $v \in V \setminus \{s\}$
5. While there exists $u \in V \setminus \{s,t\}$, $f^{\Delta}(u) > 0$:
6. If there is an edge (u,v) such that $r(u,v) > 0$ and $h(u) > h(v)$: Push (u,v)
7. else: $h(u) \leftarrow h(u) + 1$.

Goldberg algorithm (Golberg, Tarjan 1988): Push-relabel, Preflow-push

Invariant P (on a path to the source)

If $f^{\Delta}(v) \geq 0$ then there exists augmenting path from v to s.

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Let v be a vertex, $f^{\Delta}(v) \geq 0$.

$$A = \{u \in V : \text{there exists an agumenting path } v \to s\}$$

$$\sum_{u \in A} f^{\Delta}(u) = \sum_{(b,a) \in E(V \setminus A, A)} f(b,a) - \sum_{(b,a) \in E(A, V \setminus A)} f(a,b)$$

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$$\sum_{(b,a) \in E(V \setminus A, A)} f(b,a) = 0$$

$$\sum_{(b,a) \in E(A, V \setminus A)} f(a,b) \geq 0$$

So overall sum is negative: there thus must be a vertex with negative surplus (the source) □

Goldberg algorithm (Golberg, Tarjan 1988): Push-relabel, Preflow-push

Invariant H (on height)

For every vertex v it holds that $h(v) \leq 2n$.

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Assume that we increase $h(v)$ to $2n + 1$. This means that v has positive surplus. By Invariant P there exists path to source and by Invariant G every edge decrease heigh by at most 1. The path has at most $n - 1$ edges. □

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Lemma

Height is increased at most $2|V|^2$ *times.*

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Definition

We say that Push operation is full if the surplus decreased to 0. (In the opposite case it is partial and the reserve decreases to 0).

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Between each two partial Push operations $h(u)$ increases by 2. By Invariant H we thus get at most $|V|$ partial push operations for every edge. $\square$

Goldberg algorithm (Golberg, Tarjan 1988): Push-relabel, Preflow-push

Lemma

There are at most $O(|V|^2|E|)$ full Push operations.

Proof.

$$\Phi = \sum_{v \in V \setminus \{s,t\}, f^{\Delta}(v) > 0} h(v)$$

1. At beginning $\Phi = 0$.

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□

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Implementation notes:

1. Keep list P of vertices with positive surplus. During Push update P in constant time. (Every vertex can have pointer to its position in list P.)
2. For every vertex we keep list $L(u)$ containing all edges with positive reserve and that goes “down”. Again during Push updated lists in a constant time.
3. Initialization of algorithm is $O(|E|)$.
4. Choosing vertex $O(|1|)$.
5. Push $O(|1|)$.
6. Increasing of $h(u)$ in $O(|V|)$: we need to update lists $L(u)$ and lists in all neighboring vertices.

Theorem

Goldberg (PushRelabel) algorithm will find maximum flow in time $O(|V|^2|E|)$.

The runtime can be improved by always choosing a vertex with maximum height. In this case it will run in $O(|V|^2\sqrt{|E|})$.

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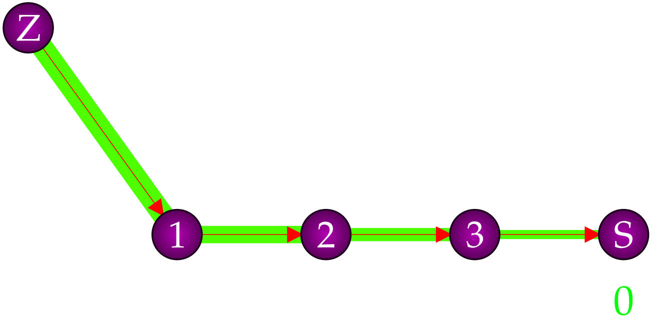
Goldberg with Max Height Rule



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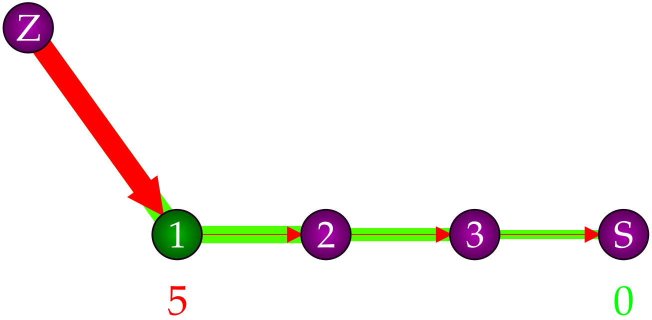

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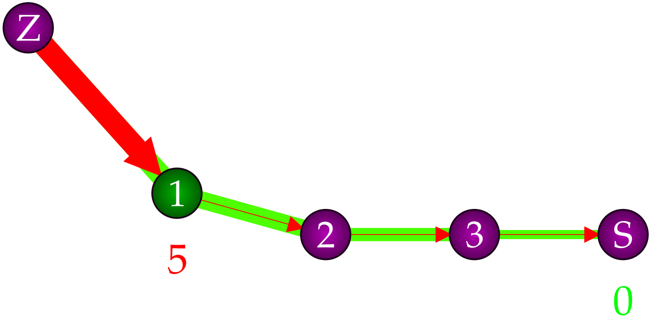

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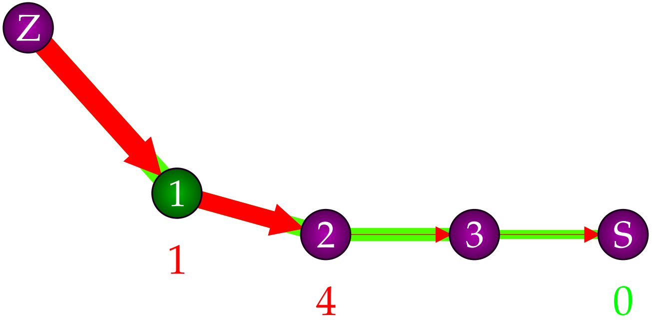

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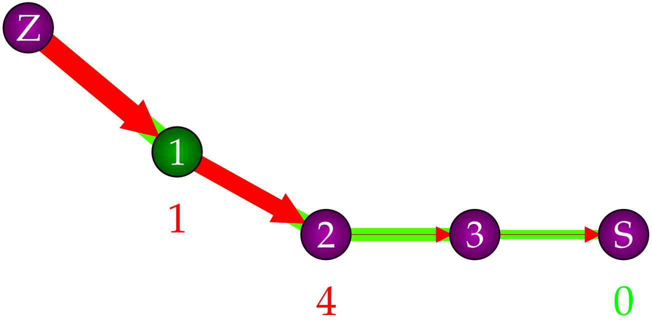

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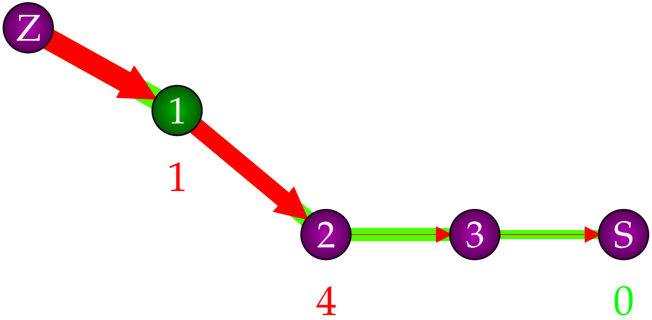

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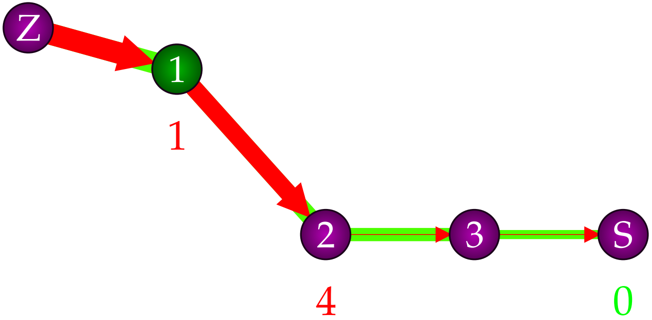

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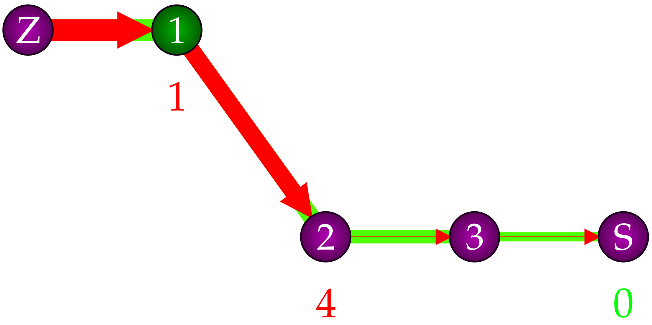

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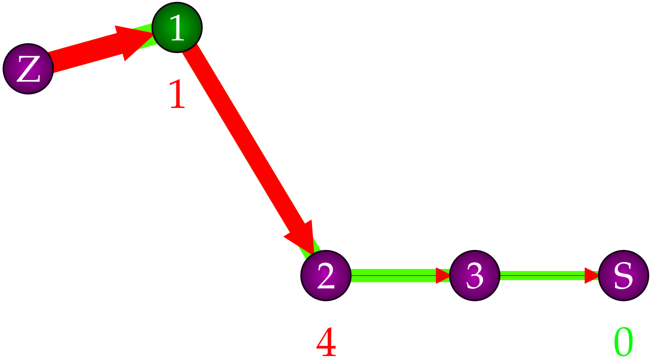

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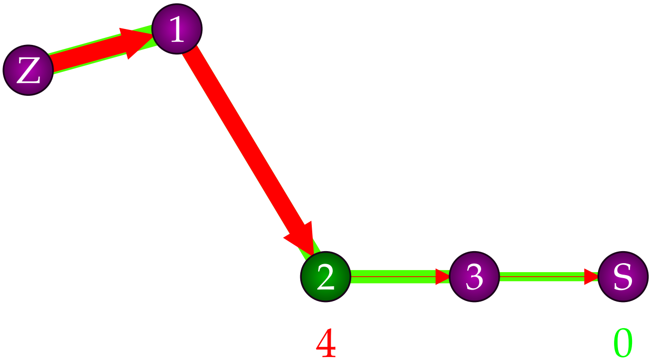

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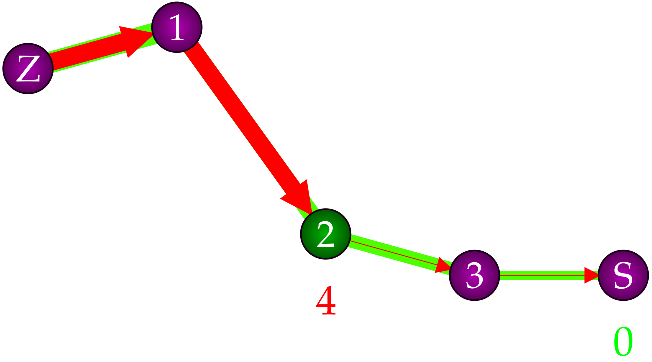

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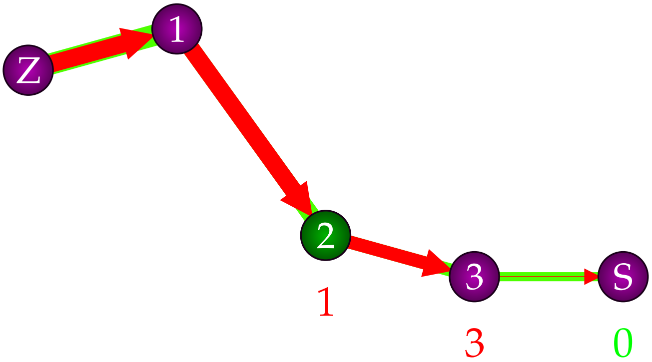

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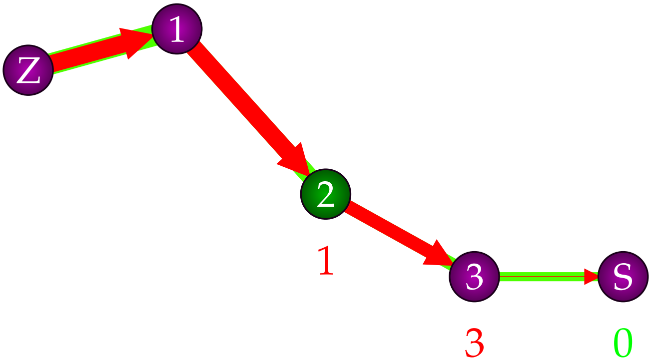

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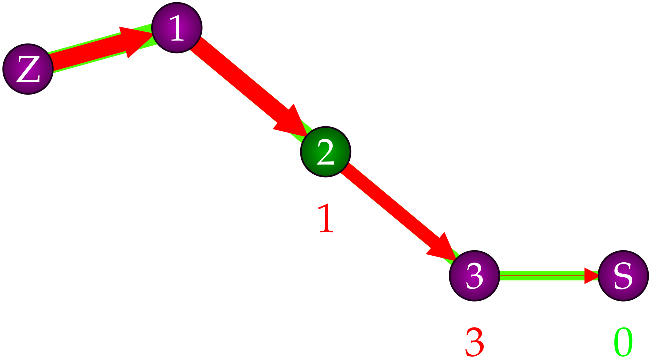

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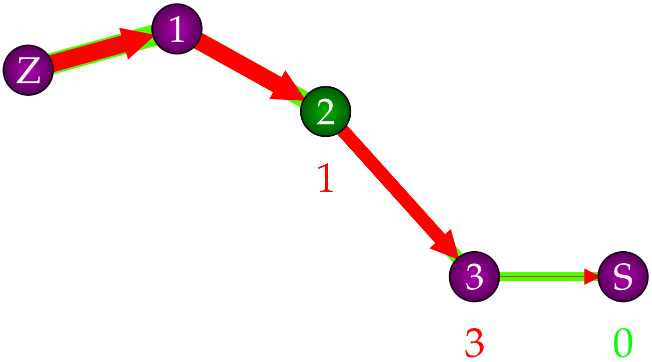

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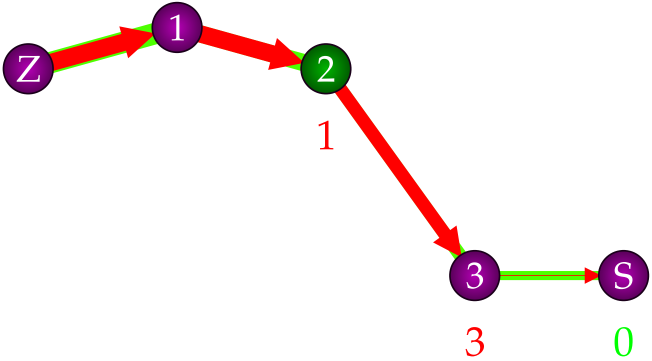

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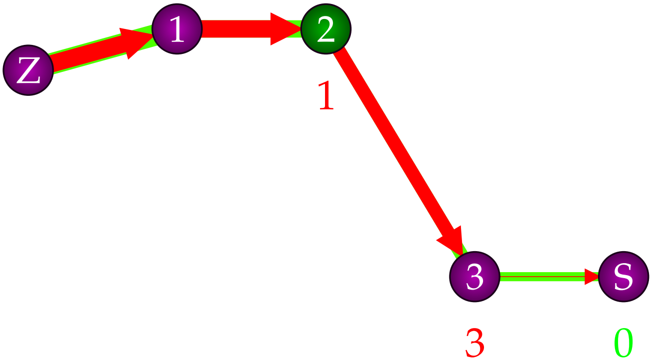

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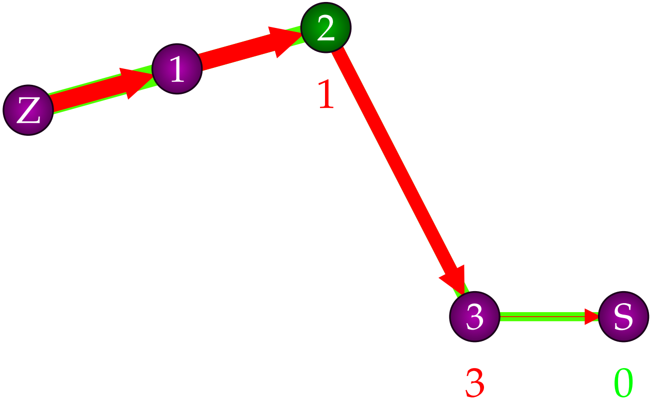

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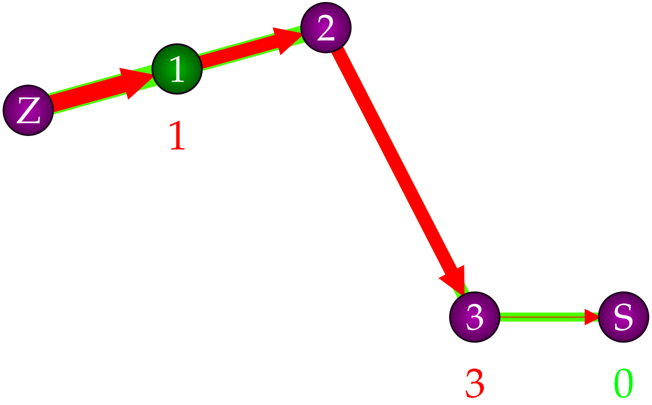

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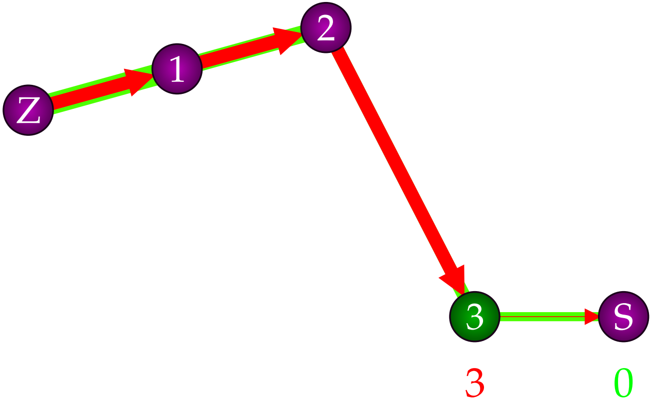

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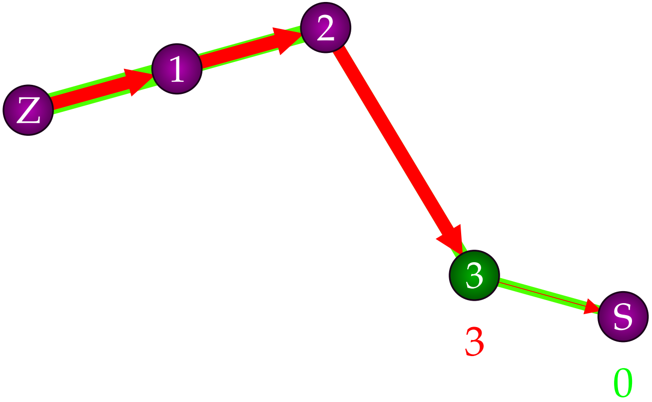

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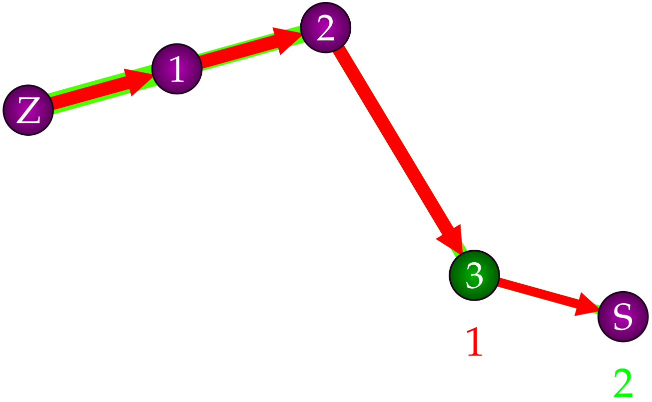

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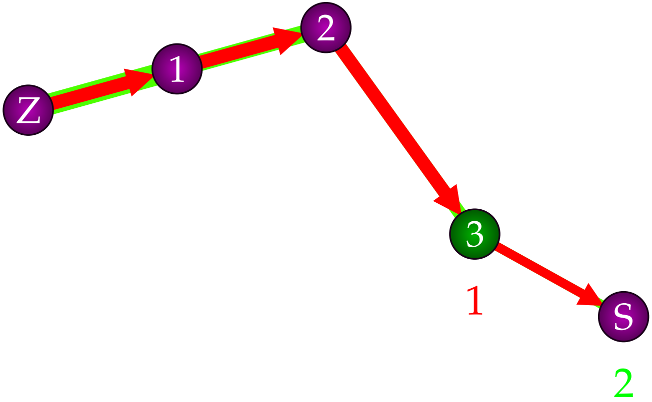

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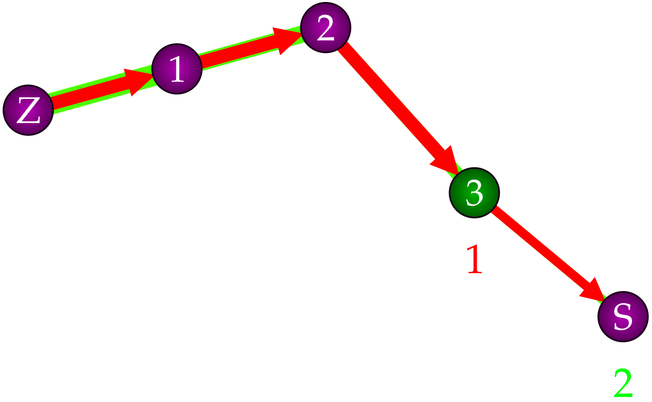

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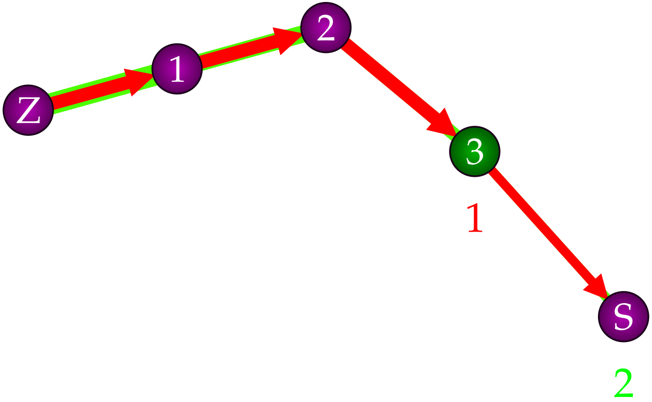

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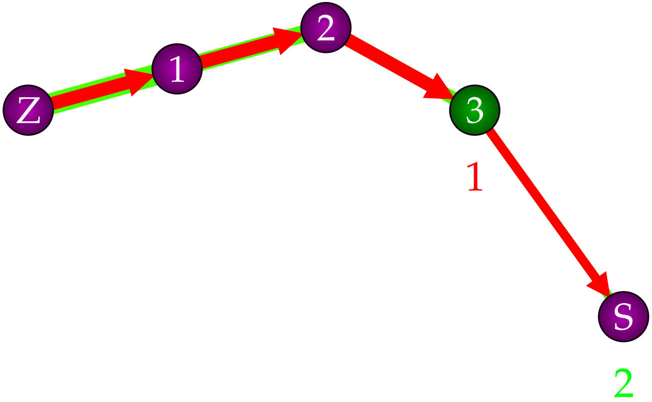

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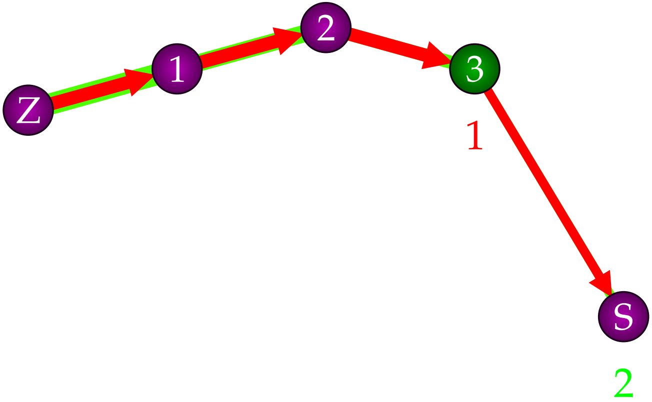

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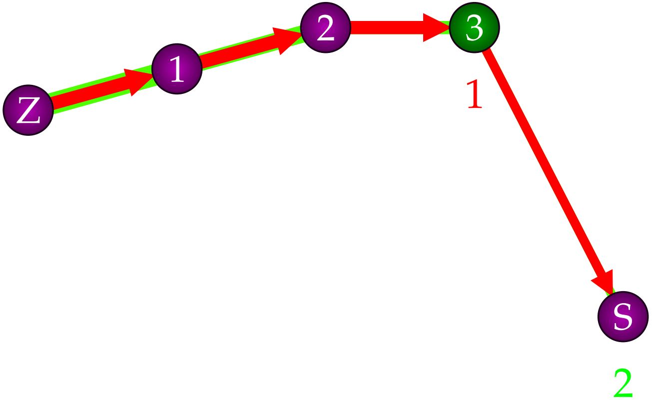

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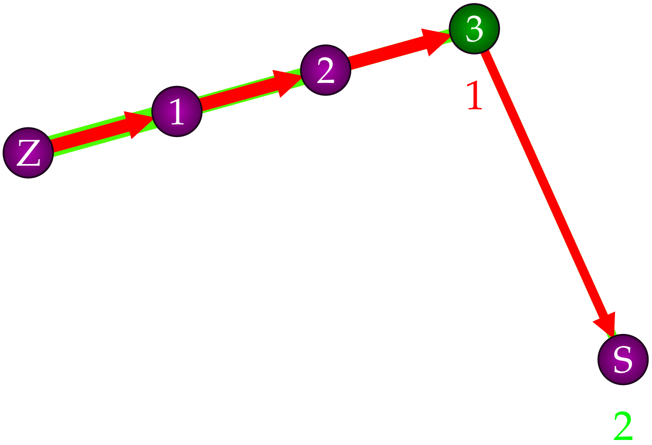

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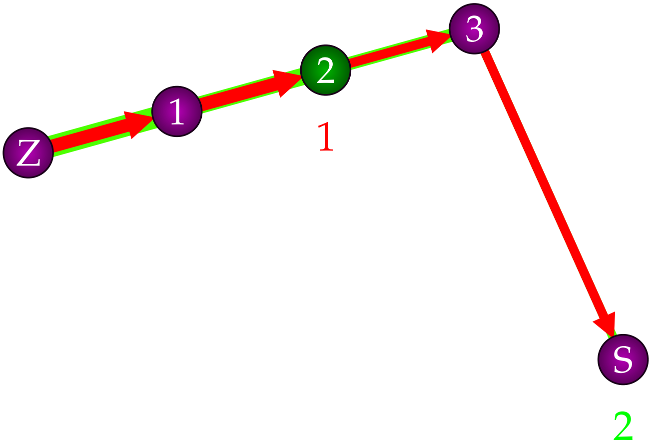

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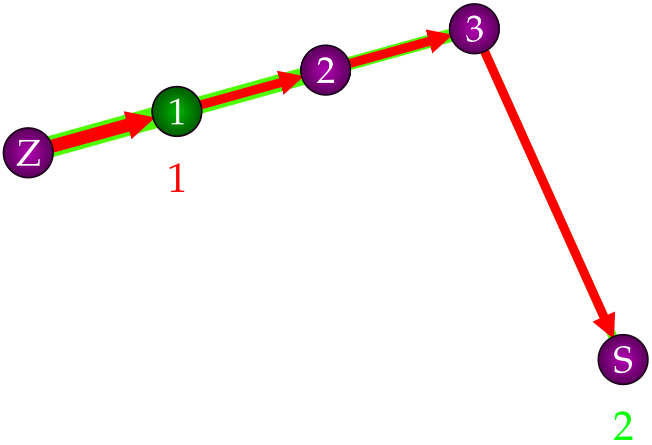

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