Algorithms and datastructures II

Lecture 10: NP-completeness

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Decision problems

Definition

A (decision) problem if a function from $\{0,1\}^*$ (the set of all possible inputs) to $\{0,1\}$.

Definition (Reduction)

Given problems A and B, we say that A is (polynomial time) reducible to B (and write $A \longrightarrow B$) if there exists function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for every $x \in \{0,1\}^*$ it holds $A(x) = B(f(x))$ and f can be computed in polynomial time relative to $|x|$. Function f is also called (polynomial time) reduction.

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Definition (P)

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$L \in P$ if and only if there exists algorithm A and polynomial f such that for every input x running $A(x)$ will finish in time at most $f(|x|)$ and $A(x) = L(x)$.

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$P = NP$ is open since 1970's.

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Problem L is NP-hard if every problem from NP can be reduced to L.

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3. Numerical problems: Finding subset of a given sum, Knapsack, $Ax = 1$, . . .

Stephen Cook, Leonid Levin

Cook-Levin theorem, 1971

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3. We thus need computer with memory of size $O(T)$. This can be represented by some boolean circuit.
4. Time of the computation can be done by using T copies of the circuit connected sequentially.

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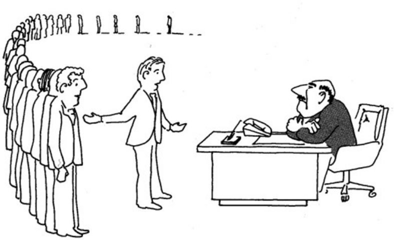
4. AND gate corresponds to CNF formula: $(z \vee \neg x \vee \neg y) \wedge (\neg z \vee x) \wedge (\neg z \vee y)$.
5. Combining formulas for all gates together leads to an input to 3-SAT.

□

Proof of Cook-Levin thoerem.

We have shown that CircruitSAT is *NP*-complete and then we gave reduction to 3-SAT. We also know that 3-SAT is in NP and equivalent to SAT. □

What to do with hard problems?



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IndSetInForest (T with root v, boolean array M indexed by vertices)

1. $M[v] \leftarrow$true.
2. If v is a leaf: Return.
3. For each son w of v:
4. $M \leftarrow$ IndSetInForest (subtree of T with root w, M).
5. If $m[w] =$ true: $M[v] \leftarrow$ false.
6. Return M.

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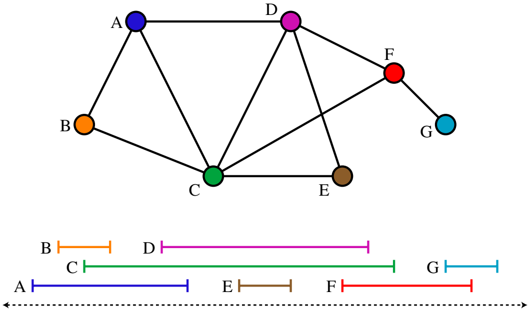

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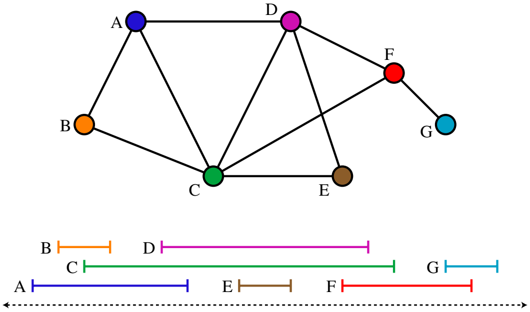

IntervalGraphColoring (intervals $[x_1, y_1] \ldots [x_n, y_n]$)

$b \leftarrow 0$.
$B \leftarrow \emptyset$.
Sort set $\{x_1, y_1, \ldots, x_n, y_n\}$.
For $\{x_1, y_1, \ldots, x_n, y_n\}$ in increasing order:
    If we process some x_i:
        If $B \neq \emptyset$: Remove color from B; store it to c_i
        else: $b \leftarrow b + 1$, $c_i \leftarrow b$
    If we process some y_i:
        Add c_i to B.
Return coloring $c_1, \ldots, c_n$.

Knapsack problem

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Given set of n objects with weights $w_1, \ldots, w_n$, costs $c_1, \ldots, c_n$ and maximum weight W your knapsack can carry. Find subset $P \subseteq \{1, 2, \ldots, n\}$ such that $w(P) = \sum_{i \in P} w_i$ is at most W and the cost $c(P) = \sum_{i \in P} c_i$ is maximum possible.

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3. Once A_n is determined we know for every possible cost the subset P of that cost minimizing the weight. It remains to find maximal c such that $A_n(c) \leq W$
4. To determine the set P one can remember how the values $A_k(c)$ was determined.

(This is pseudopolynomial algorithm).