Algorithms and datastructures I
Lecture 9: RB-trees and hashing

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Set datastructure

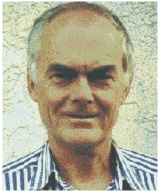
We would like to represent a set (or a dictionary) of some elements from an universe.
We expect that elements of the universum in set can be assigned and compared in $O(1)$.

- INSERT(v): Insert v to the set.
- DELETE(v): Delete v from the set.
- FIND(v): Find v in the set.
- MIN: Return minimum.
- MAX: Return maximum.
- SUCC(v): Find successor.
- PRED(v): Find predecessor.

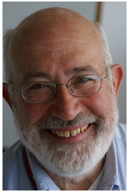
Basic implementations

	INSERT	DELETE	FIND	MIN/MAX	SUCC/PRED
Linked list	$O(n)$ or $O(1)$	$O(n)$ or $O(1)$	$O(n)$	$O(n)$	$O(n)$
Array	$O(n)$ or $O(1)$	$O(n)$ or $O(1)$	$O(n)$	$O(n)$	$O(n)$
Sorted array	$O(n)$	$O(n)$	$O(\log n)$	$O(1)$	$O(\log n)$ or $O(1)$
binary search trees	$O(n)$	$O(n)$	$O(n)$	$O(n)$	$O(n)$
AVL-trees	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$
(r, b)-trees	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(\log n)$

(a, b)-trees (Bayer, McCreight)



Rudolf Bayer



Edward M. McCreight

Definition (Generalized search tree)

Generalised search tree is a rooted tree with specified order of sons and two types of vertices:

1. Internal vertices contains non-zero number of keys. If internal vertex has keys $x_1 < \cdots < x_k$ then it has $k + 1$ sons $s_0, \ldots, s_k$. Keys separate values in sons, so: $T(s_0) < x_1 < T(s_1) < x_2 < \cdots < x_{k-1} < T(s_{k-1}) < x_k < T(s_k)$
2. External vertices contain no keys and are leaf.

Definition ((a, b)-tree)

(a, b)-tree for a given $a \geq 2$, $b \geq 2a - 1$ is a generalised search tree such that:

1. Root has 2 to b sons.
2. Other internal vertices have a to b sons.
3. All external vertices are in the level.

Lemma

Every (a, b)-tree with n keys has depth $\Theta(\log n)$.

Insert to (a, b)-tree

Insert(v,x)

Let u be the last internal vertex visited by Find(v,x).

1. If u contains x return.
2. Otherwise add x into u and insert new external vertex
3. If u has more than b sons, split it possibly recursing to father.

It is possible to split preventively if $b \geq 2a$. We will use it today.

Red-black trees (Bayer 1972; Guibas, Sedgewick 1978; Anderson 1993; Sedgewick 2008)

We can represent $(2, 4)$-trees using binary search trees with colored edges.

Leonidas J. Guibas

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Definition (Left leaning red-back tree)

LLRB-tree is binary search tree with external vertices and edges colored either red or black. It satisfies:

1. There are no two red edges adjacent to each other.
2. If there is only one red edge from a vertex then it is left.
3. Edges to leaves are always black.
4. Every path from root to leaf goes through the same number of red edges.

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Optimization: Color of edge may be stored in its destination vertex.

Depth of LLRB-trees

Lemma

Every LLRB-tree with n keys has depth $\Theta(\log n)$.

Proof.

We know that every LLRB-tree tree corresponds to an $(2, 4)$-tree of height $h = \Theta(\log n)$.
The height h' of LLRB tree is $h \leq h' \leq 2h$. □

Operations on LLRB-trees

Observation

Operations FIND, MIN, MAX, SUCC and PRED run in $\Theta(\log n)$.

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Operations INSERT and DELETE can be derived from ones on $(2, 4)$-trees.

INSERT to an LLRB-tree

Lets see how insertion to $(2, 4)$-tree with preventive splitting translates to RB-tree.

Insert(v,x)

1. If $v = \emptyset$: return newly created red vertex with key x.
2. If $x = k(v)$: Return v.

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Theorem

Operations INSERT, DELETE, FIND, MIN, MAX, SUCC *and* PRED *on LLRB-tree runs in* $\Theta(\log n)$ *time.*


And Now For Something
Completely Different

Tries

Let Σ be a fixed alphabet. Let $S \subseteq \Sigma^*$ be a set of words over alphabet Σ.

Definition (Trie: middle of retrieval, invented by René de la Briandais in 1959; named by Edward Frenklin)

Trie for some set of words S is a rooted tree where

1. vertices are all prefixes of words $W \in X$, and
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To store sets of integers one can see integers as words in some fixed base. Result is known as a radix tree.

Amortised complexity

Insertion to a (dynamically allocated) growing array.

Insert((A, s, n),x) insert element x to array A of size s containing n elements

1. if $n = s$:
2. Allocate array A' of size $2s$.
3. For $i = 0, 1, \ldots, n-1$: $A'[i] \leftarrow A[i]$.
4. Free A.
5. $A \leftarrow A'$, $s \leftarrow 2s$
6. $A[n] \leftarrow x$, $n \leftarrow n + 1$
7. Return (A, s, n).

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Theorem

Performing n operations INSERT *starting from the empty array will run in time $\Theta(n)$.*

Proof.

To insert 2^i elements one needs $2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{i-1} = 2^i - 1$ copy operations. □

Hash functions

Hash function is a function h from universe $\mathcal{U}$ to set $\mathcal{P} = \{0, 1, \ldots, p-1\}$ (of hashes).

Hash table with separate chaining for set $S \subseteq U$ with hash function $h \colon \mathcal{U} \to \mathcal{P}$.

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Corollary

Putting $p \sim |S|$ we get FIND, INSERT and DELETE is running on average approximately in $O(1)$.

Hash functions

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$$h(x) = \left(\sum_{i=1}^{|x|} x_i a^{|x|-i} \mod p\right).$$

Can be effectively computed as (Horner's method):

$$\begin{aligned}
h_1 &= x_1\\
h_2 &= (h_1 a + x_2) \mod p\\
h_3 &= (h_2 a + x_3) \mod p\\
\ldots &\quad \ldots\\
h_{|x|} &= (h_{|x|-1} a + x_{|x|}) \mod p
\end{aligned}$$

Open addressing

An alternative way of solving collisions is to use hash function $h(x, i)$ such that for every $x \in \mathcal{U}$ sequence $h(x, 0), h(x, 1), \ldots, h(x, m-1)$ is a permutation of $(0, 1, \ldots, m-1)$.

Insert(x)

1. For $i = 0, \ldots, m-1$:
2. $\quad j \leftarrow h(x, i)$
3. $\quad$ If $H[j] = \emptyset$: put $H[j] \leftarrow x$ and return.
4. Report that table is full.

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We can not remove values from the table, just mark them as removed.

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Definition (c-universal system of hash functions)

System $\mathcal{S}$ of hash functions from universe $\mathcal{U}$ to $\{0, 1, \ldots, p-1\}$ is c-universal for given $c \geq 1$ if for every $x, y \in \mathcal{U}$, $x \neq y$

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System of functions $\mathcal{S}\colon \mathbb{Z}_p^d \to \{0, 1, \ldots, p-1\}$

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$\sum_{i=1}^{d-1} a_i z_i + a_d z_d \equiv 0$ happens only if $\sum_{i=1}^{d-1} a_i z_i \equiv -a_d z_d$. This has probability $\frac{1}{p}$. □