Algorithms and datastructures I
Lecture 12: k-th smallest element; sorting

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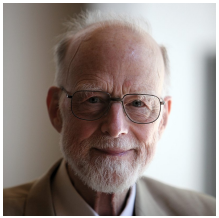
QuickSelect

Problem: Find k-th smallest element of a sequence $(x_1, \ldots, x_n)$.



Sir Tony Horae

QuickSelect($(x_1, \ldots, x_n)$, k), Sir Tony Horae 1961

1. Choose pivot p.
2. Split $(x_1, \ldots, x_n)$ to $L = \{x_i : x_i < p\}$, $E = \{x_i : x_i = p\}$, $R = \{x_i : x_i > p\}$.
3. If $k \leq |L|$: return QuickSelect(L, k).
4. If $k \leq |L| + |E|$: return x_k.
5. Return QuickSelect (L, $k - |L| - |E|$).

1. Assume that p is always maximum.
   Time complexity: $\Theta(N^2)$.
2. Assume that p is median.
   Time complexity: $T(N) = T\left(\frac{N}{2}\right) + \Theta(N) = \Theta(N)$.
3. Assume that p is "almost median" (at least $\frac{1}{4}$ of elements is smaller than p and $\frac{1}{4}$ is bigger than p).
   Time complexity: $T(N) = T\left(\frac{3N}{2}\right) + \Theta(N) = \Theta(N)$.
4. Randomized choice of almost median: Pr[random element is almost median] $\geq \frac{1}{2}$

Lemma

Let V be an event that occurs in trial with probability p.
The expected number of trials to first occurrence of V is $\frac{1}{p}$.

Expected number of trials is 2; time complexity $\Theta(N)$.

5. Random choice of pivot.
   Time complexity: $\Theta(N)$.

Median of medians algorithm

Select$((x_1, \ldots, x_n), k)$, Blum, Floyd, Pratt, Rivest, Tarjan 1973

1. If $n \leq 10$ use brute-force.
2. Divide $x_1, \ldots x_n$ to 5-tuples $P_1, \ldots P_t$, $t = \lfloor \frac{n}{5} \rfloor$.
3. For $i = 1, 2, \ldots t$: $m_i \leftarrow$Median(P_i).
4. $p \leftarrow$Select$((m_1, \ldots, m_t), \lfloor \frac{t}{2} \rfloor)$

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4. $p \leftarrow$Select($(m_1, \ldots, m_t), \lfloor \frac{t}{2} \rfloor$)
5. Split $(x_1, \ldots, x_n)$ to $L = \{x_i : x_i < p\}$, $E = \{x_i : x_i = p\}$, $R = \{x_i : x_i > p\}$.
6. If $k \leq |L|$: return Select(L,k).
7. If $k \leq |L| + |E|$: return k.
8. Return Select(L, $k - |L| - |E|$).

Median of medians algorithm

$$T(n) = T(\frac{n}{5}) + T(\frac{7}{10}n) + \Theta(n)$$

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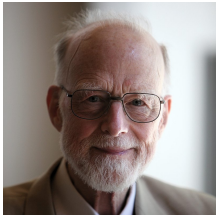
Deterministic runtime: $\Theta(n)$!

QuickSort



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1. If $n = O(1)$: use special purpose sorting algorithm.
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4. $L' \leftarrow$QuickSort (L).
5. $R' \leftarrow$QuickSort (R).
6. Return L', E, R'.

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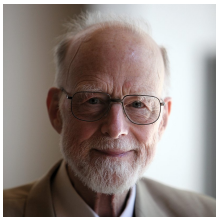

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2. If p is min: $T(n) = \Theta(n^2)$
3. If p is chosen randomly: $T(n) = \Theta(n \log n)$ (proof is coming)

Random choice of pivot

Theorem

Expected number of comparisons in QuickSort with random choice of median is $\Theta(n \log n)$.

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$$= \sum_{1 \le d \le n} \frac{2(n-d+1)}{d} \le \sum_{1 \le d \le n} \frac{2n}{d}$$

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$$= \sum_{1 \le d \le n} \frac{2(n-d+1)}{d} \le \sum_{1 \le d \le n} \frac{2n}{d} = \Theta(n \log n).$$

Lower bound on sorting

Definition (Comparsion model for sorting)

1. Elements being sorted can only be compared and assigned.
2. All decisions of the algorithm are deterministic

Theorem

Every sorting algorithm in the comparison model has time complexity $\Omega(n \log n)$.

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$$h \geq \log_2(n!) \geq \frac{1}{2} n \log n$$

Counting sort

We want to sort set of integer in range $\{1, 2, \ldots, r\}$ where r is relatively small.

CountingSort $((x_1, x_2, \ldots x_n), r)$

1. For $i = 1, \ldots, r$: $c_i \leftarrow 0$.
2. For $i = 1, \ldots, n$: $c_{x_i} \leftarrow c_{x_i} + 1$.
3. $j \leftarrow 1$.
4. For $i = 1, \ldots, r$:
5. Repeat c_i times: $y_j \leftarrow i, j \leftarrow j + 1$.
6. Return $y_1, \ldots, y_n$.

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6. Return $y_1, \ldots, y_n$.

Time complexity: $\Theta(n + r)$.

Bucket sort

Consider we want to sort records (k_i, d_i) where k_i is the key and d_i is some additional data.

BucketSort $(((k_1, d_1), \ldots, (k_n, d_n)), r)$

1. For $i = 1, \ldots, r$: $B_i \leftarrow$empty linked list.
2. For $i = 1, \ldots, n$: Append (k_i, d_i) to B_{k_i}
3. $L \leftarrow$ empty linked list.
4. For $i = 1, \ldots, r$: Append B_i to L.
5. Return L.

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Time complexity: $\Theta(n + r)$.
BucketSort is stable: relative order of elements with same keys is not changed.

Lexicographic BucketSort

LexBucketSort $((x_1, \ldots, x_n), k, r)$ where $x_i \in \{1, \ldots, r\}^k$

for every $i = 1, \ldots, n$

1. $L \leftarrow x_1, \ldots, x_n$.
2. For $i = k, k-1, \ldots, 1$:
3. Sort L using BucketSort by i-th coordinate.
4. Return L.

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After l iterations the list L is sorted lexicographically using i-th to k-th coordinate.

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Time complexity: $\Theta(k \cdot (n + r))$. Memory: $\Theta(nk + r)$.

RadixSort

Assume that our input is $x_1, x_2, \ldots x_n \in \{1, \ldots, r\}$ where r is relatively large.
Write numbers using base b and use LexBucketSort with $k = \lfloor \log_b r \rfloor + 1$.

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Putting $b = \Theta(n)$ we obtain time complexity $\Theta(\frac{\log r}{\log n} \cdot (n + b))$.

Observation

If sorted numbers are polynomially large to n, say $r \leq n^\alpha$ for some fixed α then the time complexity is linear (we get $\log r \leq \alpha \log n$).

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Excercise: adjust algorithm for sorting strings in time $\Theta(n + l)$ where l is the sum of their lengths.