WP01 – Slope number

It is well known that every planar graph has a planar embedding whose edges are non-crossing straight line segments (the so called Fáry embedding). Dujmović et al. [3] pose the question of *how many different slopes* are needed for such a representation. In particular they ask if there exists a function f such that each planar graph of maximum degree Δ has a Fáry embedding that uses at most $f(\Delta)$ slopes. Jelínek et al. [4] answer this question in affirmative for outerplanar graphs. They actually prove a stronger result by showing that every planar partial 3-tree of maximum degree Δ has a Fáry embedding which uses at most $O(2^{\Delta})$ slopes. A generalization of this result to all planar graphs was most recently proved by Keszegh et al. [5]. What remains wide open is the gap between these upper bounds and the best known lower ones. To be more explicit, let $f_{planar}(\Delta)$ denote the maximum planar slope number taken over all graphs of maximum degree Δ. The following two problems are a challenge:

Is $f_{planar}(\Delta)$ bounded from above by a function polynomial in Δ?

and

$$\text{Is } \lim_{\Delta\to\infty} \frac{f(\Delta)}{\Delta} < \infty?$$

The immediate natural question is whether the upper bound c^{Δ} could be substantially improved (possibly to a polynomial one). Other questions we want to study are exact values of $f(\Delta)$ for small Δ and tight bounds for partial 2-trees (which include outerplanar graphs). Methods of discrete geometry will be exploited in attempting these questions.

We will also address related complexity questions, in particular how difficult it is to decide if a given graph has a Fáry embedding using a small number (2, 3) of slopes.

Bounding the number of slopes proves useful not only for drawing graphs, but also for their intersection representations. For instance, it is known that recognition of intersection graphs of straight lines segments in the plane that are parallel with a collection of k directions is NP-complete for every fixed $k \geq 2$ [6, 1], while NP-membership is an open problem for intersection graphs of segments (with no restriction on the number of slopes). Similarly, the complexity of the problem of finding a largest clique in an intersection graph of segments is still open [7] (while the question is polynomial time solvable if the number of slopes is bounded). These are seemingly hard problems. We hope that recently developed techniques for drawing graphs with a small number of slopes may provide inspiration for successfully attacking them.

Very recently Chalopin and Goncalves [2] proved a very surprising result that every planar graph is the intersection graph of line segments in the plane. This remarkable result revives a question that Scheinermann asked in his thesis - is every planar graph the intersection graph of straight-line segments which are parallel with at most 4 directions only? This daring conjecture implies the celebrated Four Color Theorem, and hence one should search for a counterexample first. We will perform such a search and try to combine techniques of the so called Order Forcing Lemma developed by Kratochvíl and Matoušek [6] with recent methods of geometric graph theory and the computational power of our network.

Milestones of work package WP01: M01.1 Improve the upper bounds for planar slope number of planar graphs and/or its subclasses (partial 3-trees, outerplanar graphs, etc.).
M01.2 Determine tight upper bounds for the planar slope number of series-parallel graphs.
M01.3 Determine tight upper bounds for the planar slope number of planar graphs with small maximum degree.
M01.4 Determine the computational complexity of deciding is a planar graph has a drawing with a fixed small number of slopes.
M01.5 Attempt to generalize results on recognition of intersection graphs of segments in bounded number of slopes to larger classes of segment intersection graphs.
M01.6 Attempt to generalize results on the complexity of the CLIQUE problem for intersection graphs of segments in bounded number of slopes to larger classes of segment intersection graphs.
M01.7 Try to find (by various techniques including intelligent computer search) a planar graph that does not have an intersection representation by segments in 4 directions.

References for workpackage WP01

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