

How Many Paths Lead to a Good Beer

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Charles University, Prague, Czech Republic

(based on joint works

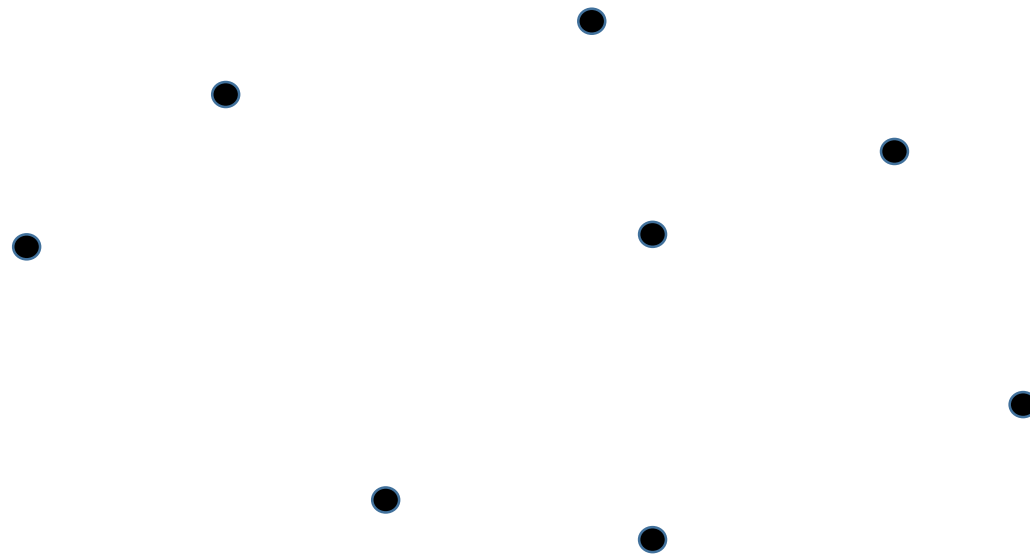
P. Kindermann, JK, G. Liotta, P. Valtr: GD 2023

T. Antič, A. Džuklevski, J. Fiala, JK, G. Liotta, M. Saghafian, M. Saumell, J. Zink: SOFSEM 2026

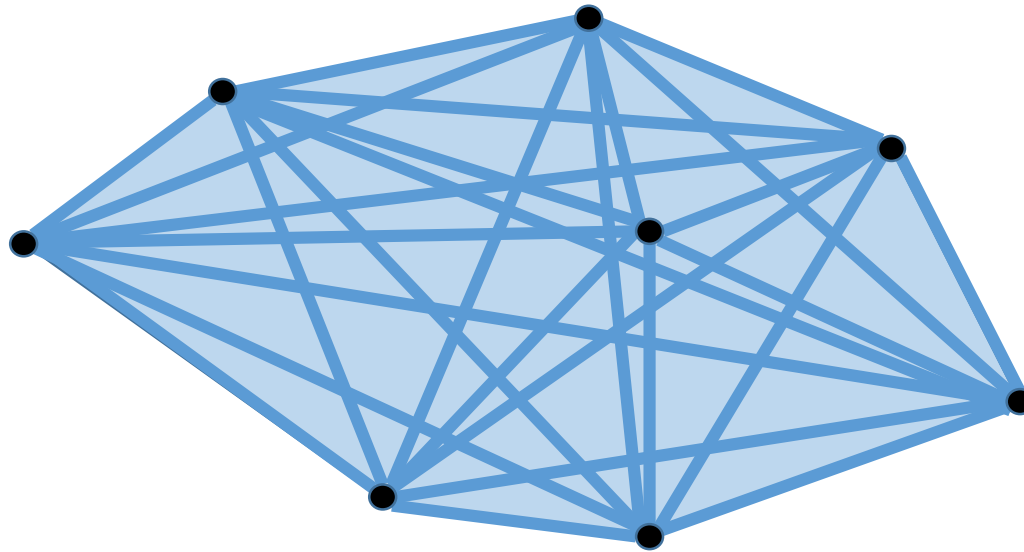
O. Aichholzer, E. Červenková, E. Katsanou, JK, D. Perz, M. Saghafian, P. Valtr: preprint)



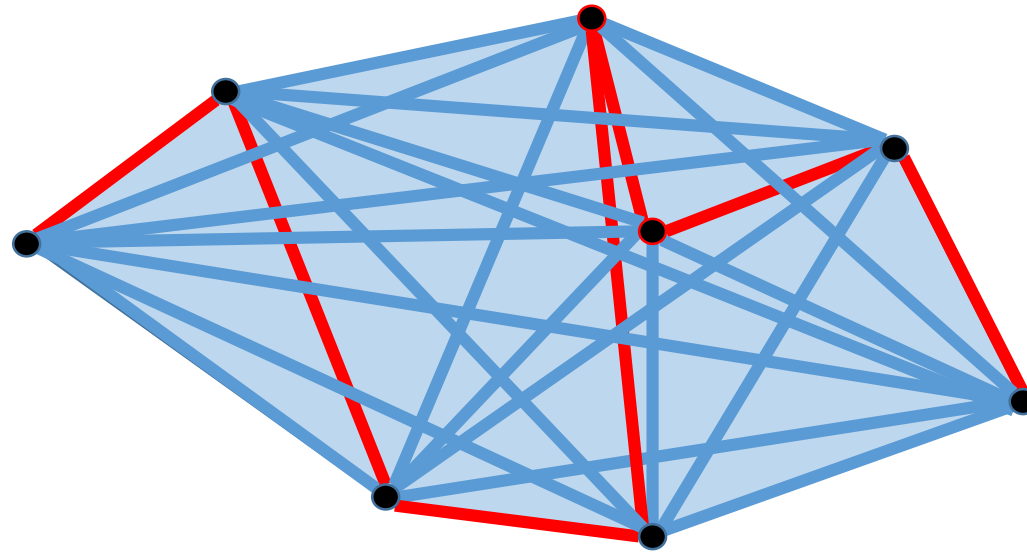
Setting: n points in the plane, no three in a line (general position)



Setting: n points in the plane, no three in a line (general position)
define a complete geometric graph (edges are straight-line segments)

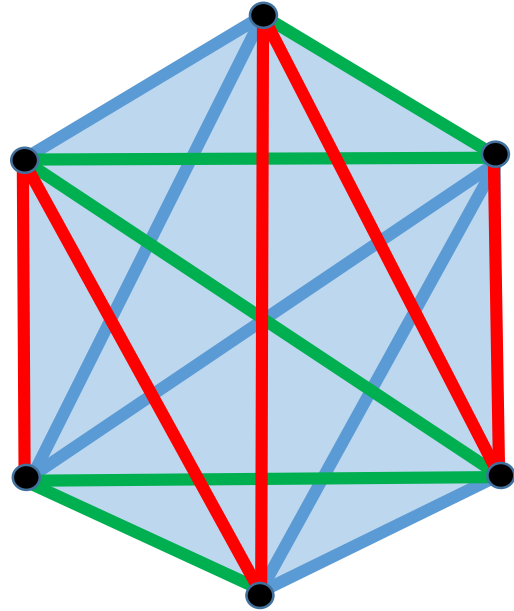


Non-crossing Hamiltonian path (plane spanning path) in the complete geometric graph determined by these points

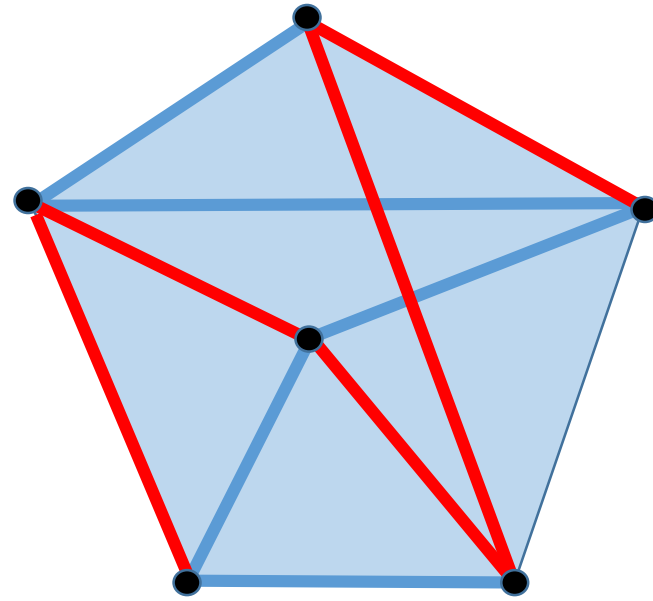
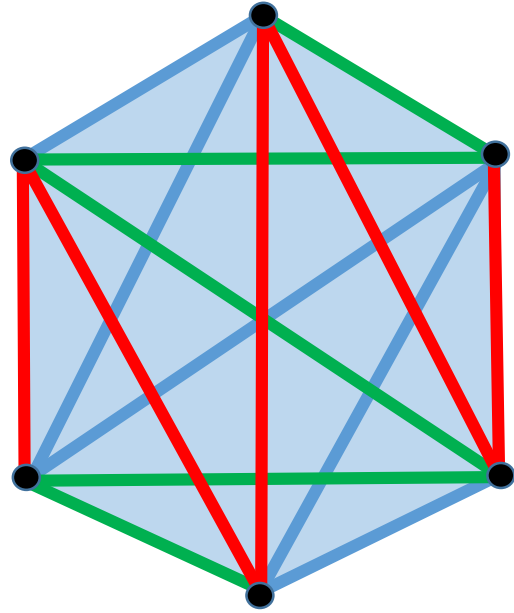


Main question: How many edge-disjoint non-crossing Hamiltonian paths can we **ALWAYS** find in **ANY** given complete geometric graph with n points?

$n = 6$



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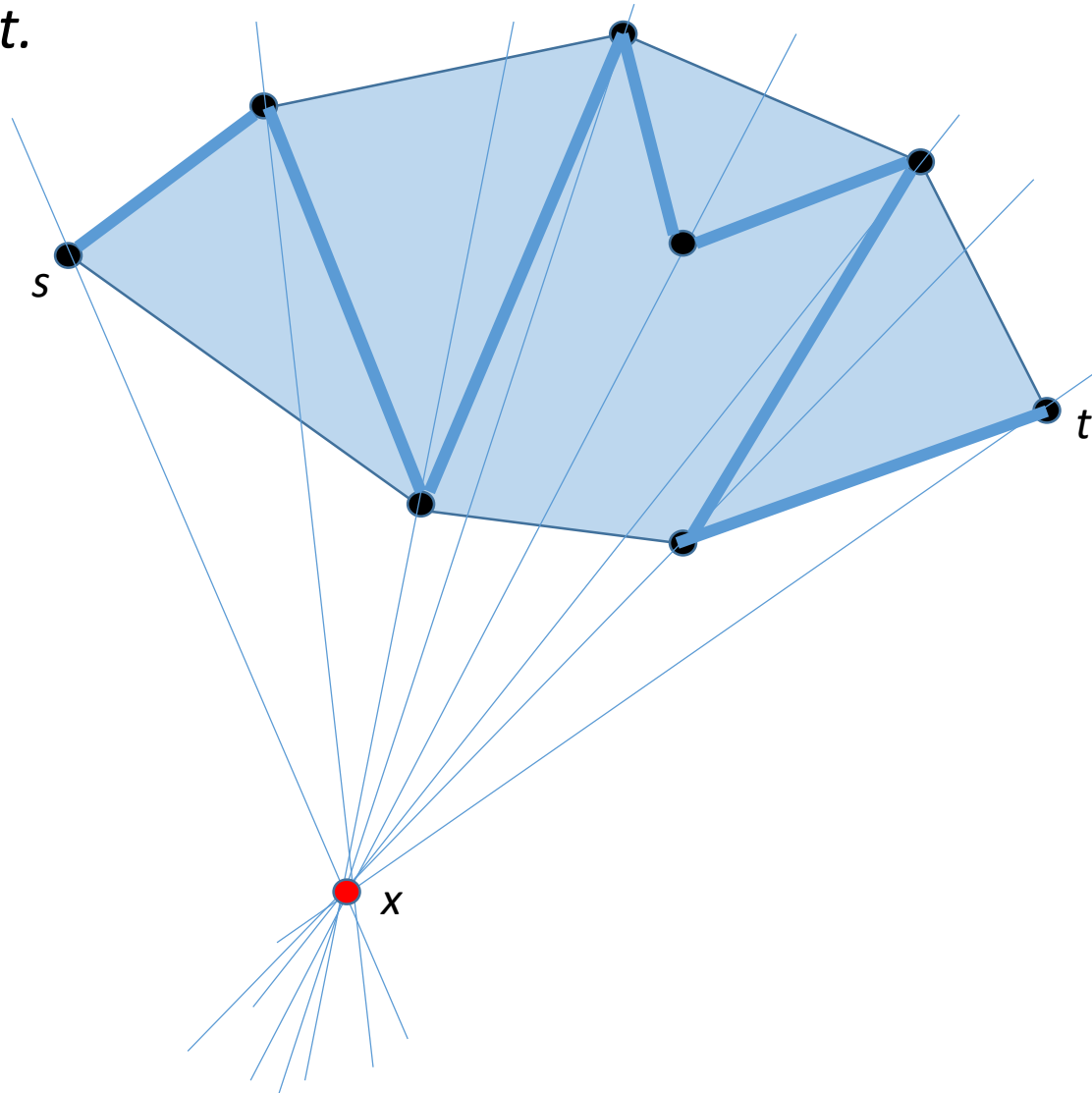
Packing edge-disjoint non-crossing Hamiltonian paths in complete geometric graphs

Known:

Folklore – 1 path

1 Path

Theorem (folklore): If s, t are distinct points of $\text{conv}(S)$, then S has a plane spanning path starting in s and ending in t .



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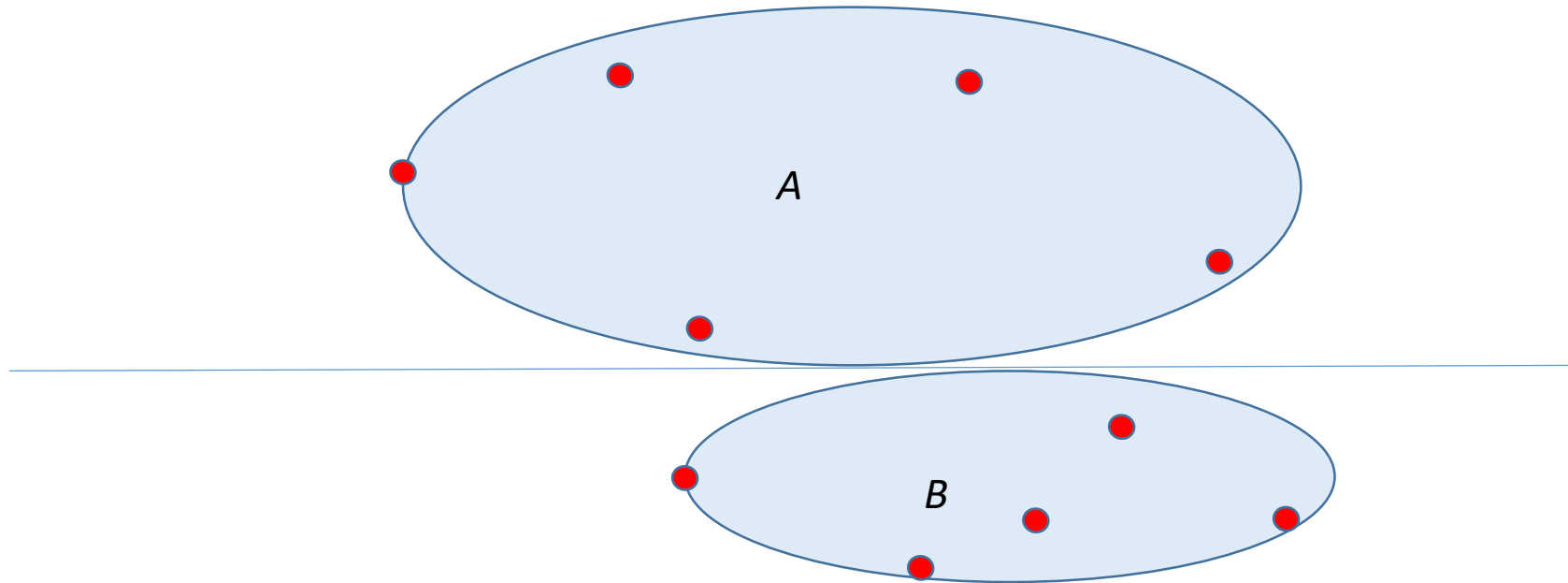
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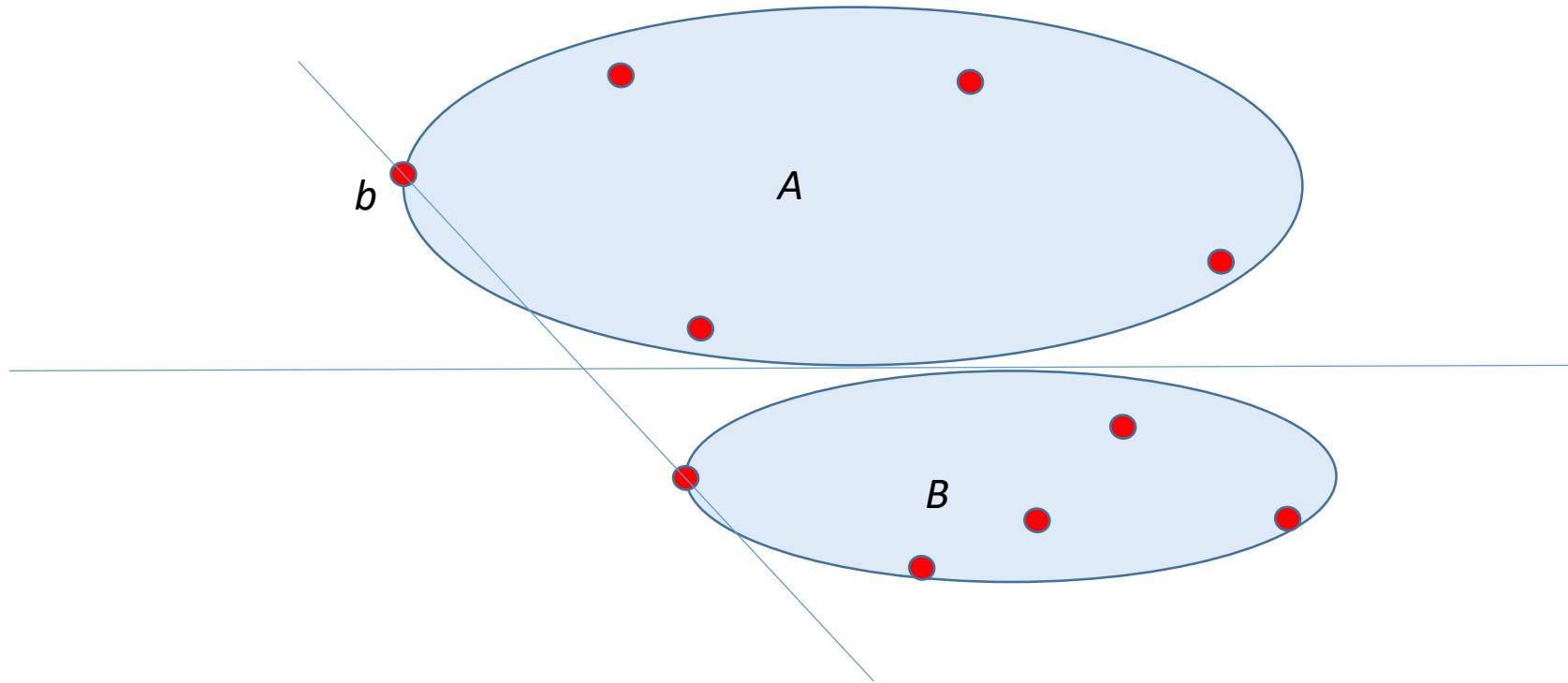
1 Path

Abellanas et al. [1999]: For every balanced separation (A,B) of S , there exists a zig-zag path starting in a bridged vertex of the larger part.



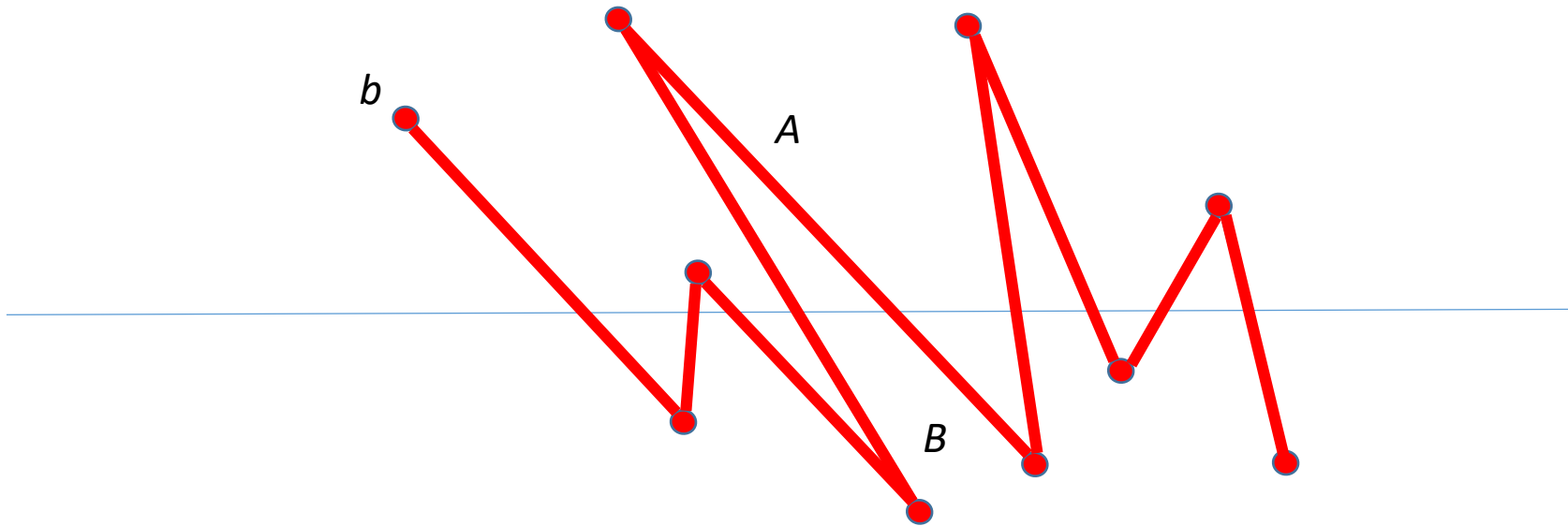
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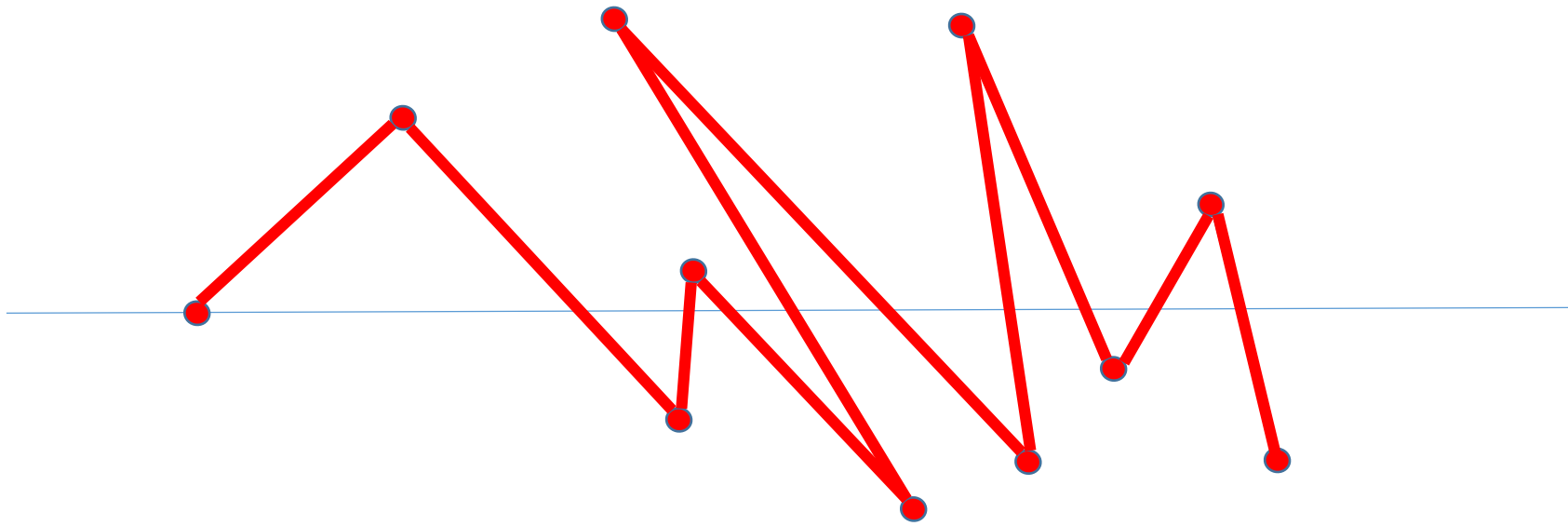
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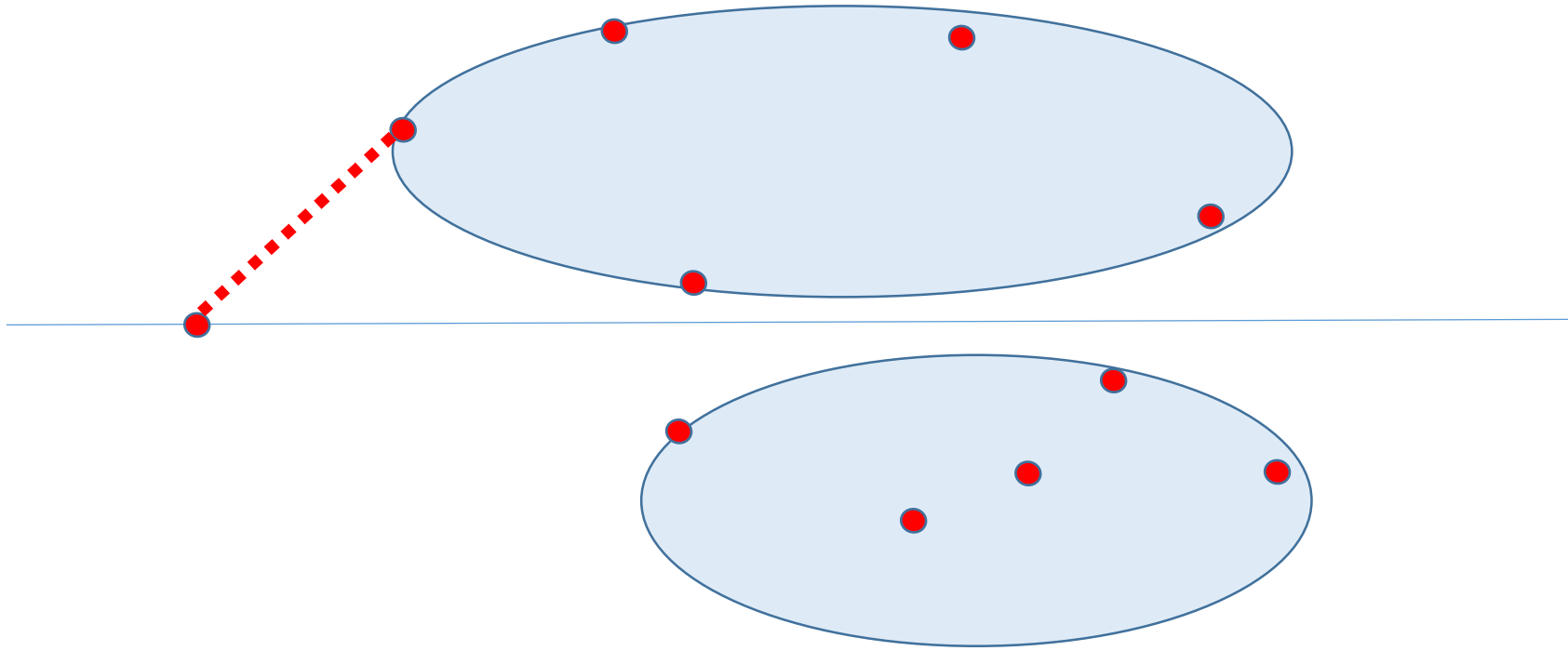
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Aichholzer et al. [2017]



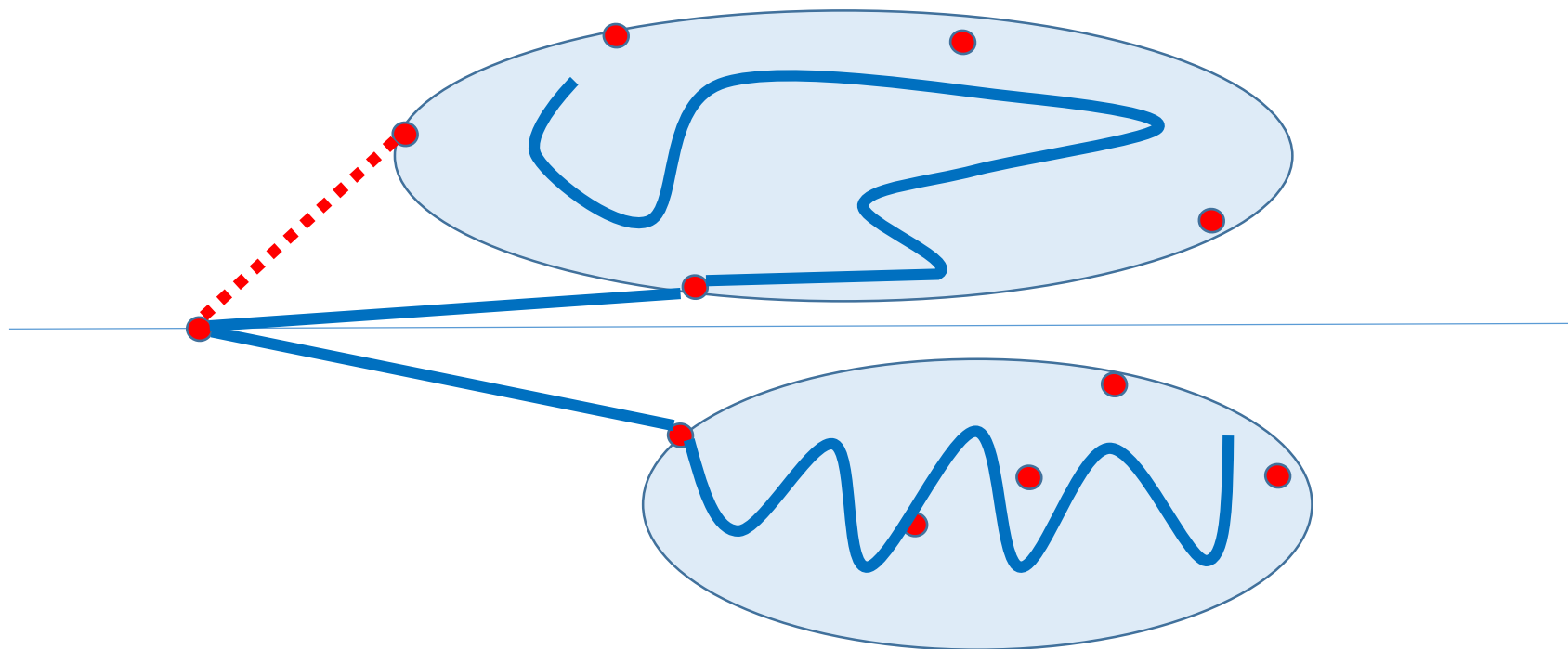
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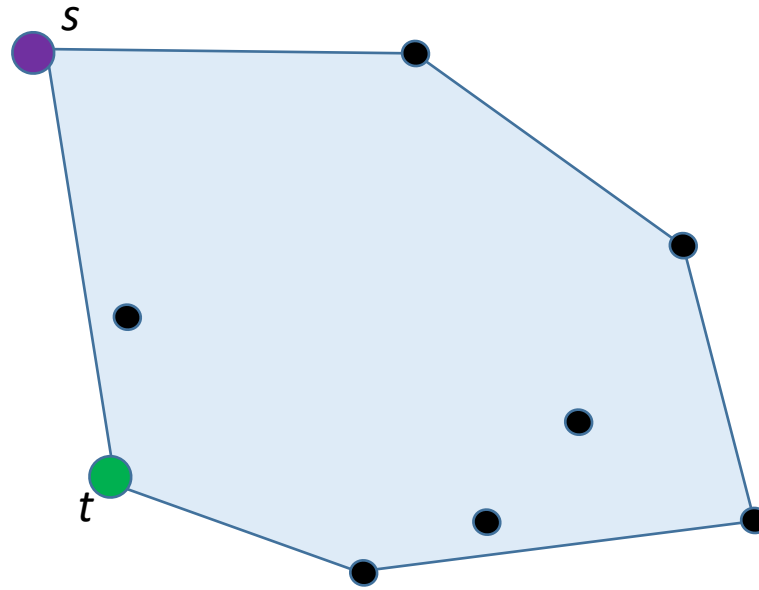
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Kindermann, Kratochvil, Liotta, Valtr [2023] – 3 paths

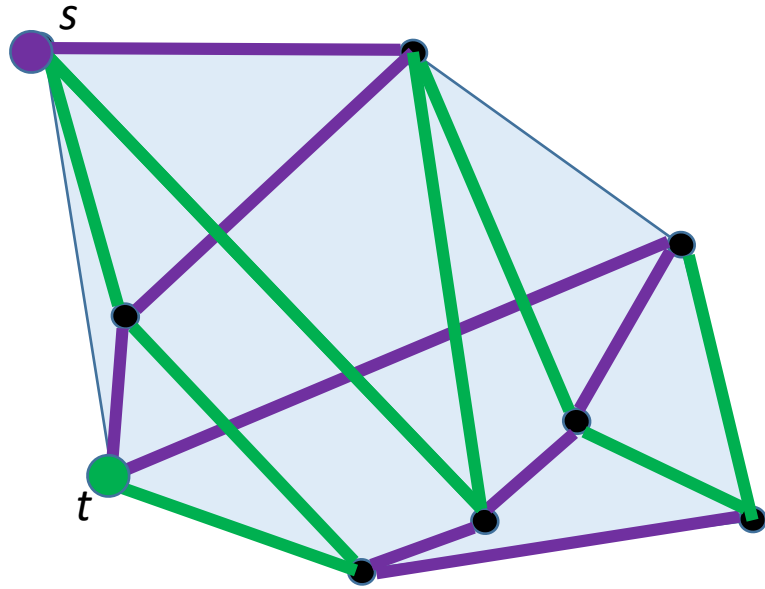
2 paths

Theorem [KKLV 23]: Let s and t be two distinct points of S lying on the boundary of $\text{conv}(S)$, and let $|S| \geq 5$. Then S admits 2 edge-disjoint plane spanning paths, one starting in s , the other one starting in t , and none of them using the edge st .



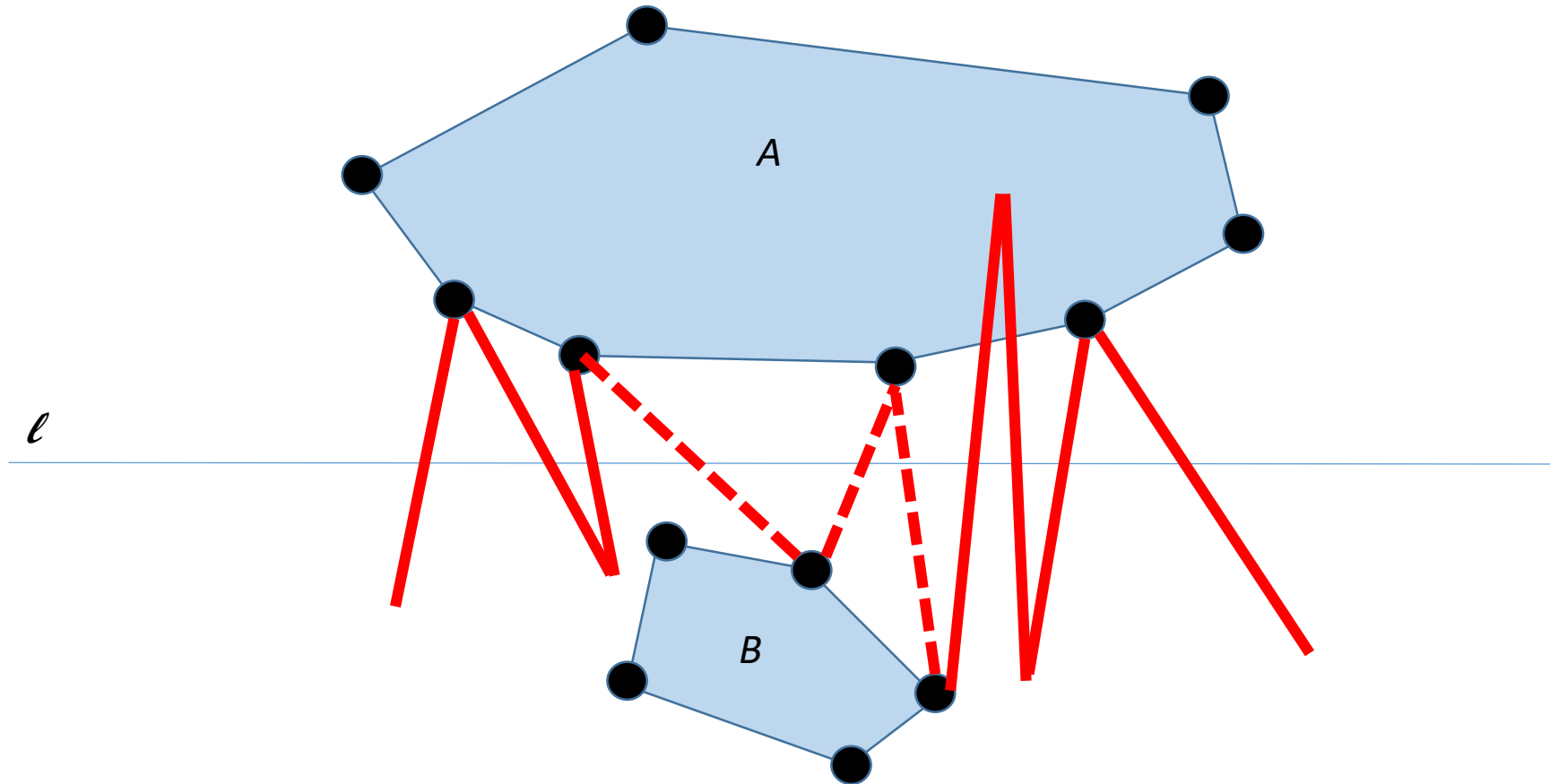
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3 Paths

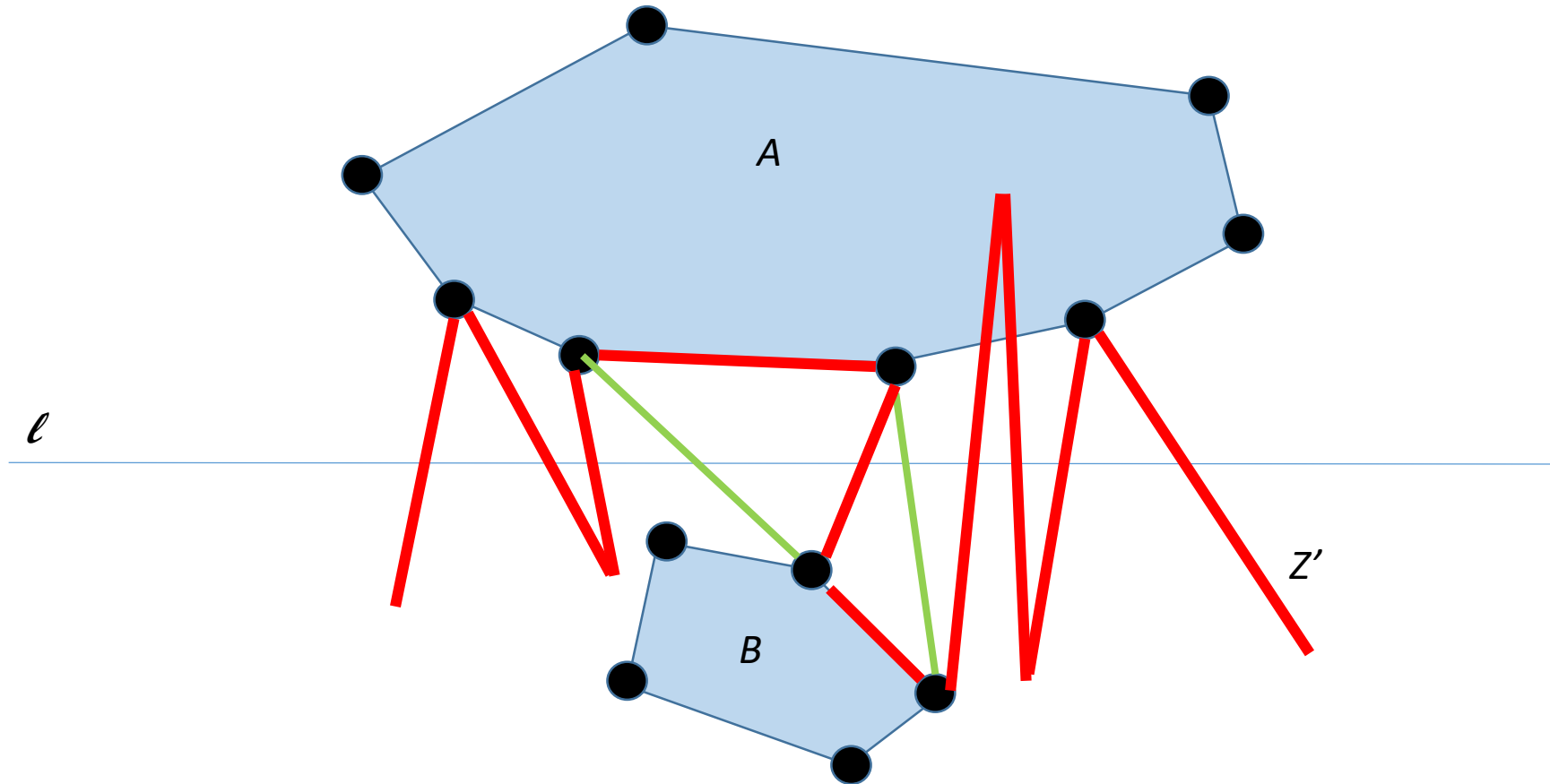
Lemma 3: Let (A,B) be a balanced separation of S of $|S| \geq 10$ points and let Z be a zig-zag path w.r.t. (A,B) . If Z uses 3 consecutive edges of $\text{Vis}(A,B)$, then S admits 3 edge-disjoint plane spanning paths.



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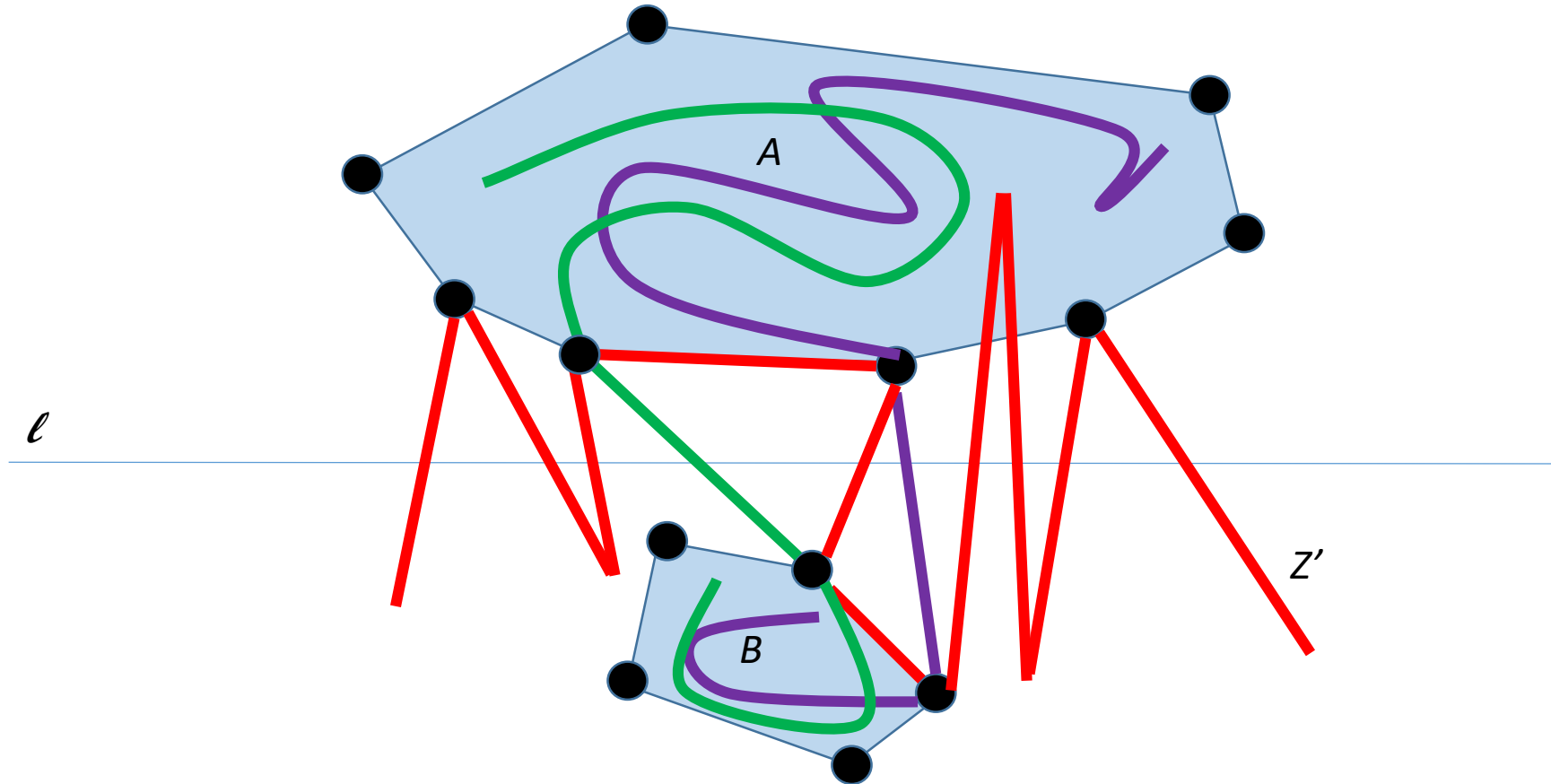
Proof:



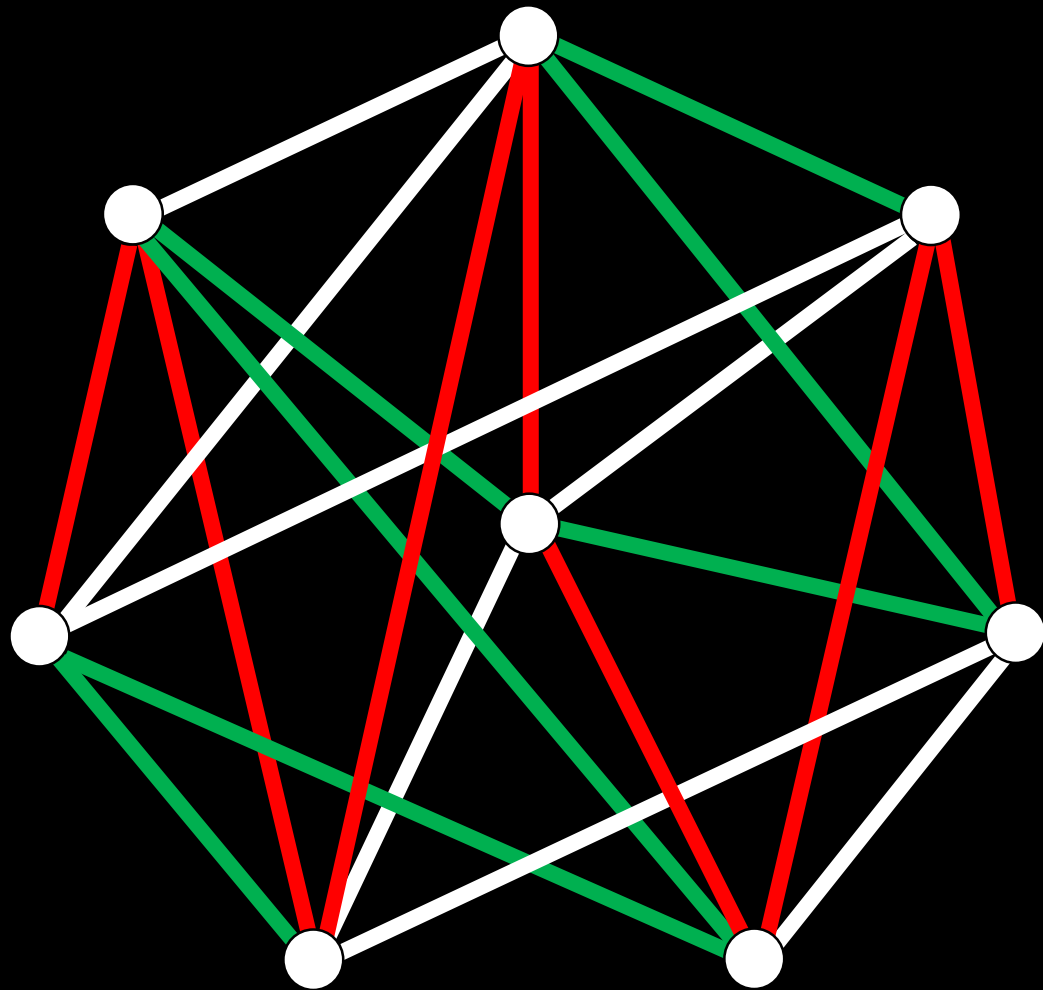
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Proof:



Let it Be⁶⁰
ppe
3 *hilipp* PATHS!
avel *onza*





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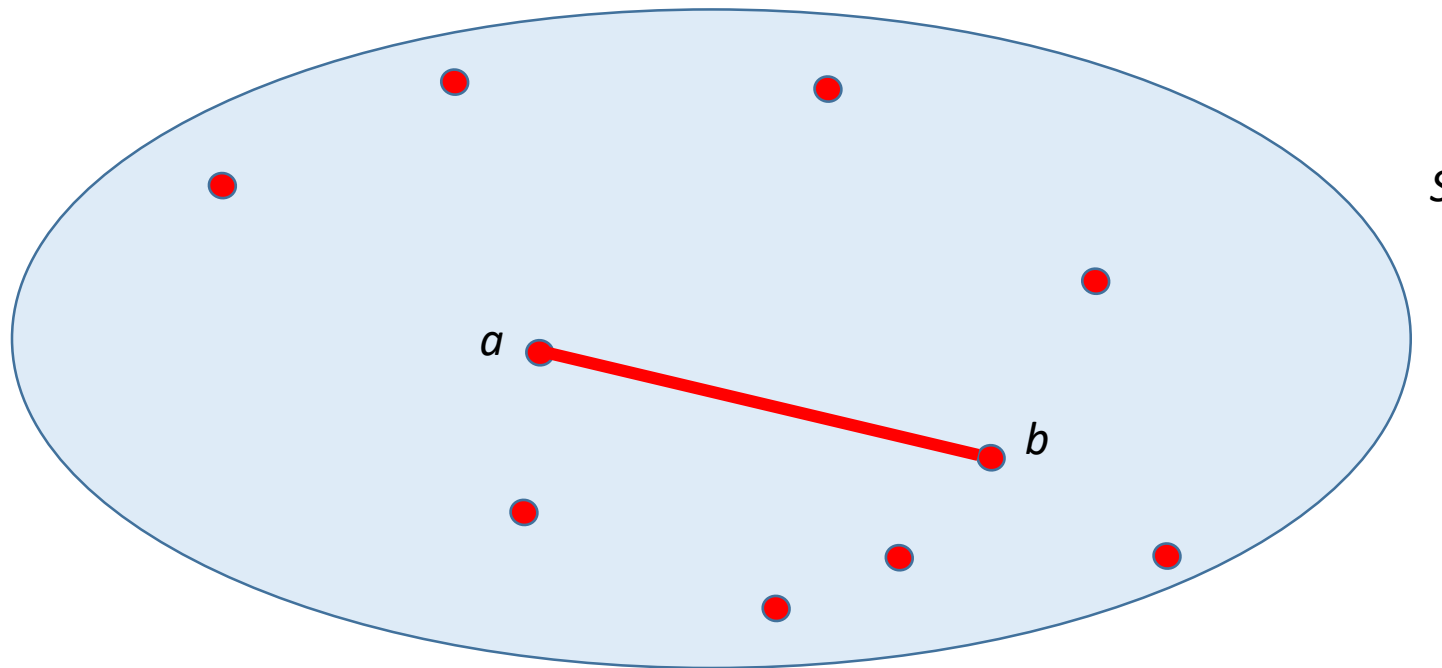
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Antič, Džuklevski, Fiala, Kratochvil, Liotta, Saghafian, Saumell, Zink [SOFSEM 2026] – 2 paths
with further constraints

2 Paths with one constrained edge

Antič, Džuklevski, Fiala, Kratochvíl, Liotta, Saghafian, Saumell, Zink [SOFSEM 2026]: Given a point set S and two distinct points a, b . Do there exist two paths such that

- Both contain the edge ab and are otherwise edge-disjoint
- Exactly one contains the edge ab and the paths are edge-disjoint
- None of them contains the edge ab and they are edge-disjoint



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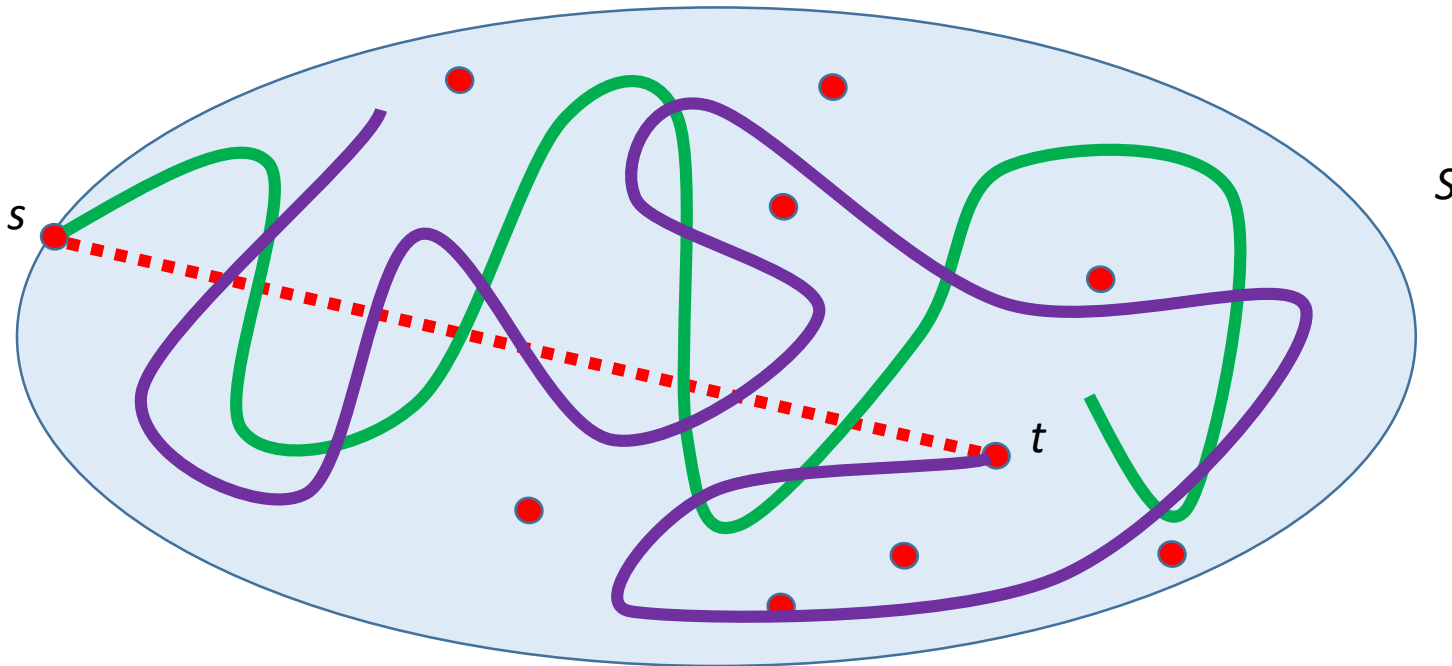
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Aichholzer, Červenková, Katsanou, Kratochvil, Perz, Saghafian, Valtr [2026+] – 2 paths with
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2 Paths with prescribed starting points

Aichholzer, Červenková, Katsanou, Kratochvíl, Perz, Saghafian, Valtr [2026+]: Given a point set S with at least 5 points and two of its points s, t , exactly one of them lying on the boundary of $\text{conv}(S)$. Then there exist two edge-disjoint plane spanning paths, one starting in s , the other one starting in t , and none of them using the edge st .

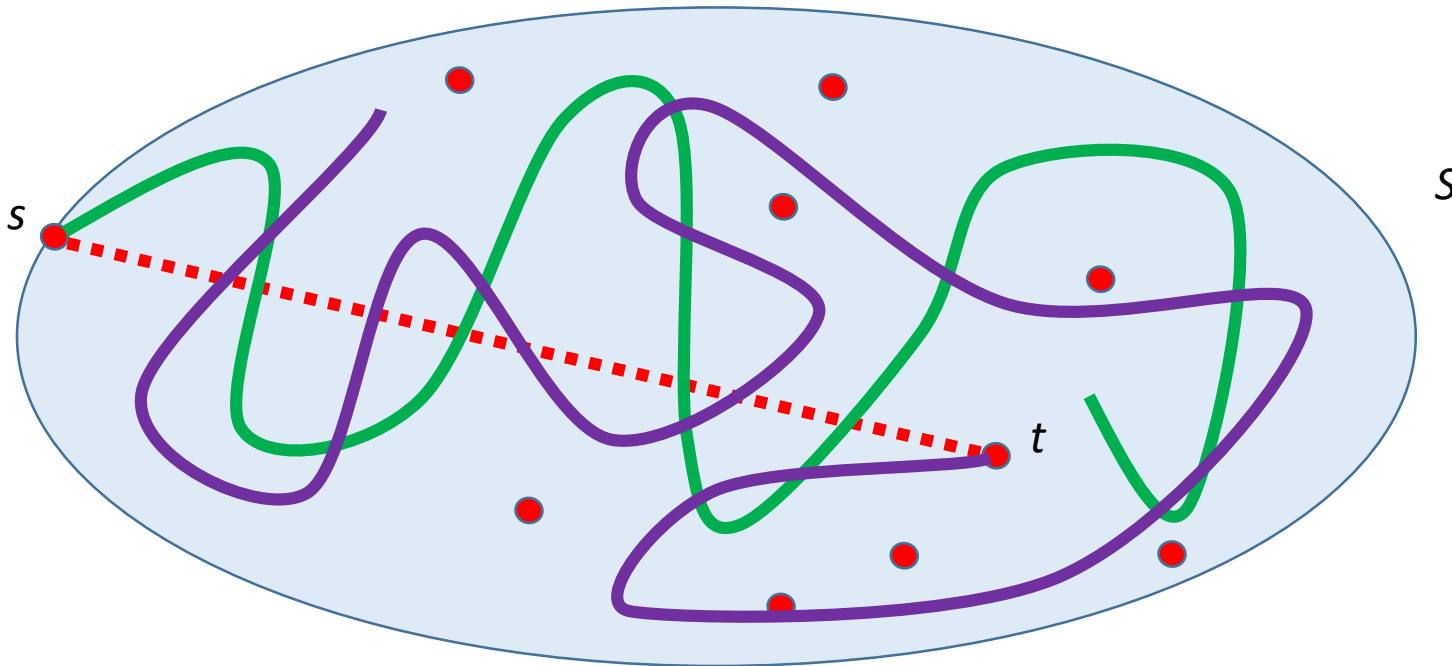


2 Paths with prescribed starting points

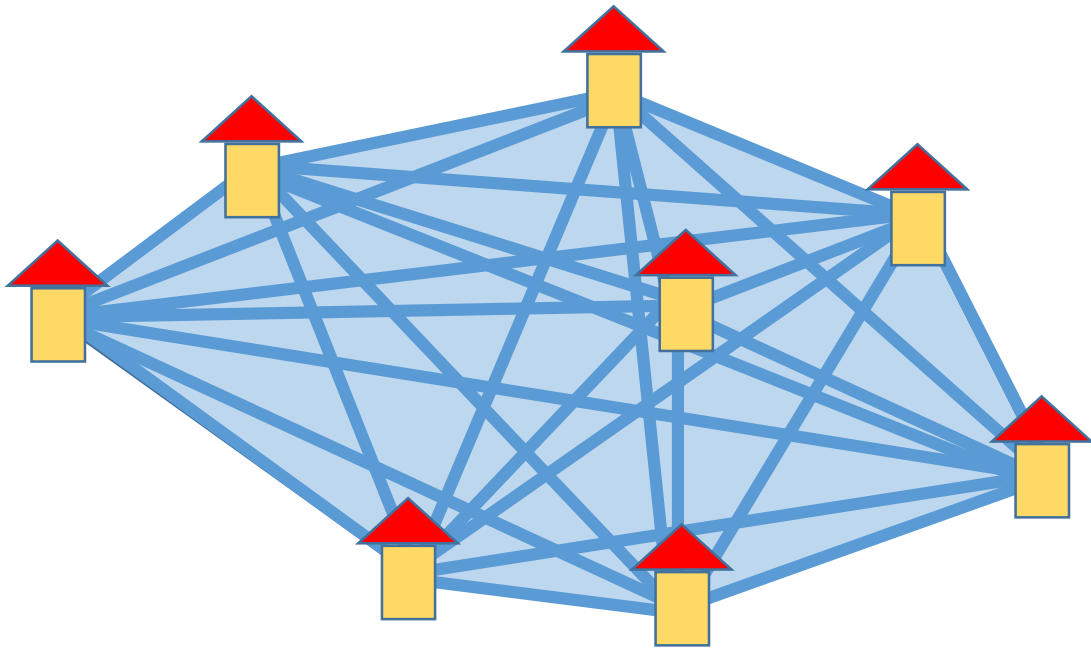
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Open: If both s and t lie inside $\text{conv}(S)$ and are distinct.

Known: If s and t coincide and are inside $\text{conv}(S)$ (then 2 edge-disjoint paths may not exist).

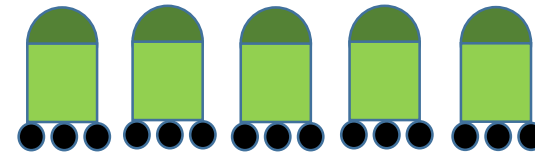
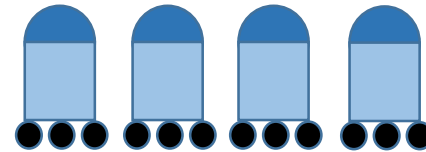
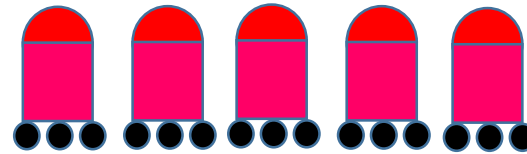


Searching and serving motivation

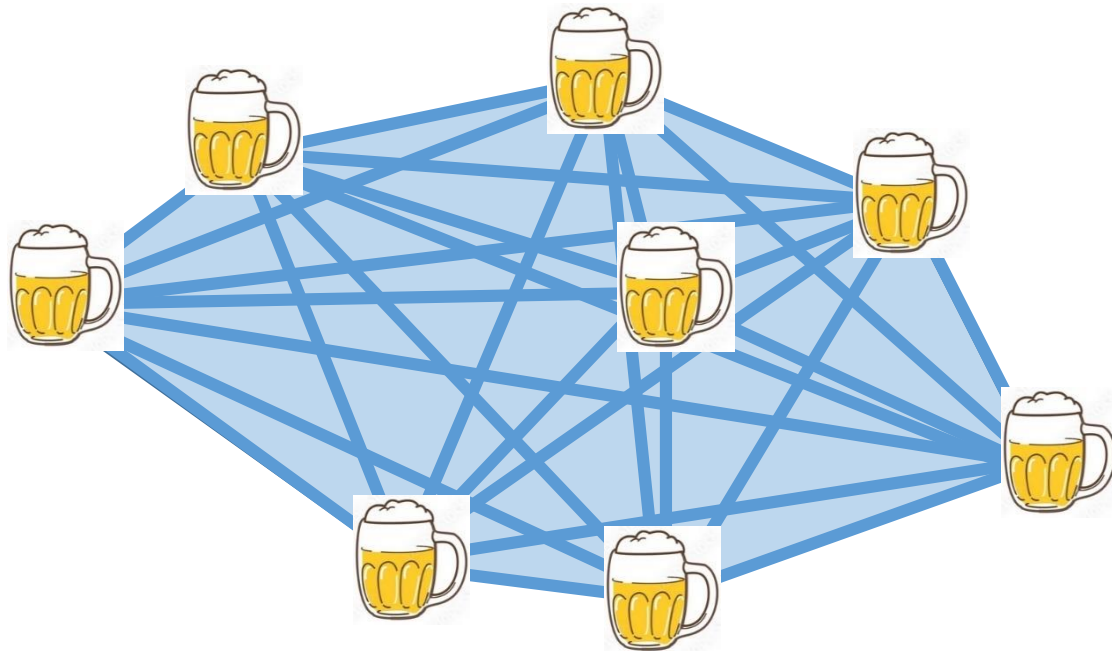


Robots

- Move on edges
- Simple
- Hierarchical
- One type of robots cooperate in serving all sites
- In edge crossings, robots give way to robots of higher class
- The paths of robots of the same class should not cross



How many paths lead to a good beer



Workshop participants

- Move on edges between pubs
- Want to taste beer in all pubs every evening (spanning paths)
- Move at different speed and do not want to bump into each other (non-self-crossing paths)
- Want to explore as many edges as possible and hence do not want to repeat edges (paths are edge-disjoint)



**Many
happy beers,
Pavel!**

Happy birthday!