NMAG403 - Combinatorics

September 29, 2025 – Basics and Hall's theorem

In class problems

1. Prove a slightly stronger version of Menger theorem: For every positive integer k, a graph with at least $k + 1$ vertices is vertex-k-connected if and only if every two distinct **non-adjacent** vertices are connected by at least k internally vertex-disjoint paths.
2. Prove that for every positive integer k, a graph with at least $k + 1$ vertices is vertex-k-connected if and only if every two k-element sets X, Y of vertices are connected by k completely disjoint paths.
3. Prove that in a vertex-2-connected graph, for any two vertices, there is a cycle that passes through both of them.
4. Prove that for every positive integer k, every k vertices in a vertex-k-connected graph lie on a cycle.
5. Does there exist a graph with at least 2 vertices such that every two distinct vertices have different degrees?
6. Prove that every planar vertex-2-connected graph has an $s - t$-numbering (an orientation with a single source and a single sink) which allows an upward plane drawing.
7. Prove Brooks theorem which says that for every graph G, $\chi(G) \leq \Delta(G)$ holds true, unless G is a complete graph or a cycle of odd length.
8. * For which k is the following statement true? Every legal filling of the first k lines of a SUDOKU can be extended to a legal completion of the entire 9×9 table. Prove your answer.
9. Prove or find a counterexample to the following statement: Let I and X be **infinite** sets and let $\mathcal{M} = \{M_i\}_{i \in I}$ be a set system such that $\bigcup \mathcal{M} = X$. If $\mathcal{M}$ satisfies the Hall condition ($\forall J \subseteq I : |\bigcup_{j \in J} M_j| \geq |J|$), then $\mathcal{M}$ has an SDR.
10. Let G be a bipartite graph with 42 vertices such that whenever you pick 31 vertices, they will contain at least one edge. Show that G has a matching with at least 12 edges.
11. Prove, for every integer k, that a graph has an orientation of maximum outdegree at most k if and only if each of its subgraphs H satisfies $|E(H)| \leq k \cdot |V(H)|$.
12. Dilworth's theorem says that if a finite poset $(P, \prec)$ has the largest antichain of size r, then P can be decomposed into r chains. Prove that Dilworth's theorem implies the harder implication of Hall's theorem.
13. Prove Birkhoff's theorem which says that for every n, the set of all bistochastic matrices of order n is exactly the convex hull of the set of permutation matrices of the same order (matrices being viewed as points in n^2-dimensional Euclidean space).