NMAG403 - Combinatorics

November 29, 2024 – Finite projective planes

In class problems

41. For a graph G, let $\zeta(G)$ be the smallest integer k such that G has an $\mathcal{L}$-list coloring for every list assignment $\mathcal{L} = \{L(u), u \in V(G)\}$ satisfying

    (a) $|L(u)| = k$ for every $u \in V(G)$, and

    (b) $|L(u) \cap L(v)| \leq 1$ for every edge $uv \in E(G)$.

    Prove that for every $n \geq 3$, $\zeta(K_n) \leq \lfloor\sqrt{n - \frac{11}{4}} + \frac{3}{2}\rfloor$.

42. Consider bipartite graphs with both classes of bipartition having the same number n of vertices and which do not contain cycles of length 4.

    (a) Prove that the number of edges in such a graph is at nost $(1 + o(1))n\sqrt{n}$.

    (b) Show that there are infinitely many values n for which there exist such graphs with at least $n\sqrt{n}$ edges.

43. Recall the axioms of projective planes from the lecture and prove that if (A1) and (A2) hold, then (A3) is equivalent to (A3').

44. A *finite affine plane* is a set system $\mathcal{A} = (X, \mathcal{L})$ satisfying the following axioms:

    A1 For every $L \in \mathcal{L}$ and every $x \in X \setminus L$ there is a unique $L' \in \mathcal{L}$ such that $x \in L'$ and $L' \cap L = \emptyset$.

    A2 For every two distinct $x, y \in X$, there is a unique $L \in \mathcal{L}$ such that $x, y \in L$.

    A3 For every $L \in \mathcal{L}$ it holds that $|L| \geq 2$.

    A4 There exist distinct points $x, y, z \in X$ such that $\{x, y, z\} \not\subseteq L$ for every $L \in \mathcal{L}$ (i.e., x, y, z are not colinear).

    (a) Prove that if one removes a line and all of its points from a projective plane, one gets an affine plane.

    (b) Prove that if $\mathcal{A}$ is an affine plane, then one can add some new points and one new line (and extend the existing lines by some of the new points) to get a projective plane.