NMAG403 - Combinatorics

November 22, 2024 – Hamiltonicity

Homework

Deadline: **December 18, 2024**
Send to: `honza@kam.mff.cuni.cz` (in PDF)

1. Let Q_n be the n-dimensional hypercube graph, i.e. the vertices of Q_n are all $\{0,1\}$-strings of length n and two strings are connected by an edge if and only if they differ in exactly one position. Prove that Q_n has a Hamiltonian cycle, provided $n \geq 2$.
2. Prove that in a bipartite graph whose all vertices of one class of bipartition have odd degrees, the total number of Hamiltonian cycles is even.
3. Show that if $c_v(G) \geq \alpha(G) - 1$ holds true for a graph G with at least 3 vertices, then G has a Hamiltonian path.
4. * Construct a vertex-3-connected graph with exactly one Hamiltonian cycle. (An original solution of this problem will gain you the "zápočet".)
5. ** Does there exist a planar vertex-3-connected graph with exactly one Hamiltonian cycle? (A solution of this problem will gain you an A-grade from this course.)

In class problems

38. Show that the Petersen graph does not contain a Hamiltonian cycle.
39. Show that if $c_v(G) \geq \alpha(G)$ holds true for a graph G with at least 3 vertices, then G has a Hamiltonian cycle.
40. Prove that a plane triangulation with more than 3 vertices cannot contain exactly one Hamiltonian cycle. Hint:
    (a) Let $G = (V, E)$ be such a graph and let $H = x_1, x_2, \ldots, x_n$ be a Hamiltonian cycle in G. For every edge e of H, let $M(e)$ be the set containing the two triangular faces incident with e. Prove that $\mathcal{M} = \{M(e) : e \in E(H)\}$ has a system of distinct representatives.
    (b) Let F be the domain of this SDR (i.e., the assigned faces) and consider the bipartite graph G' with one partition V, the other partition F and vertex $v \in V$ being adjacent to a face-vertex $f \in F$ if and only if $v \in f$. Observe that each Hamiltonian cycle in G' gives rise to a Hamiltonian cycle in G.
    (c) Prove that G' has at least two Hamiltonian cycles. If at least one of them does not correspond to H, we are done, otherwise do some extra work to prove the main statement anyway.