NDMI028 - LAK

October 03, 2024 – Warm-up

In class problems

1. Show that the number of labelled trees on n vertices is n^{n-2} using the theorem about Laplace matrix.
2. Why did we cross out a row and a column of L_G before computing the determinant? Determine $\det L_G$.
3. Why did we cross out a row and a corresponding column? Prove or disprove: For every graph G and every pair of vertices u, v of G, $\det L_G^{(u,v)}$ is the number of spanning trees of G, where $L_G^{(u,v)}$ is the Laplace matrix of G with u-th row and v-th column crossed out.
4. Let A and B be square matrices of the same order over a field F. Prove that
$$\mathrm{rank}(AB) \leq \mathrm{rank}(A).$$
5. Let A and B be square matrices of the same order over a field F. Prove that
$$\mathrm{rank}(AB) \leq \mathrm{rank}(A) \cdot \mathrm{rank}(B).$$
6. Let A and B be matrices of the same order over a field F. Prove that
$$\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B).$$
7. Suppose you have a software package for fast multiplication of matrices. How can you use it for determining if an input graph is connected?
8. And how can you use it for deciding whether the input graph is triangle-free in time faster than $\Omega(n^3)$?
9. Formulate and prove a theorem about counting walks between all pairs of vertices in directed graphs.
10. Formulate and prove a theorem about counting walks between all pairs of vertices in multigraphs (which allow loops and multiple edges).
11. Let $A \in \{0, 1\}^{n \times n}$ be a square 0-1 matrix. Consider fields $GF(2), GF(p), Q, R, C$. For which pairs F_1, F_2 among them the following holds true: If A is regular over F_1, then it is regular over F_2?