Graph Covers: A journey from Topology via Computational Complexity to Generalized Snarks

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joint work with
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University of Oregon

Eugene, March 25, 2024

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A journey
from Eugene via Bergen and Prague and Nova Louka
back to Eugene
and beyond

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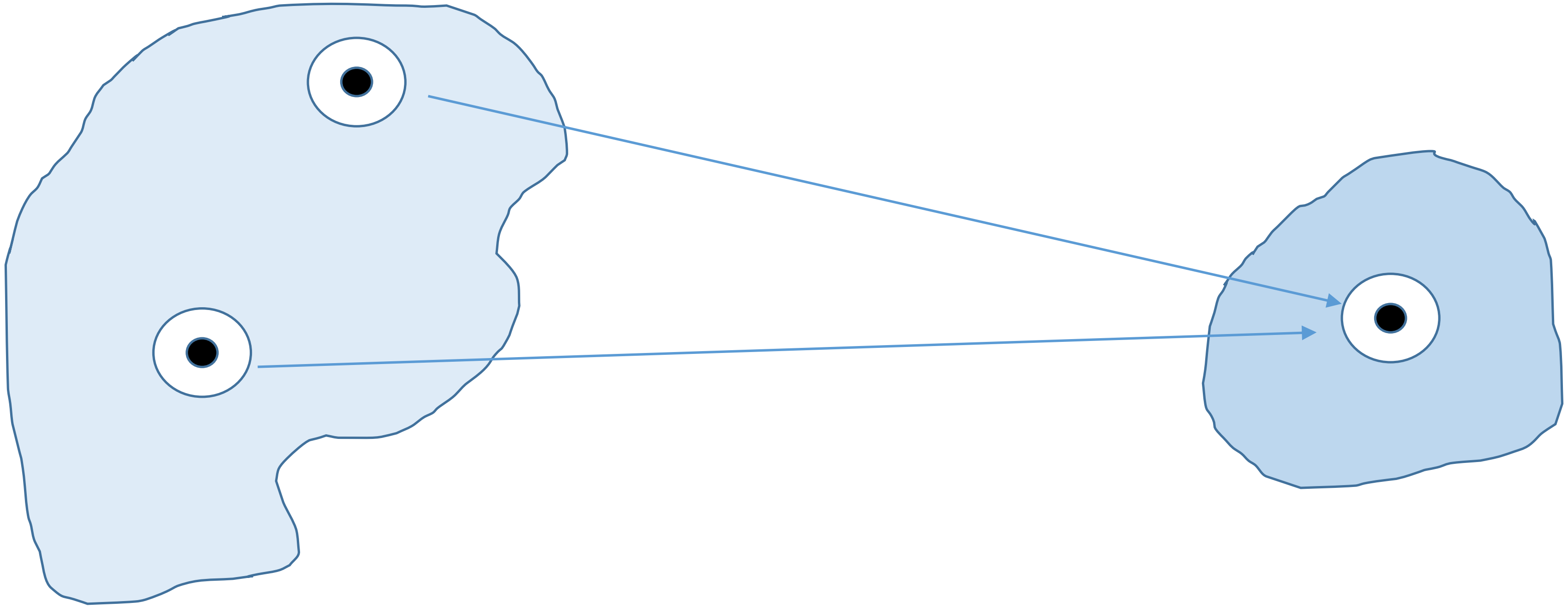
Eugene, March 25,


81

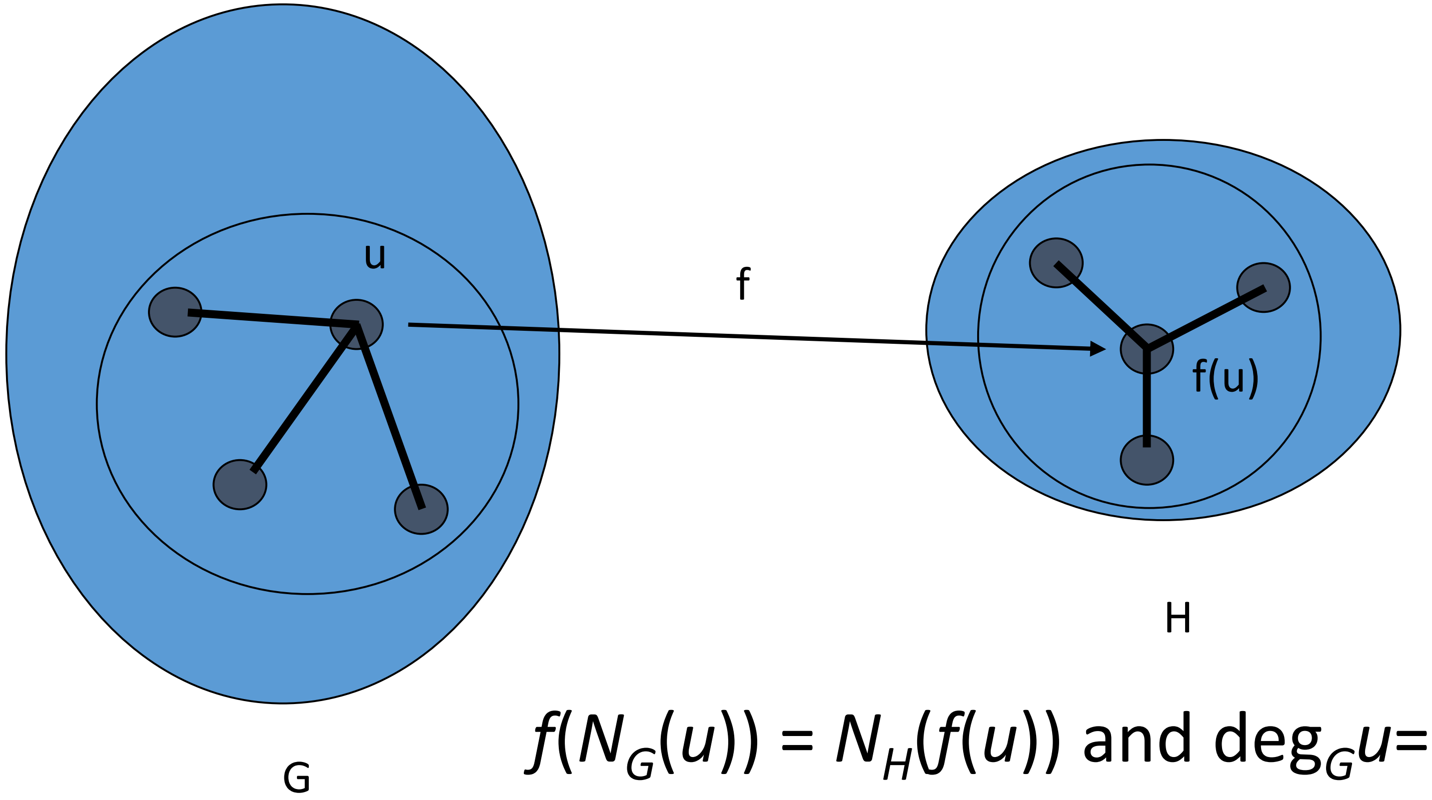
Covering spaces in topology

Euclidean and projective planes – the Euclidean plane is a double cover of the projective one

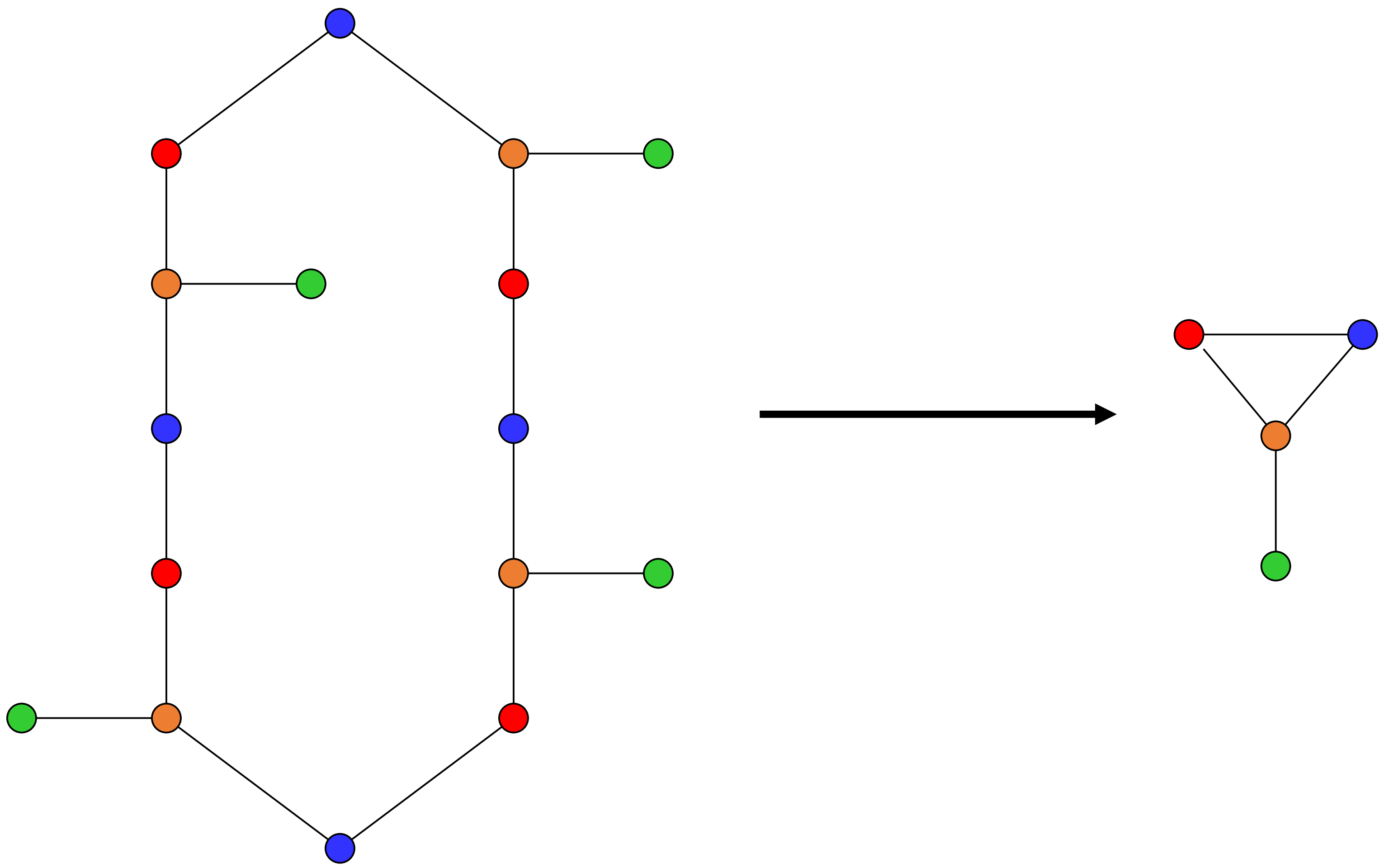


Definition of graph covering
(for connected simple graphs)

Definition: Mapping f: $V(G) \to V(H)$ is a *graph covering projection* if for every $u \in V(G)$, $f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$



$f(N_G(u)) = N_H(f(u))$ and $\deg_G u = \deg_H f(u)$



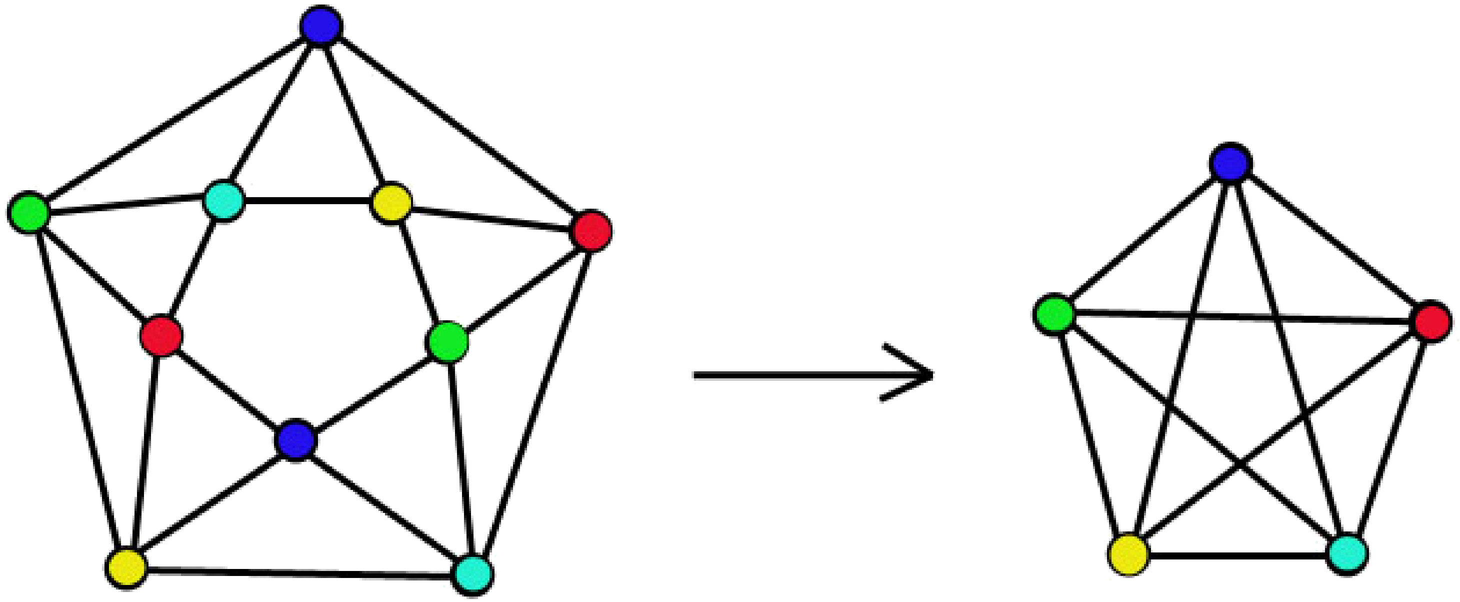
A bit of the history

- Topological graph theory, construction of highly symmetric graphs (Biggs 1974, Djokovic 1974, Gardiner 1974, Gross et al. 1977)
- Local computation (Angluin STOC 1980, Litovsky et al. 1992, Courcelle et al. 1994, Chalopin et al. 2006)
- Common covers (Angluin et al. 1981, Leighton 1982)
- Finite planar covers (Negami's conjecture 1988, Hliněný 1998, Archdeacon 2002, Hliněný et al. 2004)

Outline of the talk

- Negami’s conjecture
- Computational complexity
- Multigraphs with semi-edges
- Strong dichotomy conjecture
- Covers of disconnected graphs
- Generalized snarks
- Covers of directed graphs

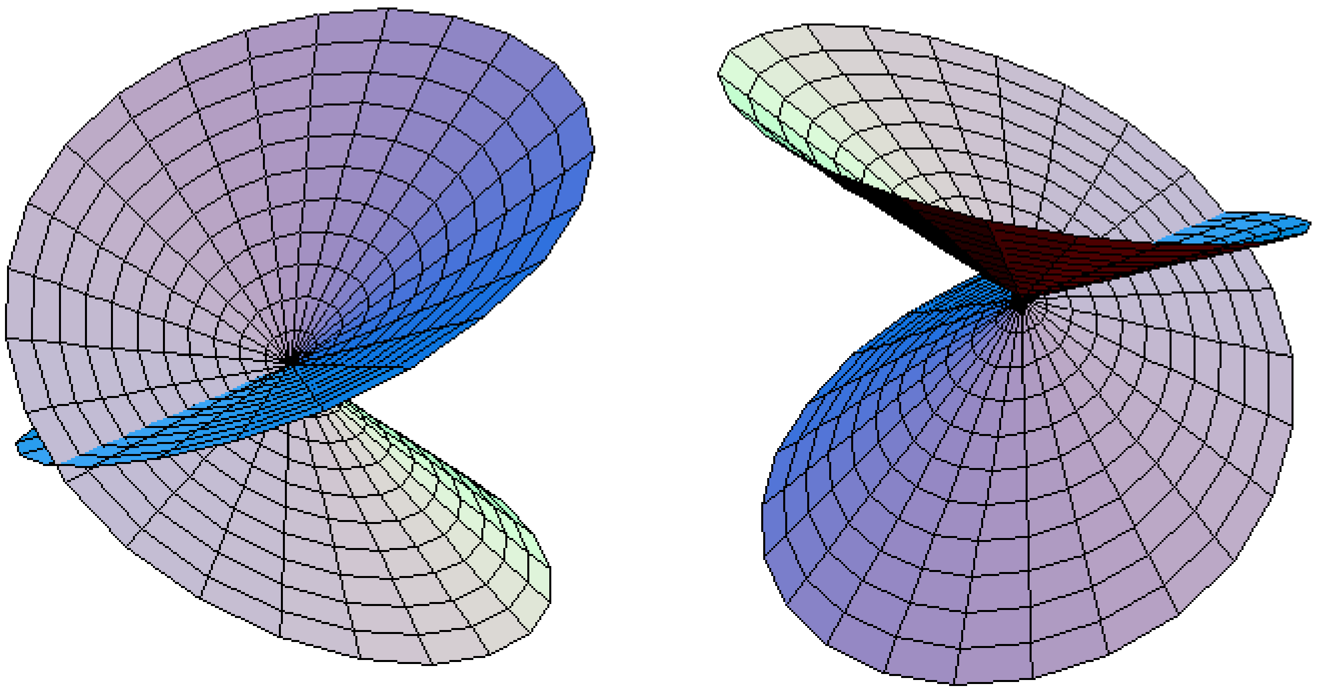
Negami’s conjecture



Negami's conjecture

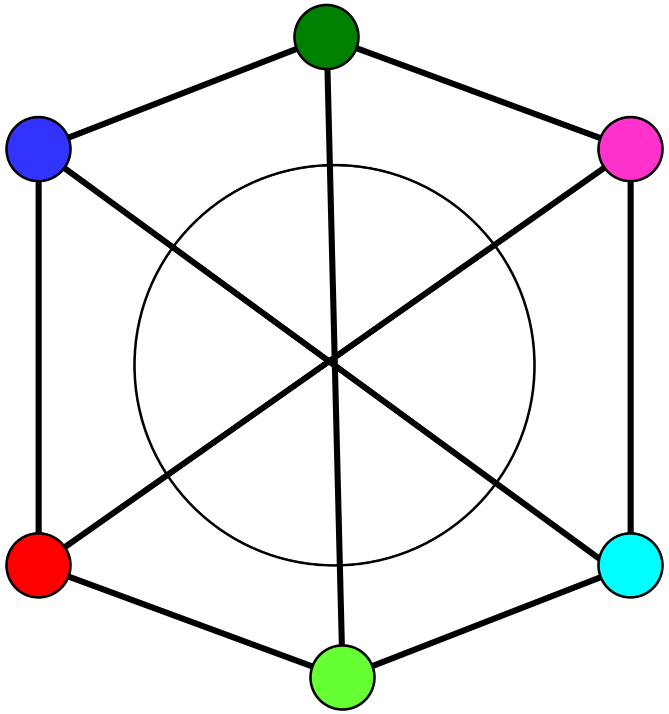
Conjecture (Negami 1988): A graph has a finite planar cover if and only if it is projective planar.

The projective plane

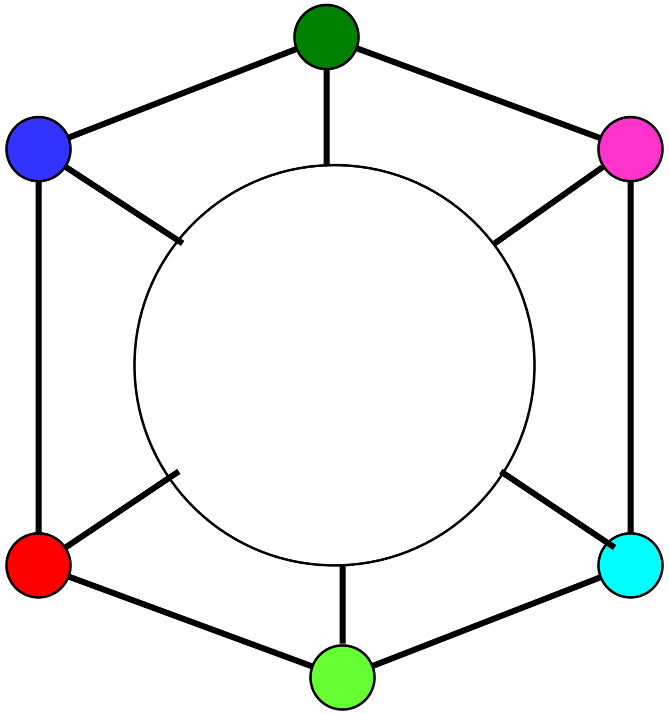


Wikipedia – Wikimedia commons

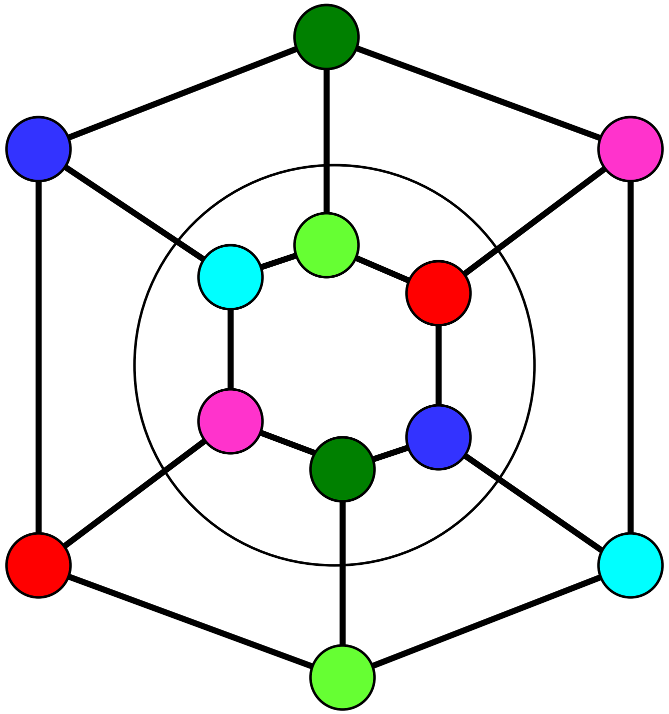
The cross-cap description of the projective plane



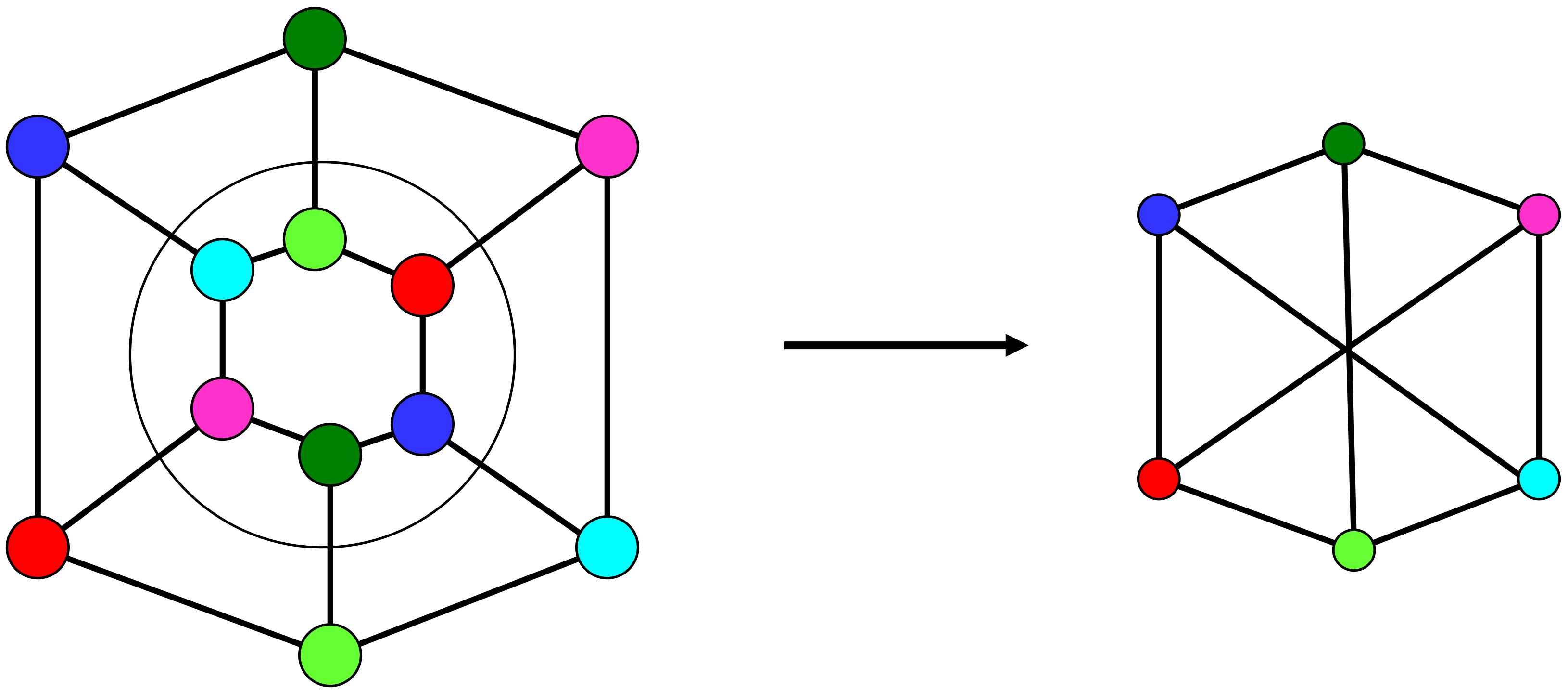
$K_{3,3}$



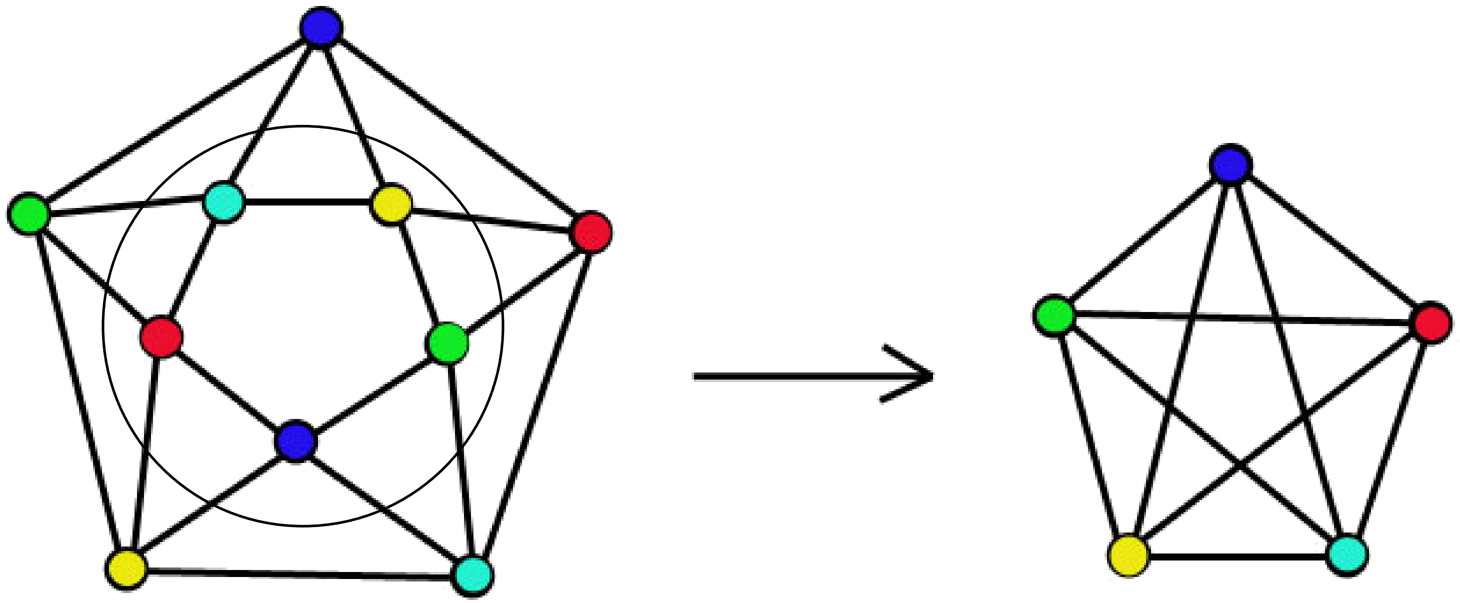
$K_{3,3}$



A planar cover of $K_{3,3}$



A planar cover of $K_{3,3}$



A planar cover of K_5

Negami's conjecture

Attempts to prove via Robertson-Seymour theory of minors, namely forbidden minors for projective planar graphs: Both *PlanarCoverable* and *ProjectivePlanar* are classes closed in the minor order. Moreover,

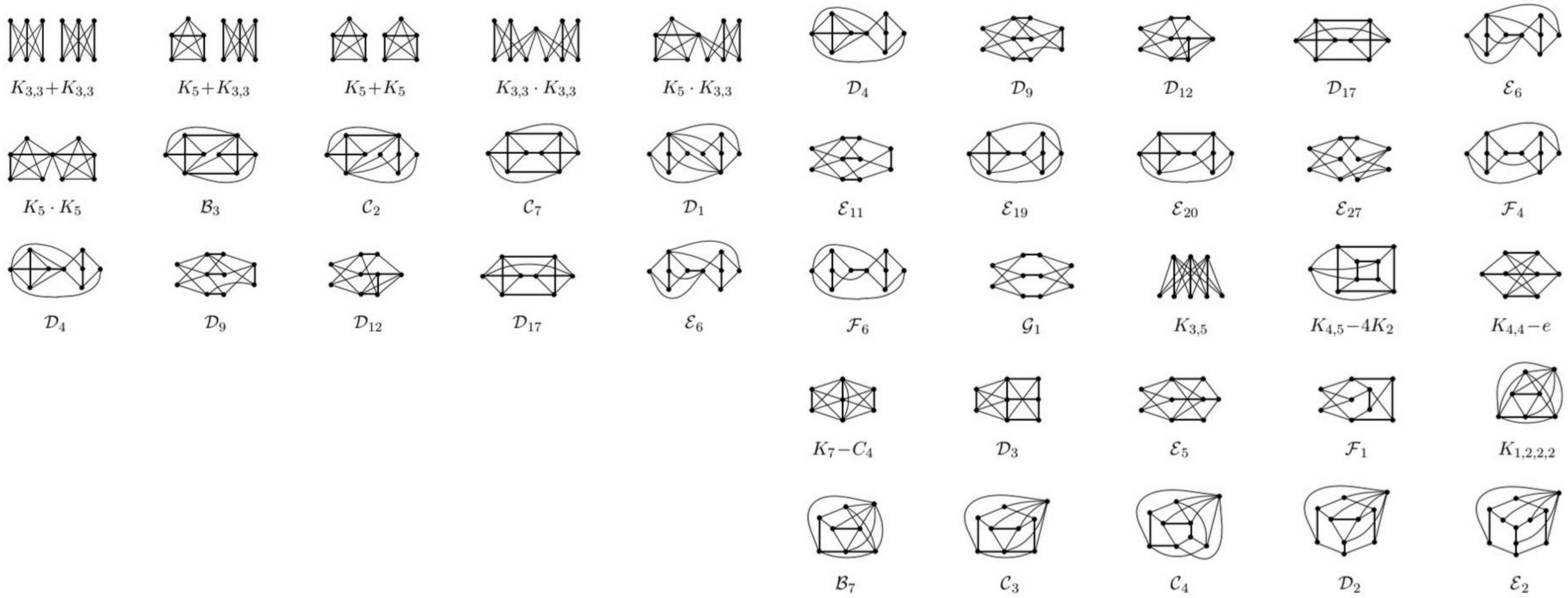
$$\textit{ProjectivePlanar} \subseteq \textit{PlanarCoverable}.$$

Need to show that no forbidden minor for the projective plane has a finite planar cover.

Projective Planar Graph

Download Wolfram Notebook

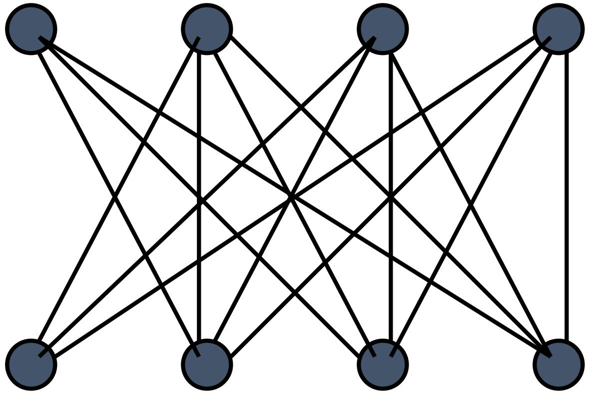
A graph with projective plane crossing number equal to 0 may be said to be projective planar. Examples of projective planar graphs with graph crossing number ≥ 2 include the complete graph K_6 and Petersen graph P.



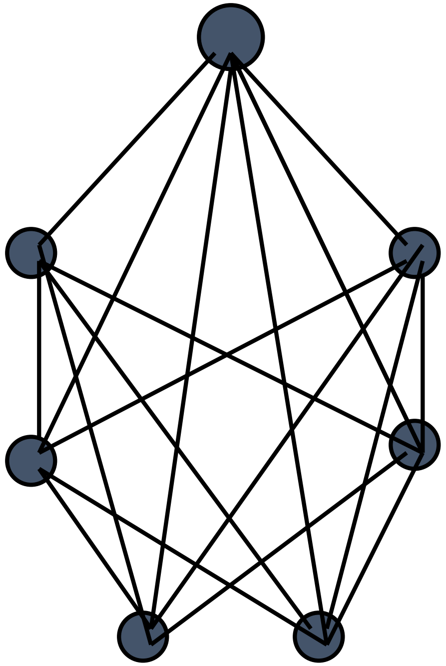
Negami's conjecture

Thm (Negami, Fellows, Archdeacon 1990): Conjecture is true for graphs not containing $K^{--}_{4,4}$ and $K_{1,2,2,2}$ as minors.

The terrible two



$K^{--}_{4,4}$



$K_{1,2,2,2}$

Negami's conjecture

Thm (Negami, Fellows, Archdeacon 1990): Conjecture is true for graphs not containing $K^{--}_{4,4}$ and $K_{1,2,2,2}$ as minors.

P. Hliněný (1998): $K^{--}_{4,4}$ does not have a finite planar cover.

P. Hliněný, R. Thomas (2002): Only finite number of counterexamples exist (if any).

HBO

Computer Science

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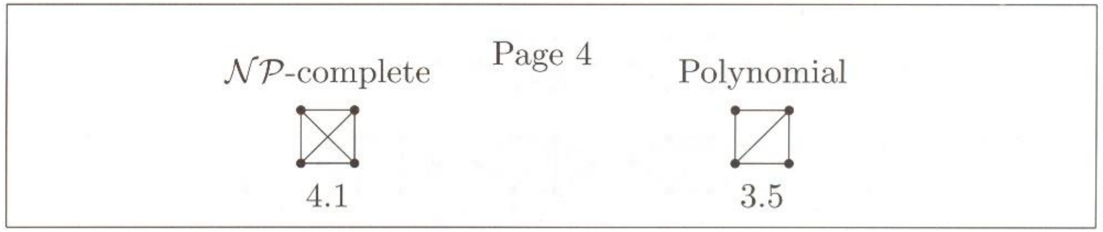
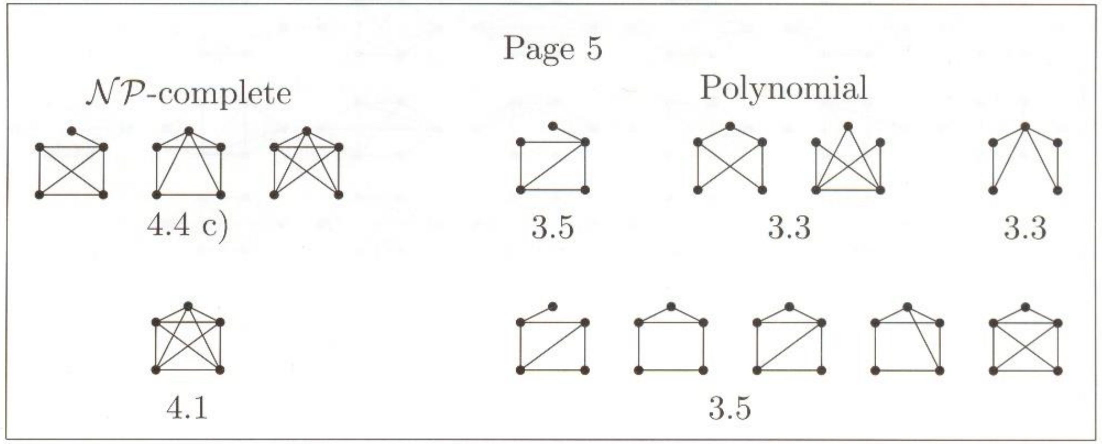
Computational complexity of graph covers

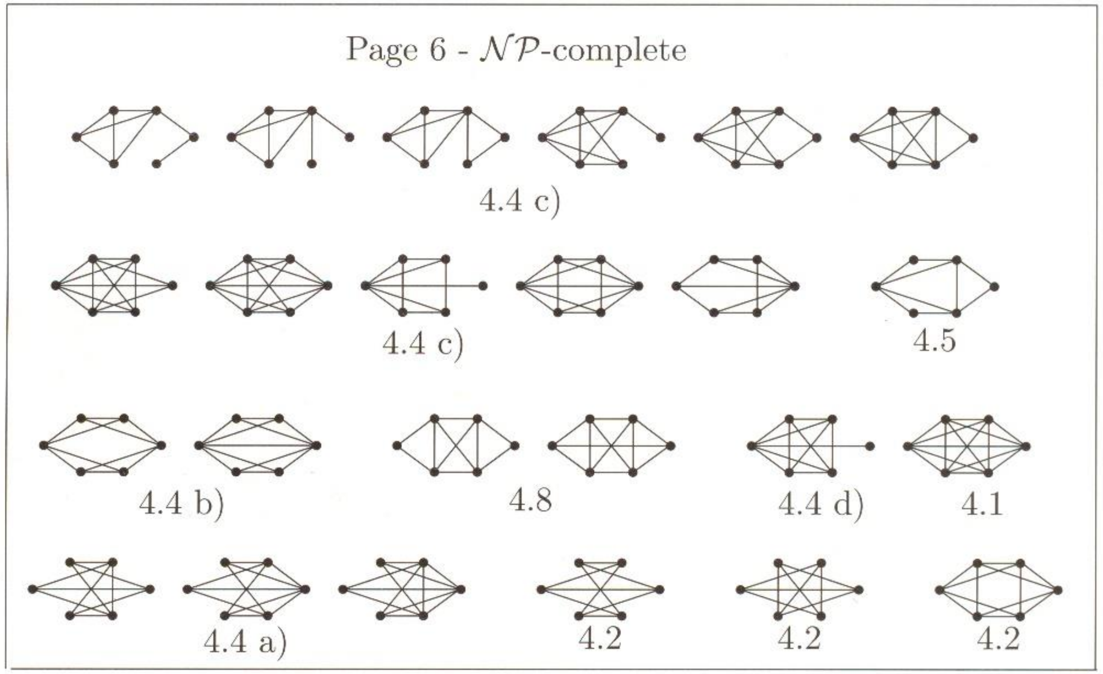
H-COVER
Input: A graph *G*
Question: Does *G* cover *H*?

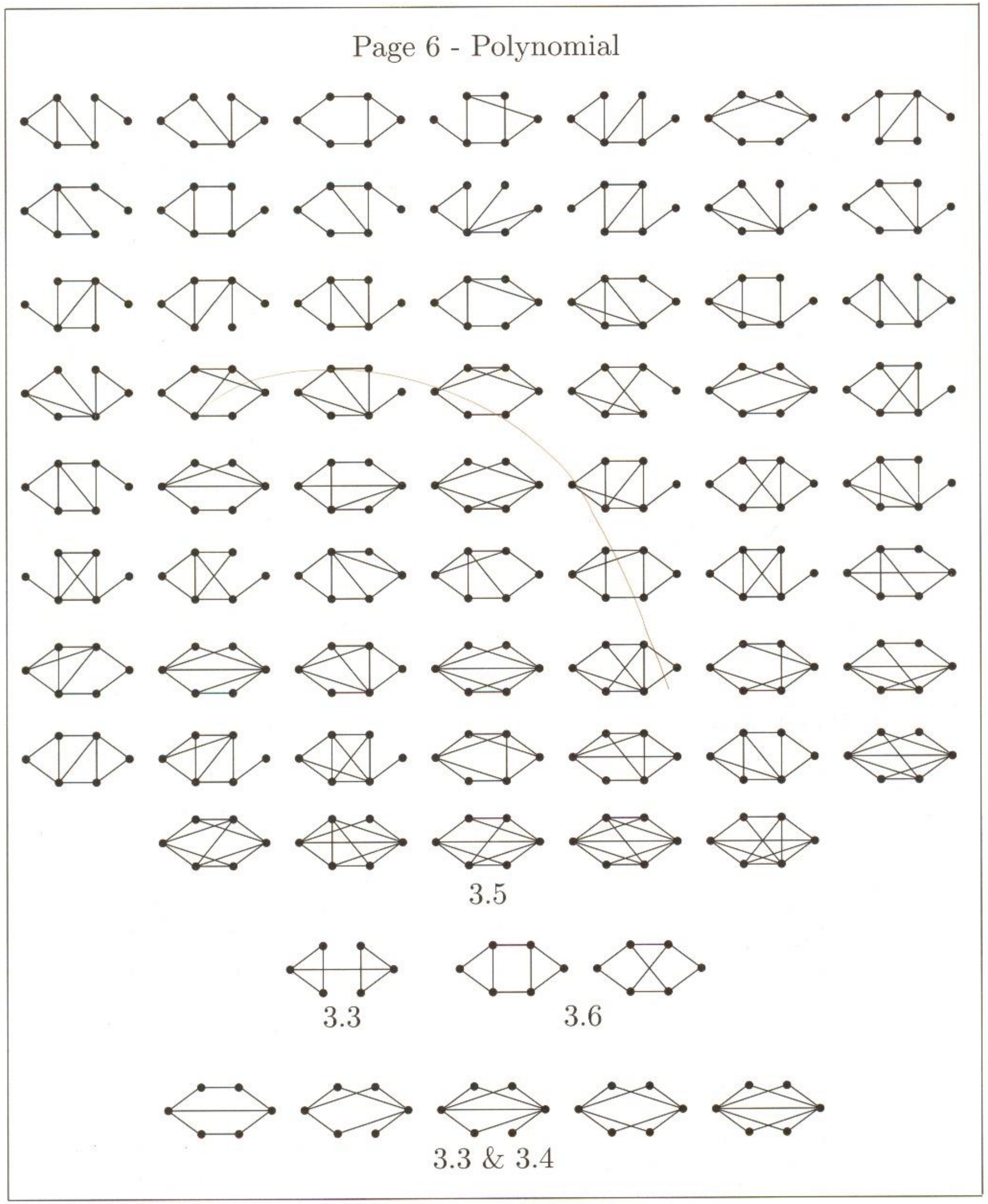
Computational complexity of graph covers

- Thm (Bodlaender 1989): *H*-COVER is NP-complete if *H* is also part of the input.
- Abello, Fellows, Stilwell 1991: Initiated the study of computational complexity of the *H*-COVER problem for fixed *H*.
- Thm (Kratochvil, Proskurowski, Telle 1994): *H*-COVER is polynomial time solvable for every simple graph with at most 2 vertices per equivalence class in its degree partition.
- Thm (Fiala, Kratochvil, Proskurowski, Telle 1998): *H*-COVER is NP-complete for every simple regular graph of valency at least 3.
- Fiala, Kratochvil 2008: Relation to CSP
- Bílka, Jirásek, Klavík, Tancer, Volec 2011: NP-hardness of covering small graphs by planar inputs.


Page 4
$\mathcal{NP}$-complete
Polynomial
4.1
3.5

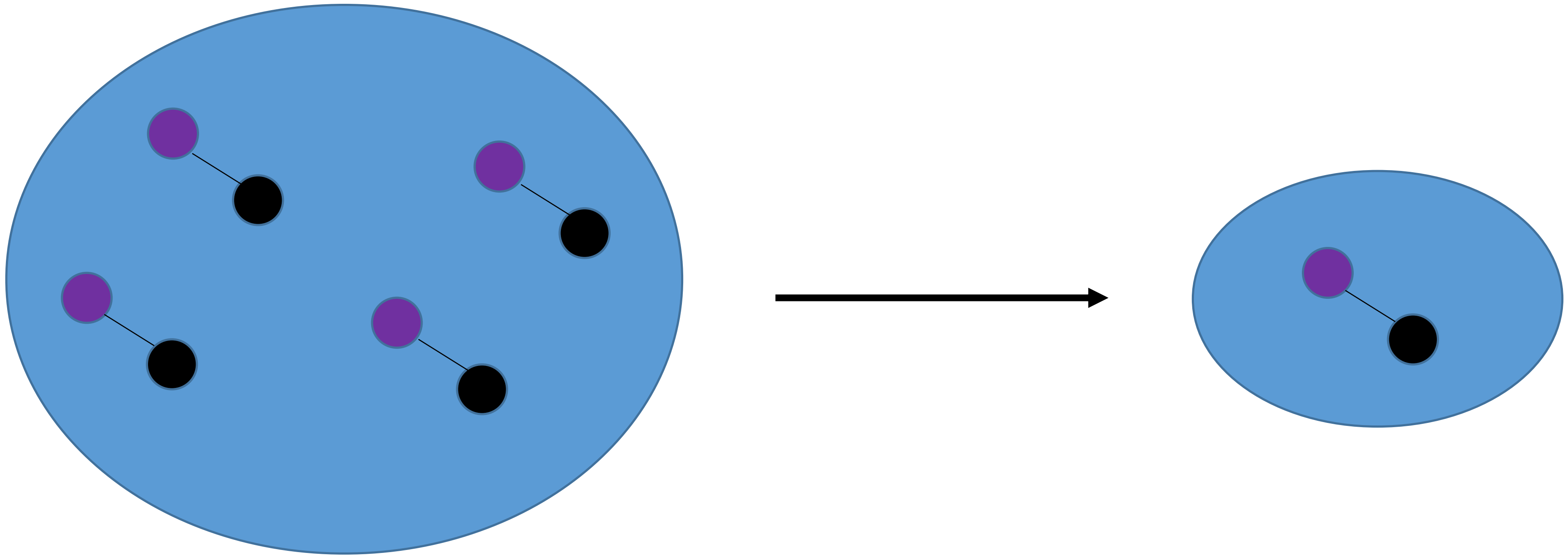

Page 5
$\mathcal{NP}$-complete
Polynomial
4.4 c)
3.5
3.3
3.3
4.1
3.5


Page 6 - $\mathcal{NP}$-complete
4.4 c)
4.4 c)
4.5
4.4 b)
4.8
4.4 d)
4.1
4.4 a)
4.2
4.2
4.2

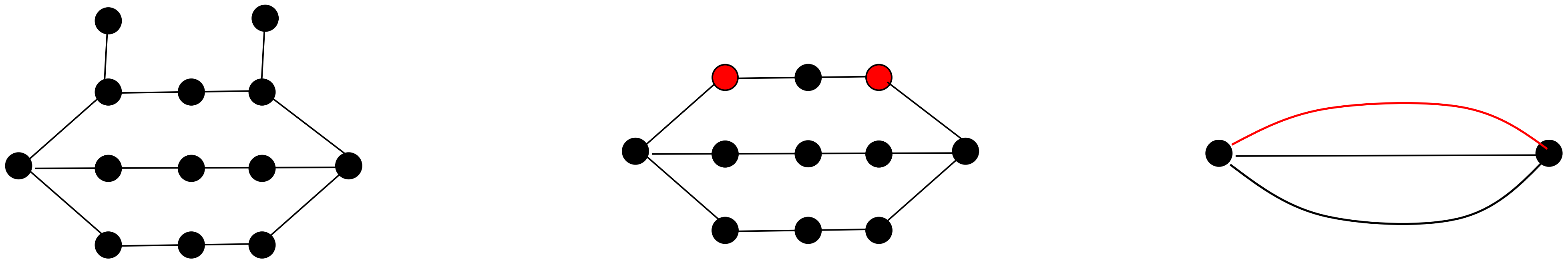

Page 6 - Polynomial
3.5
3.3
3.6
3.3 & 3.4

A few facts on graph covers

- Every covering projection to a connected graph is equitable
- A (rooted) tree is covered only by an isomorphic tree
- A path is covered only by a path of the same length



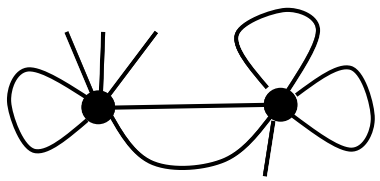
Reduction to colored graphs



Kratochvil, Proskurowski, Telle 1997: Apply the same reductions to *G* and *H*. Every covering projection must respect the colors. To fully understand the complexity of *H*-COVER for all simple graphs, it is necessary and suffices to understand its complexity for colored mixed multigraphs of minimum degree ≥ 3.

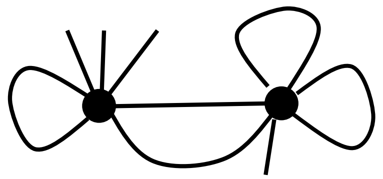
General graphs

(with multiple edges, loops and semi-edges allowed)



General graphs

(with multiple edges, loops and semi-edges allowed)



Why semi-edges?

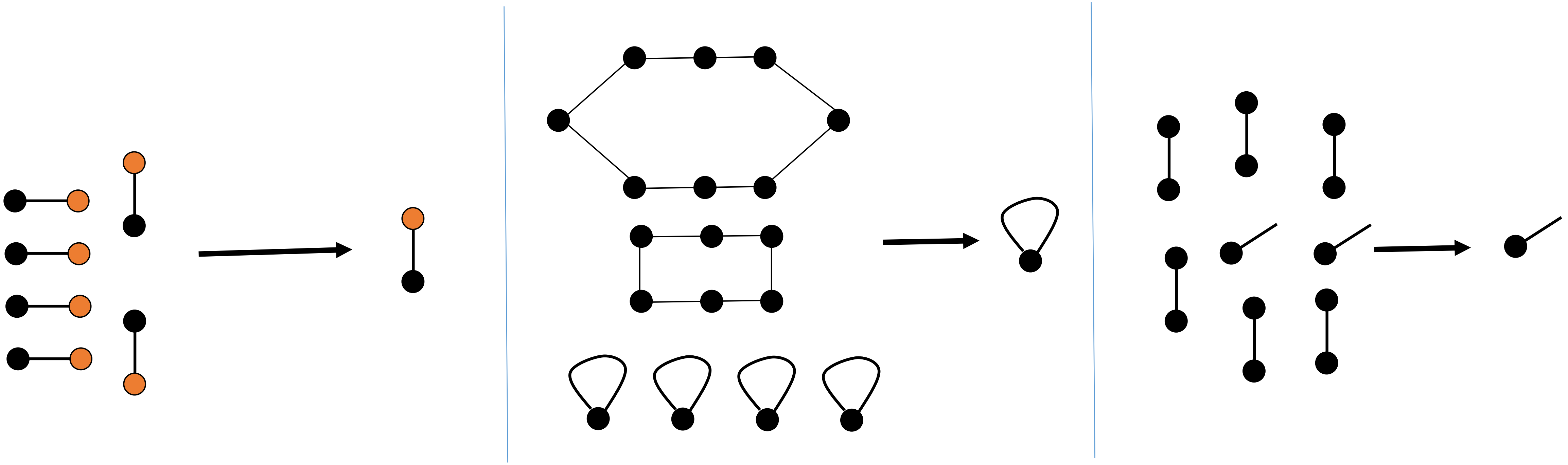
- Appear naturally as quotients of automorphism groups
- Recently became standard in topological graph theory and mathematical physics
- Are reasonable in the local computation model
- Capture interesting and standard graph theoretical invariants

Covers of general graphs

(with multiple edges, loops and semi-edges)

Definition: A pair of mappings $f = (f_V, f_E): G \to H$ is a graph covering projection if

- $f_V: V(G) \to V(H)$ is a homomorphism,
- $f_E: E(G) \to E(H)$ is compatible with f_V, and it is a bijection of {edges incident with u} onto {edges incident with $f_V(u)$} for every $u \in V(G)$



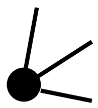
Complexity of covering multigraphs

- Kratochvil, Proskurowski, Telle 1997: Complete characterization of the computational complexity of *H*-COVER for colored mixed 2-vertex multigraphs (without semi-edges) *H.*
- Kratochvil, Telle, Tesař 2016: Complete characterization of the computational complexity of *H*-COVER for 3-vertex multigraphs *H* (monochromatic, undirected, without semi-edges).

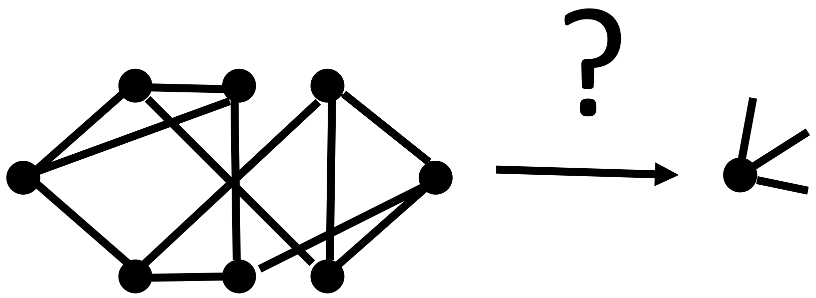
- Bok, Fiala, Hliněný, Jedličková, Kratochvíl MFCS 2021: First results on the computational complexity of *H*-COVER for (multi)graphs with **semi-edges**. Full classification for 1-vertex and 2-vertex graphs *H*.
- Bok, Fiala, Jedličková, Kratochvíl, Rzazewski IWOCA 2022: If *H* is a *k*-regular (multi)graph, $k \geq 3$, with at least one semi-simple vertex, then List-*H*-COVER is NP-complete for simple input graphs.

Some examples

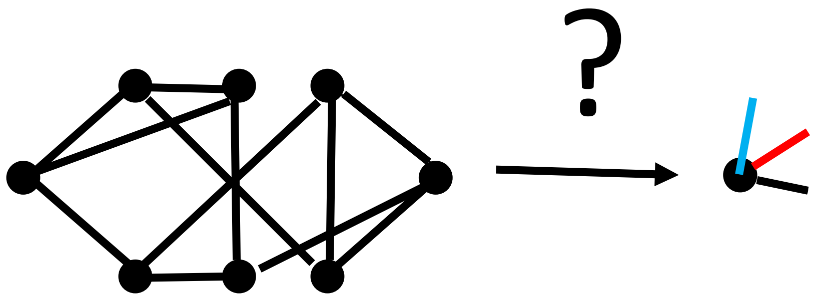
Some examples



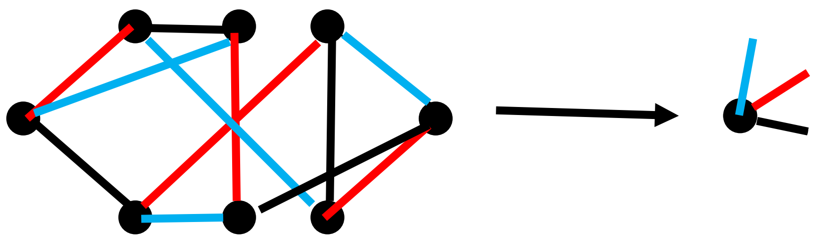
Some examples

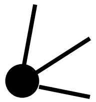


Some examples

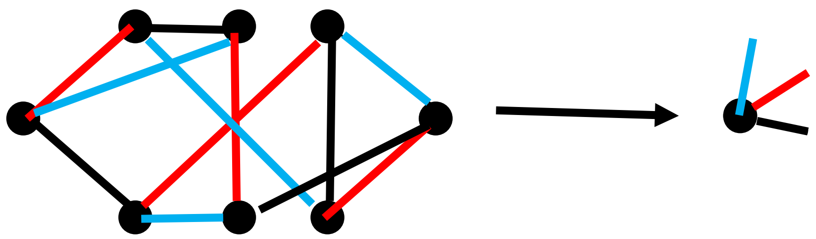


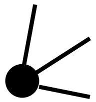
Some examples



A graph covers  iff it is cubic and 3-edge-colorable.

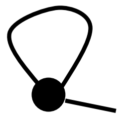
Some examples



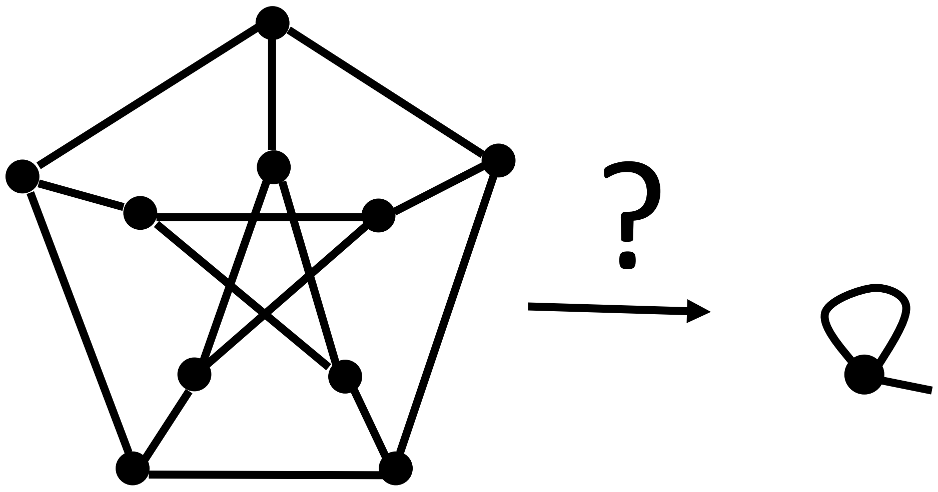
A graph covers  iff it is cubic and 3-edge-colorable.

NP-complete

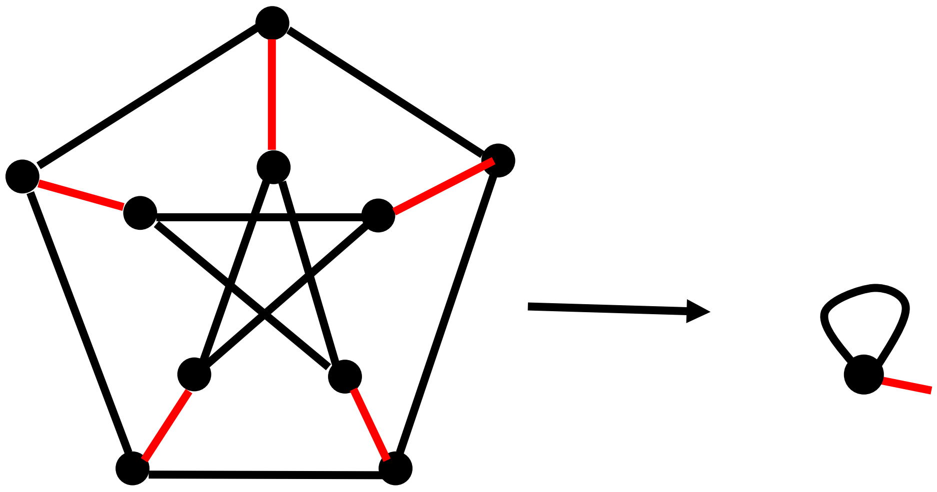
Some examples

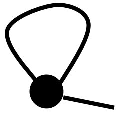


Some examples

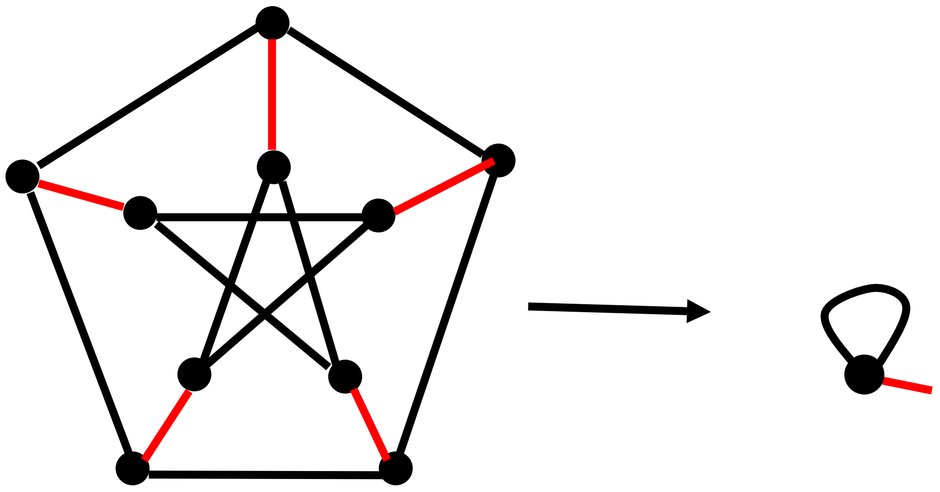


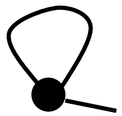
Some examples



A graph covers  iff it is cubic and has a perfect matching.

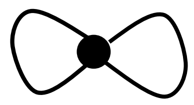
Some examples



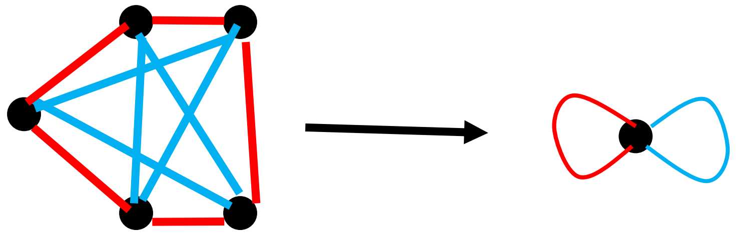
A graph covers  iff it is cubic and has a perfect matching.

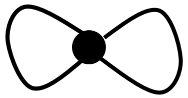
Poly time (Edmonds)

Some examples



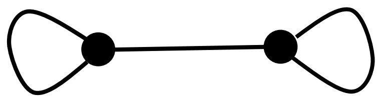
Some examples



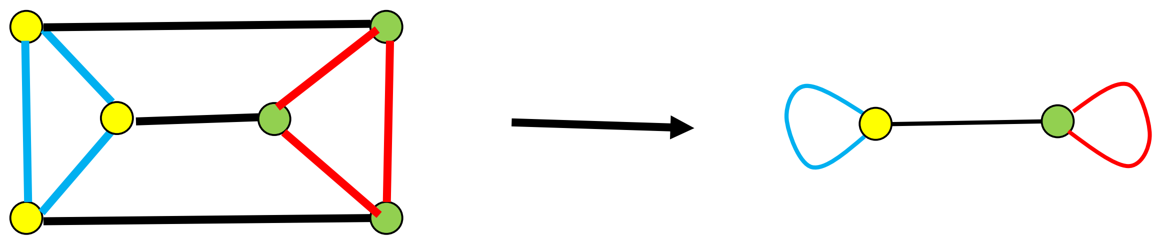
A graph covers  iff it is 4-regular (Petersen/Konig-Hall thm).

Poly time

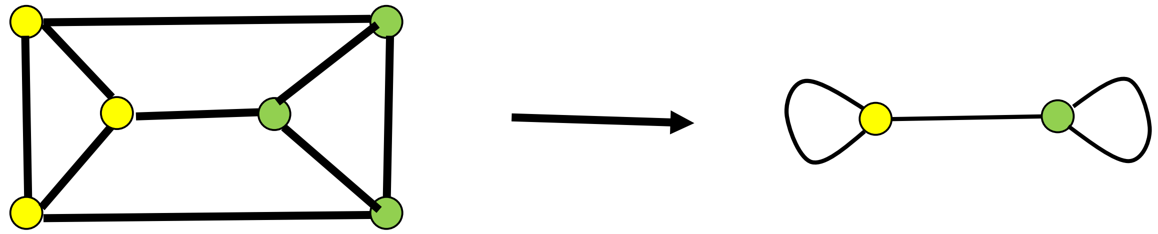
Some examples

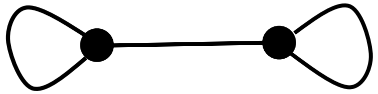


Some examples



Some examples



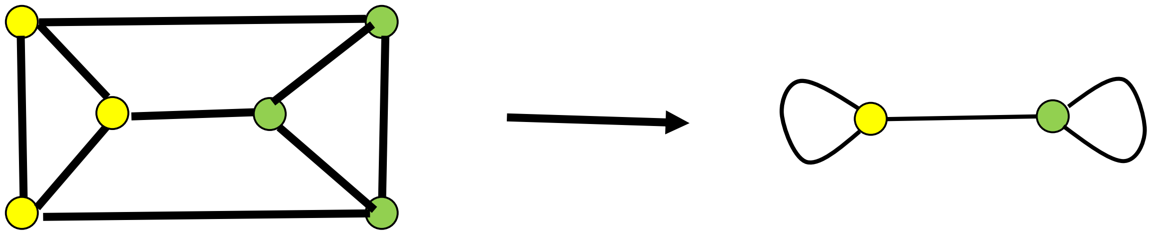
A graph covers  iff it is cubic and its vertices can be 2-colored so that every vertex has two neighbors of its own color and one neighbor of the other color.

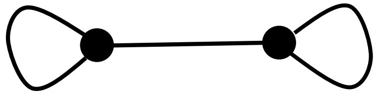
Some examples

NP-complete 1991 Abello et al (loops on input)

2011 Bilka et al (simple graphs)

2021 Bok et al (simple bipartite graphs)



A graph covers  iff it is cubic and its vertices can be 2-colored so that every vertex has two neighbors of its own color and one neighbor of the other color.

Strong Dichotomy Conjecture

2022 Bok et al: For every fixed graph *H*, the *H*-COVER problem is either polynomial time solvable for arbitrary input graphs (loops, multiple edges, semi-edges allowed), or NP-complete for simple input graphs.

Dichotomy Theorem for CSP

Várdy, Bulatov, Zhuk et al.: For any fixed relational structure *S*, the homomorphism problem to *S* is either polynomial-time solvable or NP-complete.

Dichotomy Theorem for CSP

Várdy, Bulatov, Zhuk et al.: For any fixed relational structure *S*, the homomorphism problem to *S* is either polynomial-time solvable or NP-complete.

Graph covers are expressible as CSP, but dichotomy for graph covers does not follow from the dichotomy of CSP!

Covers of disconnected graphs

Covers of disconnected graphs

WG 1994:

Without loss of generality, we will consider only connected graphs, because of the following observations (whose proofs are left to the reader.)

Fact 2. *(a) A disconnected graph G covers a connected graph H if and only if every connected component of G covers H.*

(b) For a disconnected graph H, the H-cover problem is polynomially solvable ($\mathcal{NP}$-complete) if and only if the H_i-cover problem is polynomially solvable ($\mathcal{NP}$-complete) for every (for some) connected component of H.

Covers of disconnected graphs

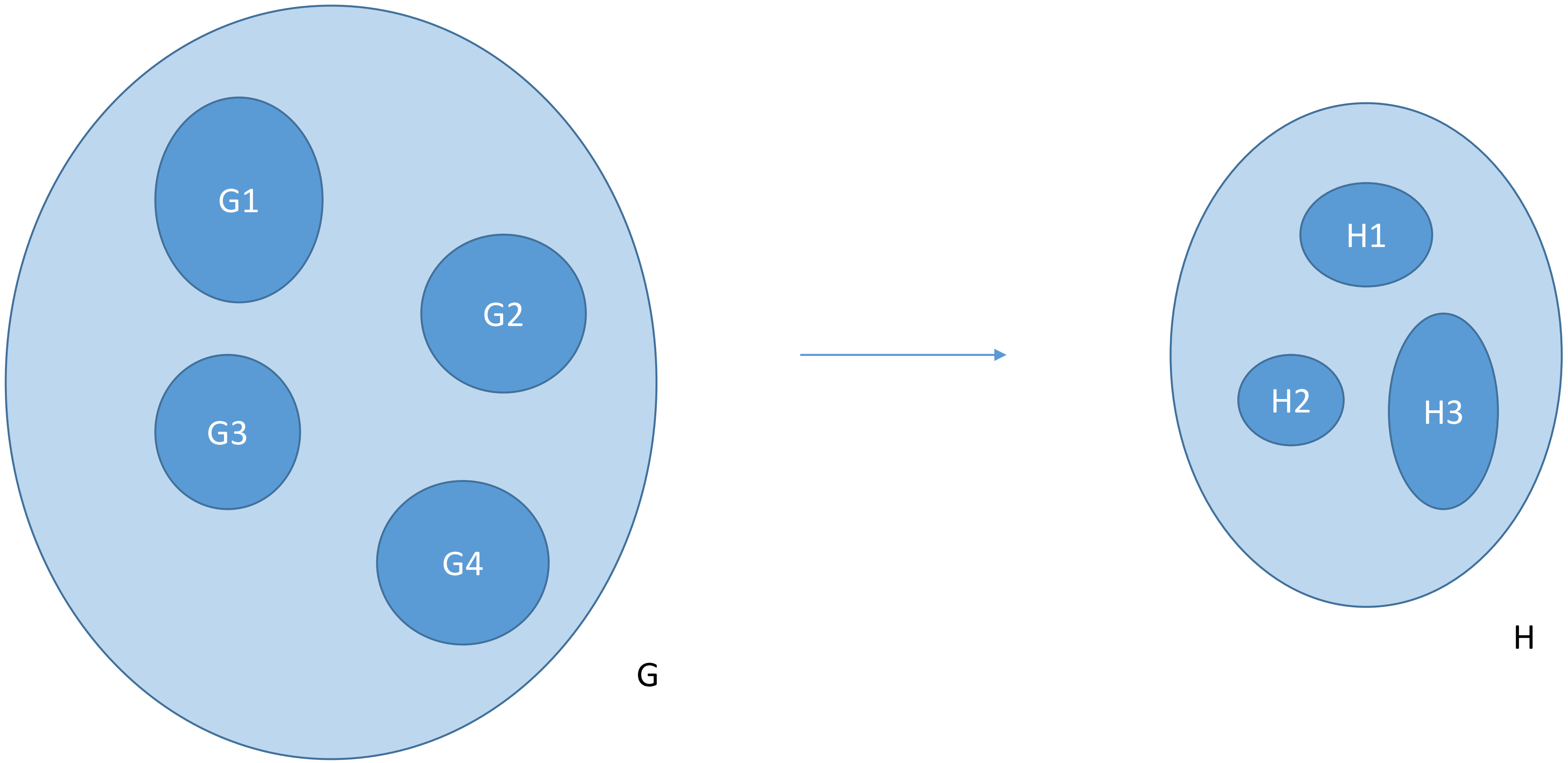
Complexity of Graph Covering Problems

Jan Kratochvíl[1], Andrzej Proskurowski[2] and Jan Arne Telle[2]

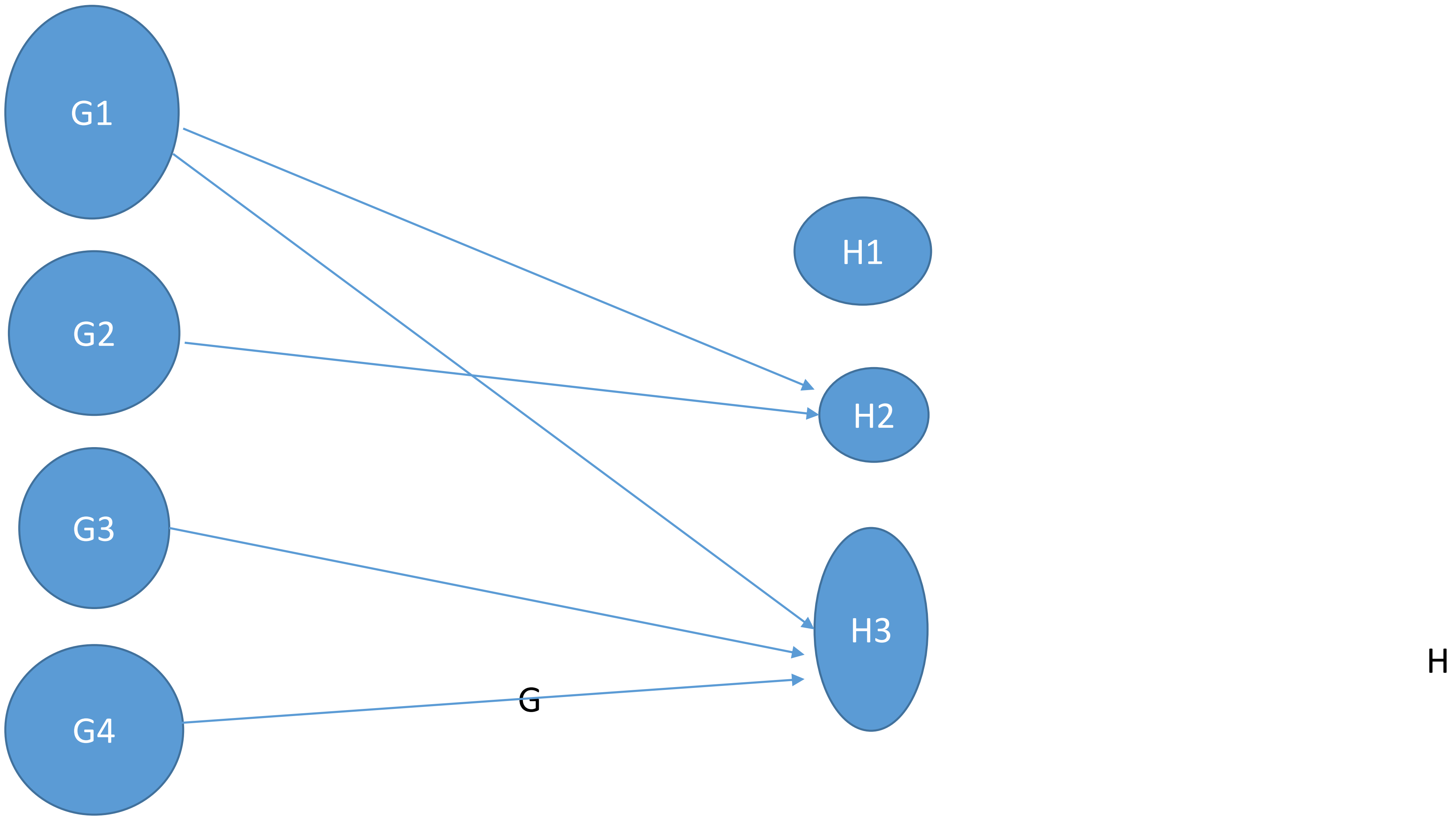
[1] Charles University, Prague, Czech Republic
[2] University of Oregon, Eugene, Oregon

Abstract. Given a fixed graph H, the H-cover problem asks whether an input graph G allows a degree preserving mapping $f : V(G) \to V(H)$ such that for every $v \in V(G)$, $f(N_G(v)) = N_H(f(v))$. In this paper, we design efficient algorithms for certain graph covering problems according to two basic techniques. The first one is a reduction to the 2-SAT problem. The second technique exploits necessary and sufficient conditions for the existence of regular factors in graphs. For other infinite classes of graph covering problems we derive $\mathcal{NP}$-completeness results by reductions from graph coloring problems. We illustrate this methodology by classifying all graph covering problems defined by simple graphs with at most 6 vertices.

Covers of disconnected graphs

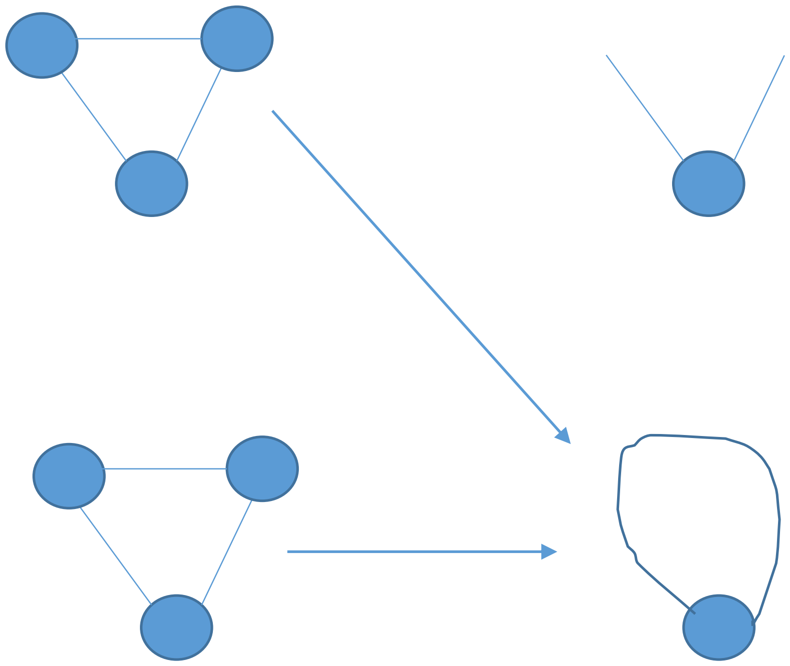


Covers of disconnected graphs



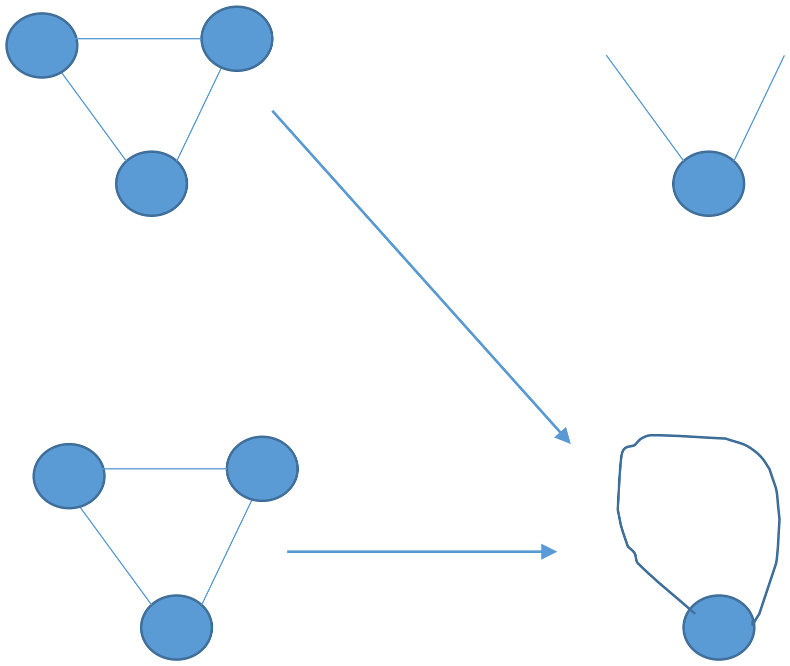
Covers of disconnected graphs

Locally bijective homomorphism

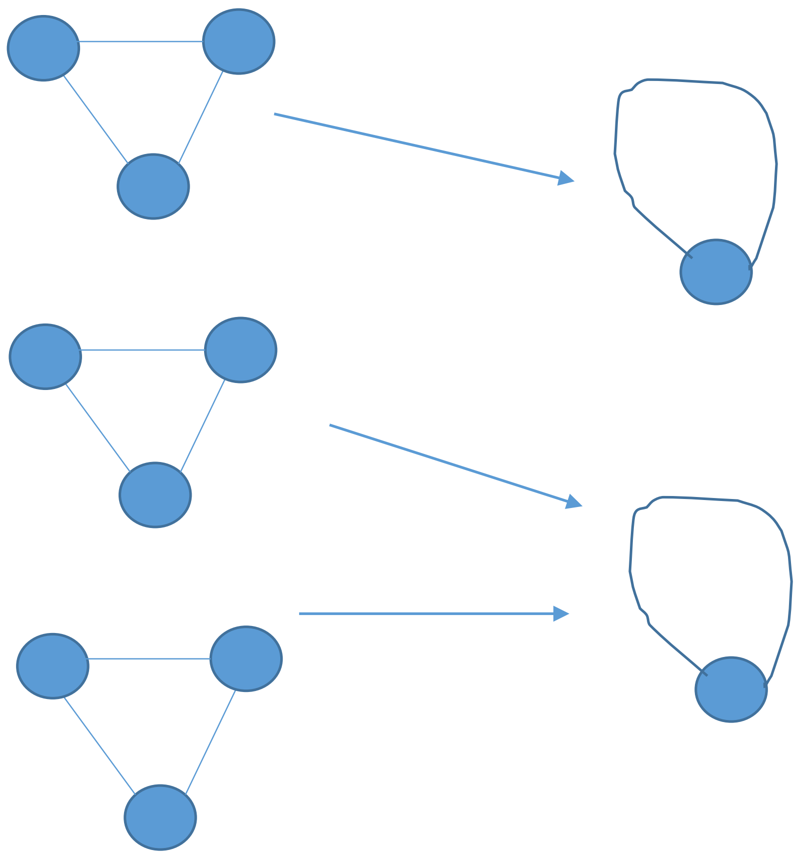


Covers of disconnected graphs

Locally bijective homomorphism

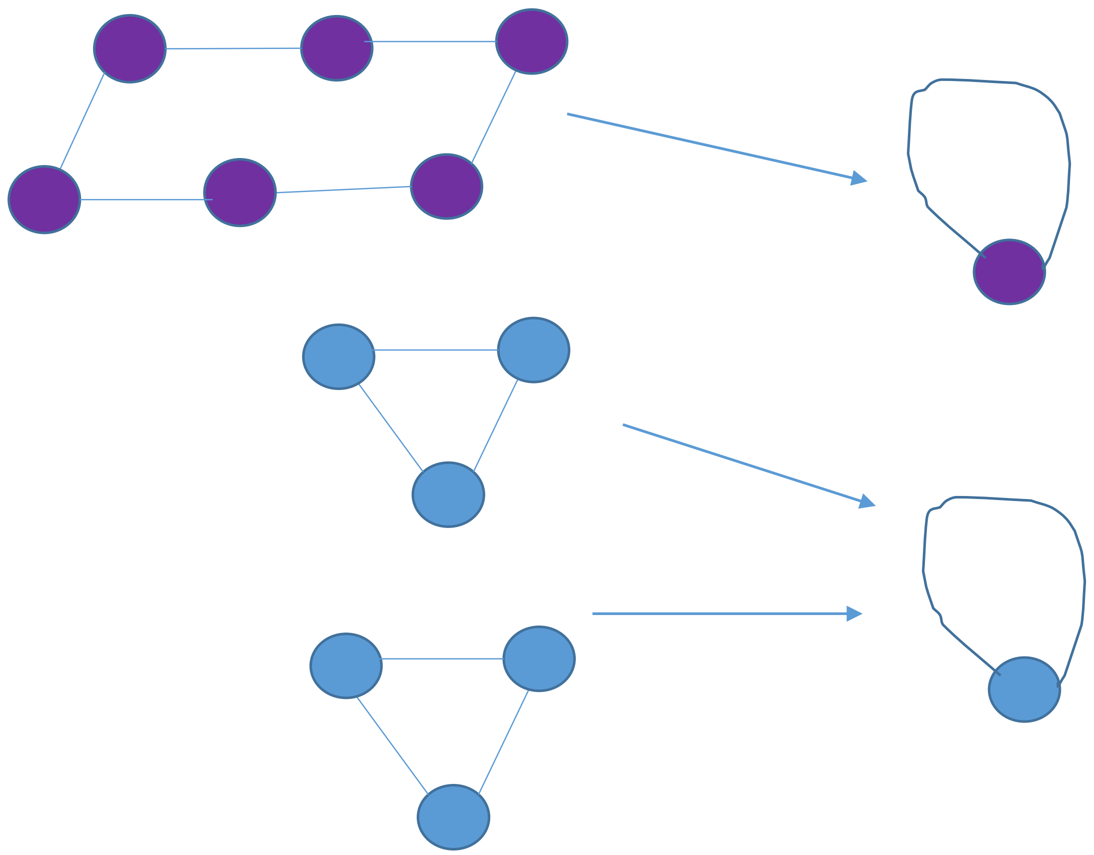


Surjective cover



Covers of disconnected graphs

Equitable cover



Computational complexity of covering disconnected graphs

Thm (Bok, Fiala, Jedlickova, Kratochvil, Seifertova FCT2021):
For a disconnected graph *H*,

- both the *H*-SURJECTIVE-COVER and *H*-EQUITABLE-COVER problems are polynomially solvable if the H_i-COVER problem is polynomially solvable for every connected component H_i of *H*, and
- both the *H*-SURJECTIVE-COVER and *H*-EQUITABLE-COVER problems are NP-complete for simple input graphs if the H_i-COVER problem is NP-complete for simple input graphs for some connected component H_i of *H*.

Computational complexity of covering disconnected graphs

Proof of “the *H*-SURJECTIVE-COVER problem is NP-complete for simple input graphs if the H_i-COVER problem is NP-complete for simple input graphs for some connected component H_i of *H*.“

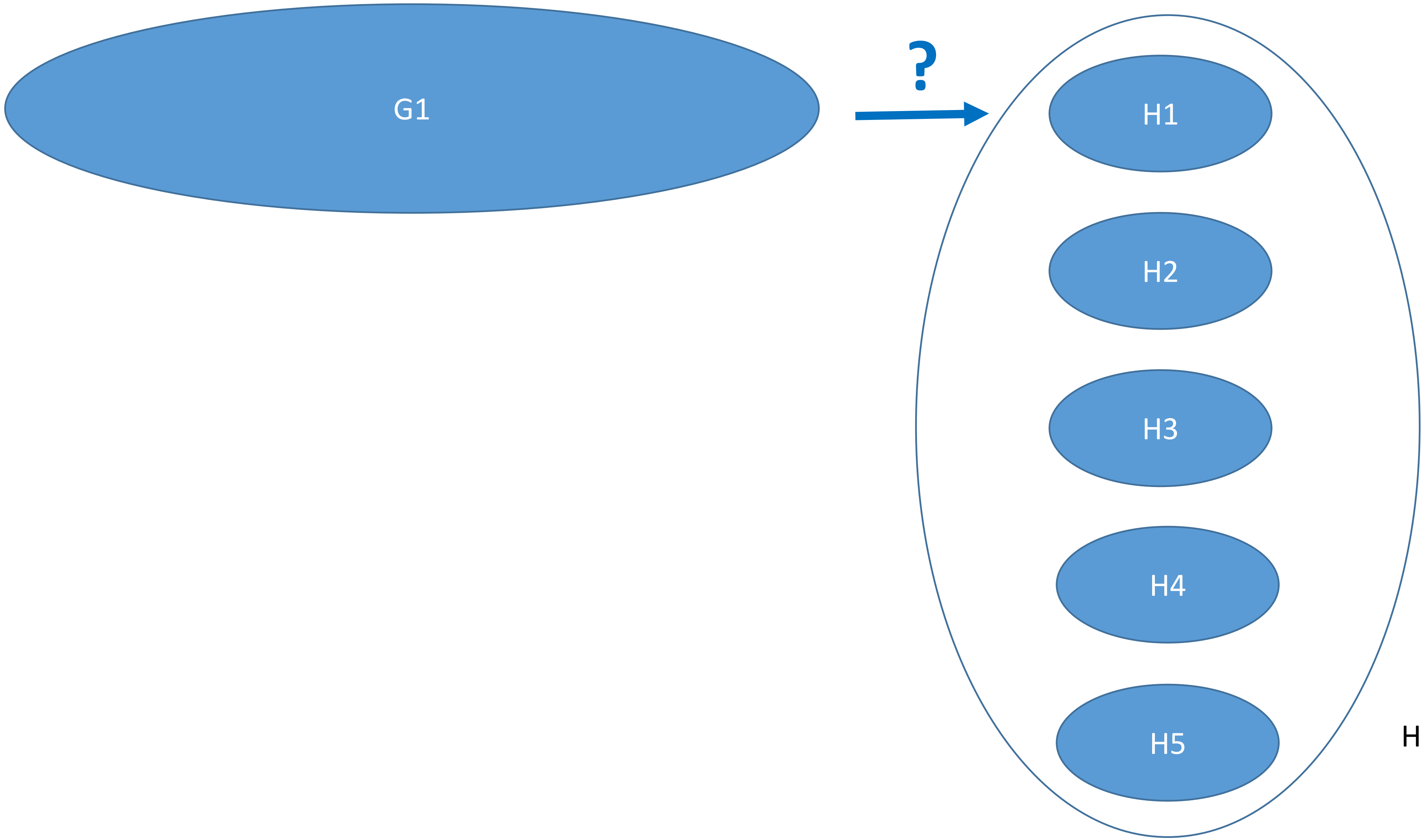
Computational complexity of covering disconnected graphs

Proof of "the *H*-SURJECTIVE-COVER problem is NP-complete for simple input graphs if the H_i-COVER problem is NP-complete for simple input graphs for some connected component H_i of *H*."

Let *H=H1+H2+…+Hk*. Suppose that *H1*-COVER is NP-complete for simple input graphs, and let *G1* be a simple graph whose covering of *H1* is to be tested. For each *j* =2,3,…,*k*, fix a simple graph *Gj* such that *Gj* covers *Hj*, and moreover *Gj* does not cover *H1*, unless *Hj* is such that every simple graph that covers *Hj* also covers *H1*.

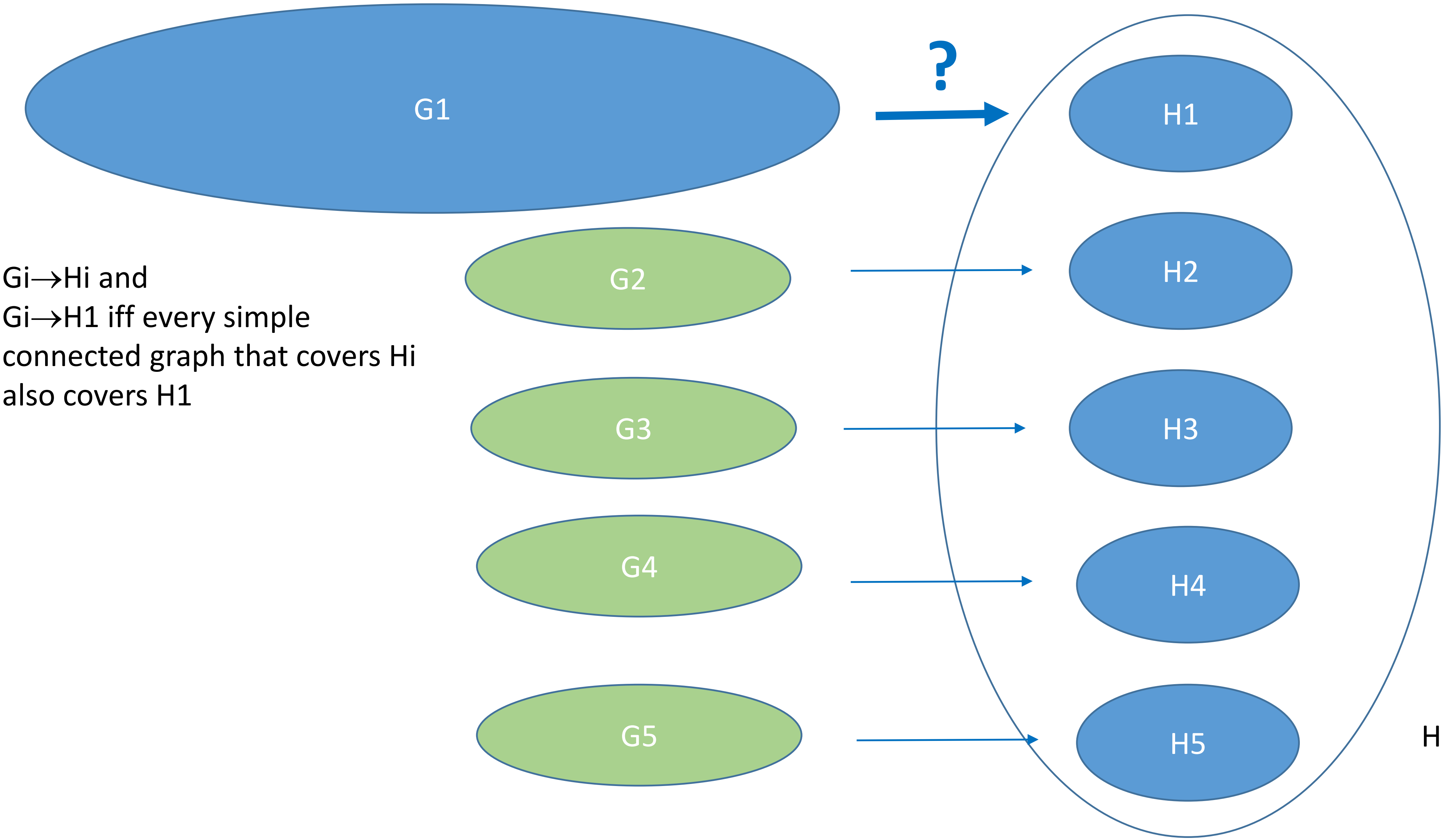
Then *G=G1+G2+…+Gk* surjectively covers *H* if and only if *G1* covers *H1*.

Computational complexity of covering disconnected graphs



Computational complexity of covering disconnected graphs

Gi→Hi and
Gi→H1 iff every simple connected graph that covers Hi also covers H1



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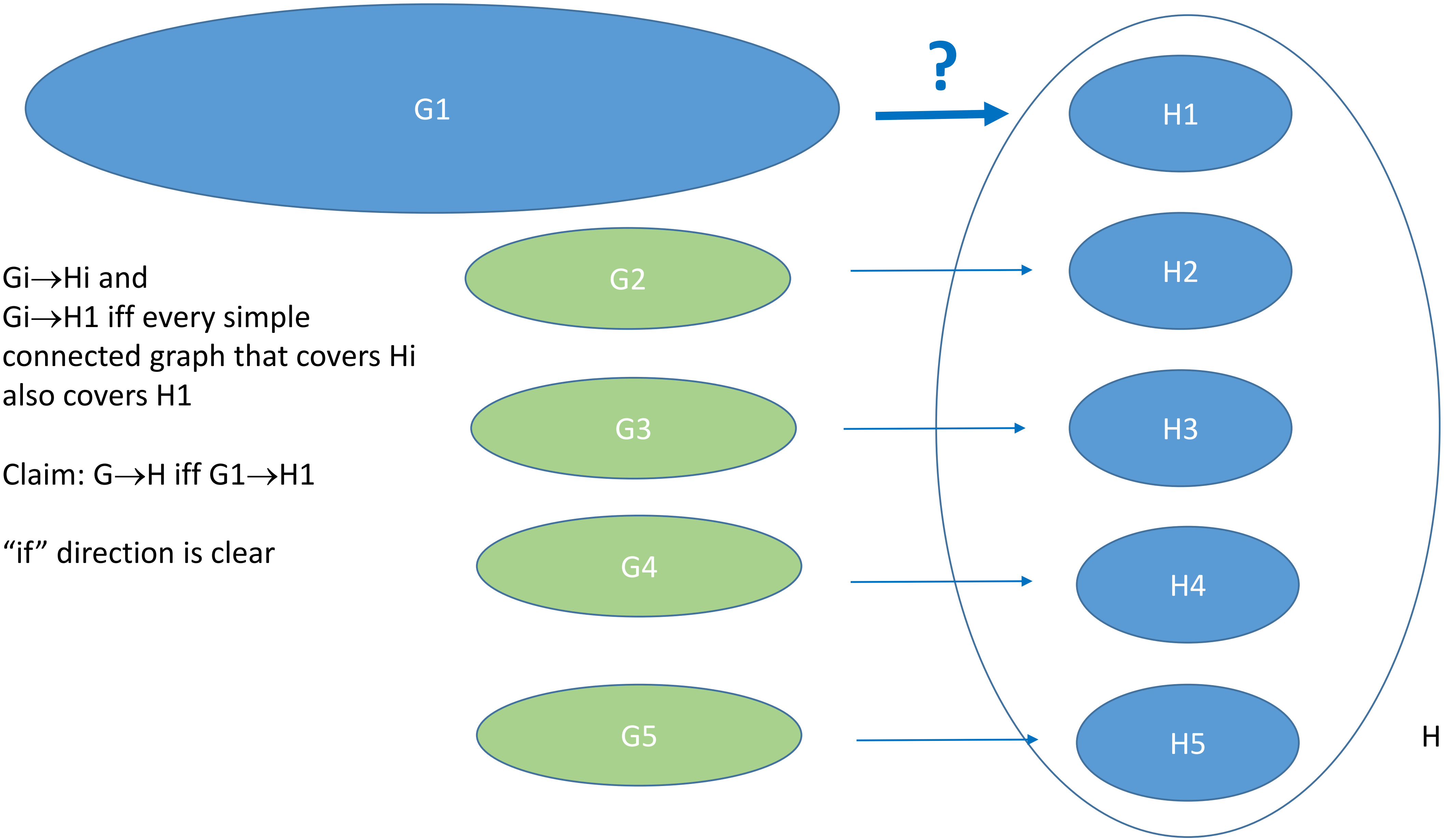
Claim: G→H iff G1→H1

"if" direction is clear



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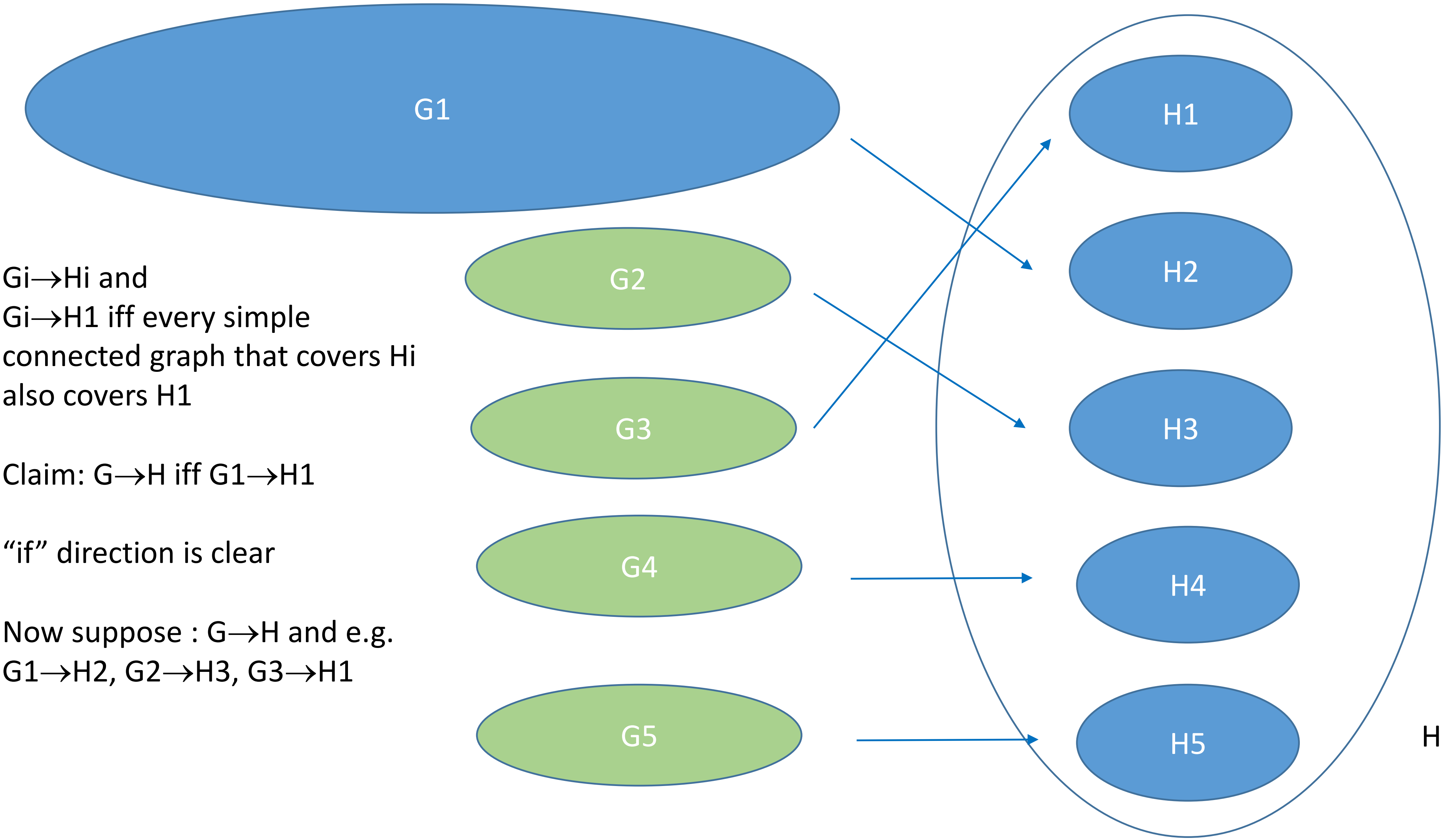
G1→H2, G2→H3, G3→H1



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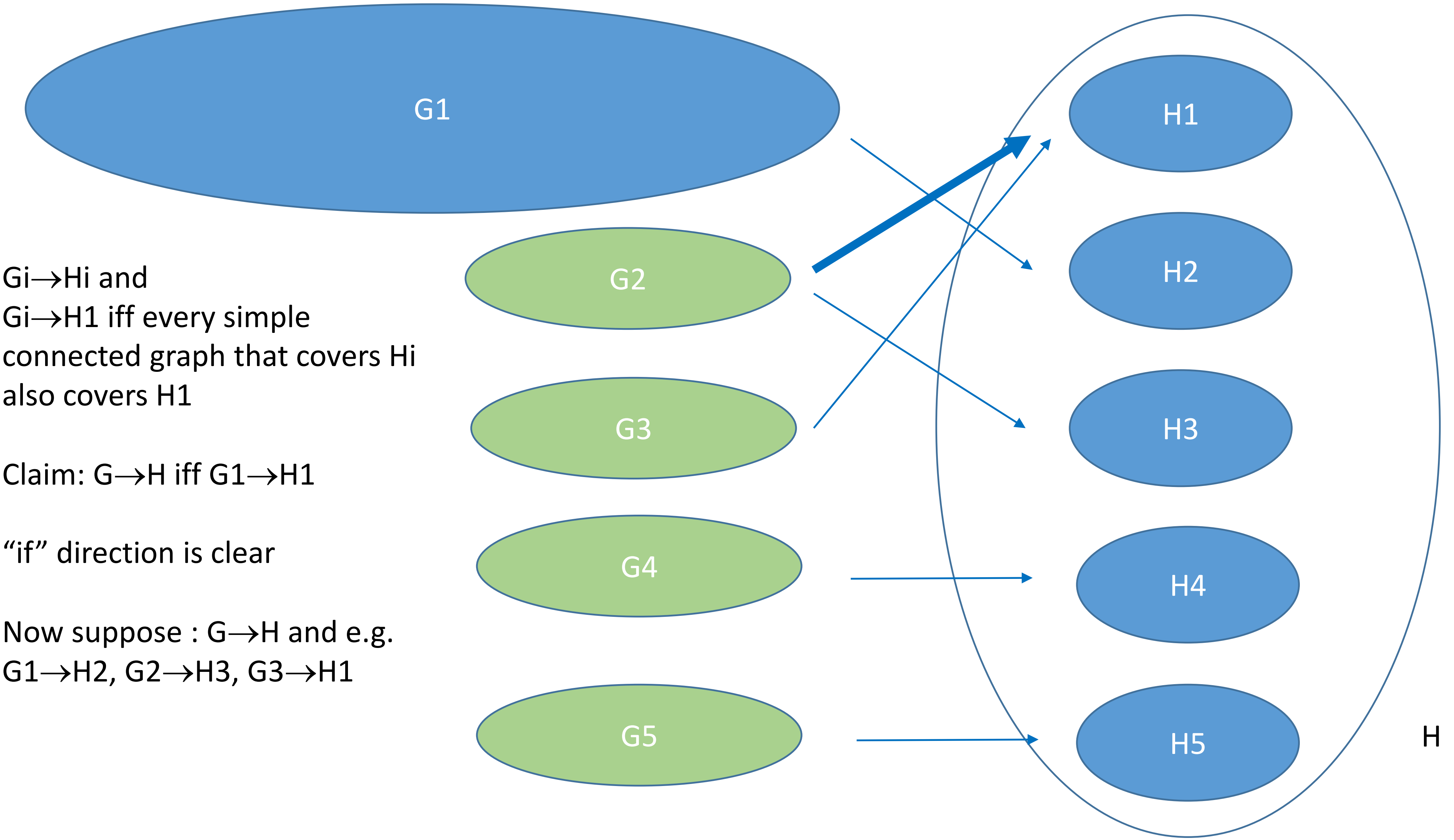

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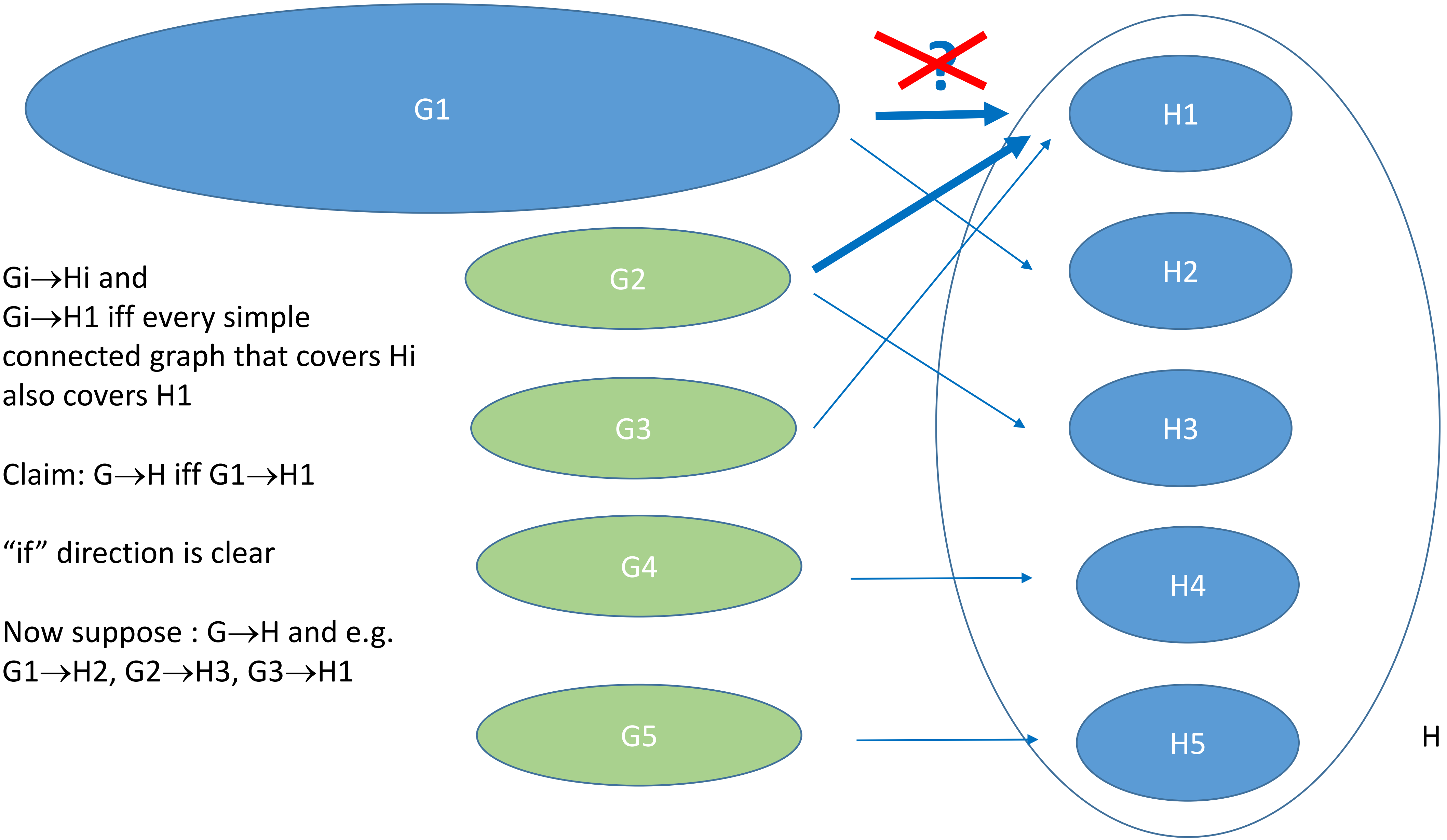

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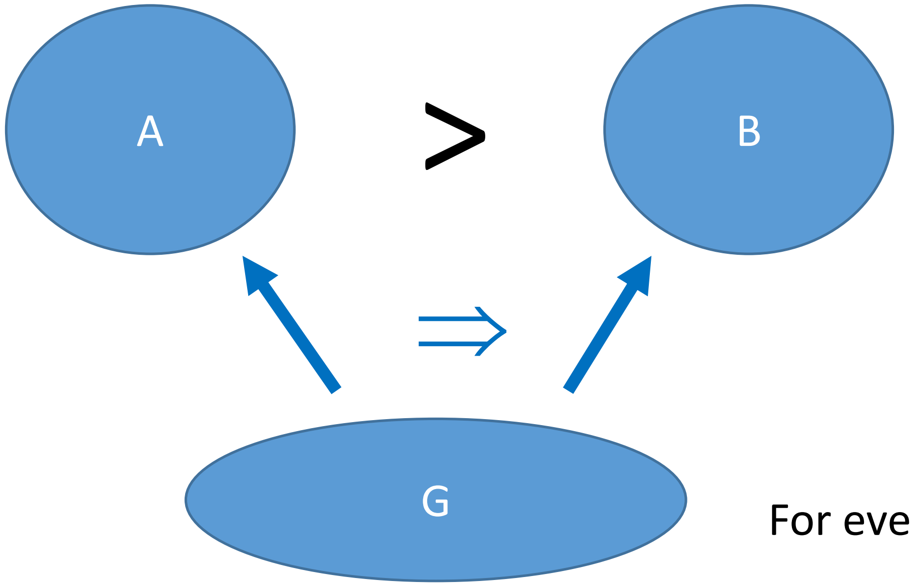
> relation on connected graphs

Definition: Given connected graphs A and B, we say that $A > B$ if for every simple graph G, it is true that G covers B whenever G covers A.



For every simple connected *G*

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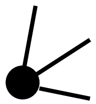
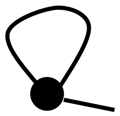
Example 2:  > 

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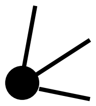
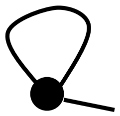
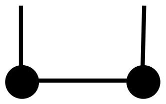
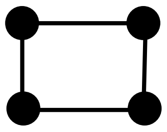
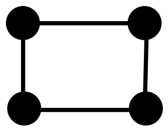
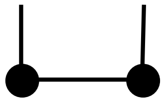
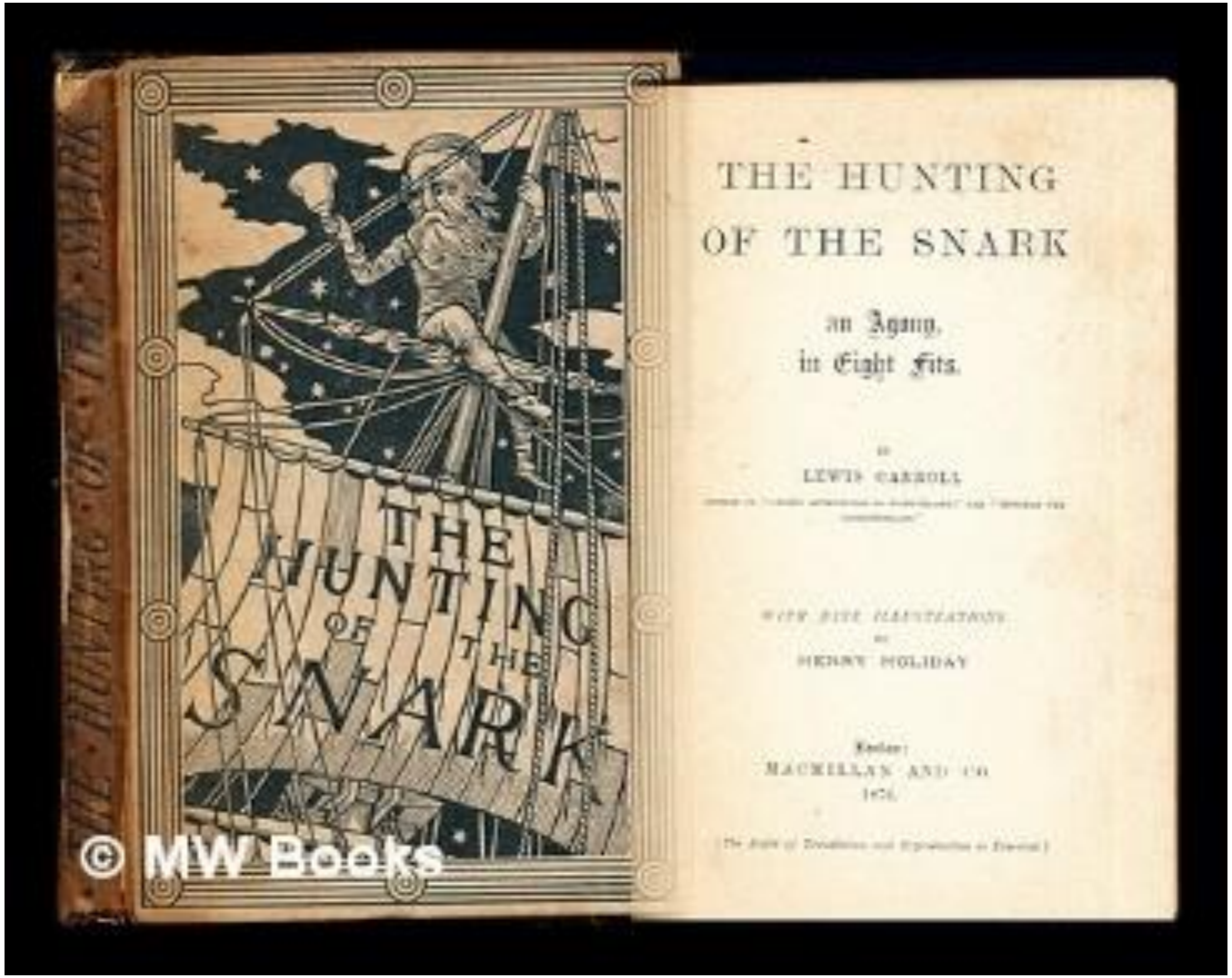
Example 2:  > 

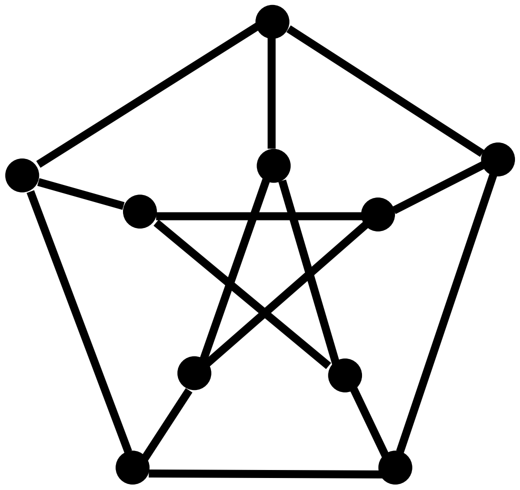
Example 3:  >  and  > 

Hunting for Snarks



Snarks

Definition: A cubic 2-connected graph is a **snark** if it is not 3-edge-colorable.



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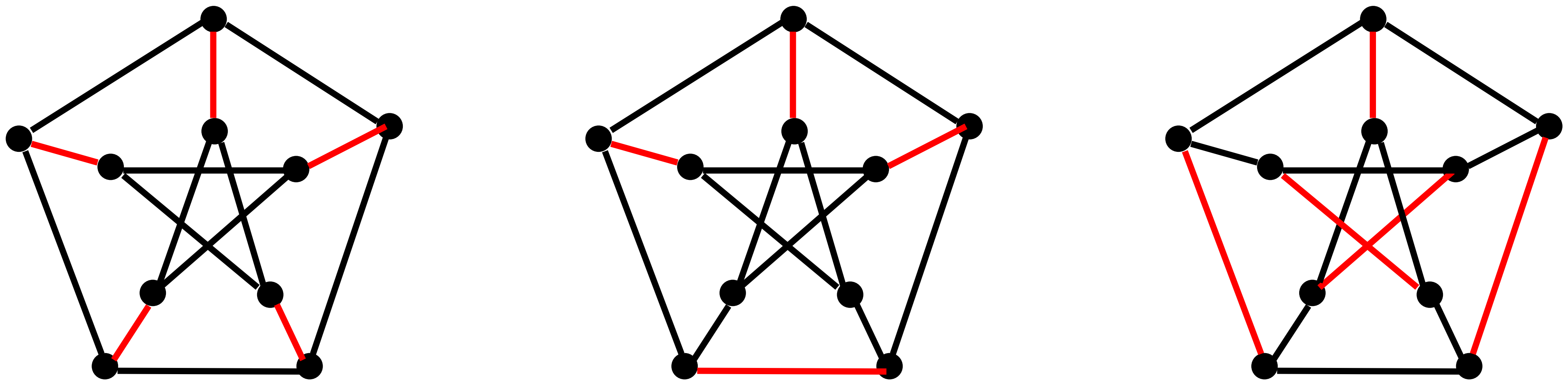

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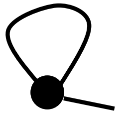
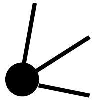
We know that ¬ ( > ). 2-connected witnesses are **snarks**.

Four Color Theorem

1852 Francis Guthrie: Can every planar map (planar graph) be colored by 4 colors?
Augustus De Morgan, Arthur Cayley
Proofs:
1879 Alfred Kempe, showed incorrect by Percy Heawood in 1890
1880 Peter Guthrie Tait, showed incorrect by Julius Petersen in 1891
1890 Percy Heawood **Five colors suffice**
1880 P. G. Tait **FCT is equivalent to non-existence of planar snarks**
1976 Appel, Haken computer assisted proof
1996 Robertson, Sanders, Seymour, Thomas simplified still computer assisted proof

Generalized snarks

Question: If $\neg(A>B)$, then there is a witness G (a simple graph) such that G covers A but G does not cover B. How big would such a witness be? Can such a witness be constructed easily?

> relation on connected graphs

Open problem: Describe all pairs of connected graphs A and B such that $A > B$ and A does not cover B.

Conjecture (Bok et al. 2022): If A has no semi-edges, then
$A > B$ if and only if A covers B.

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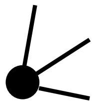
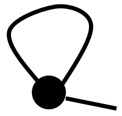
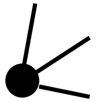
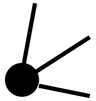
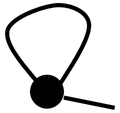
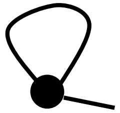
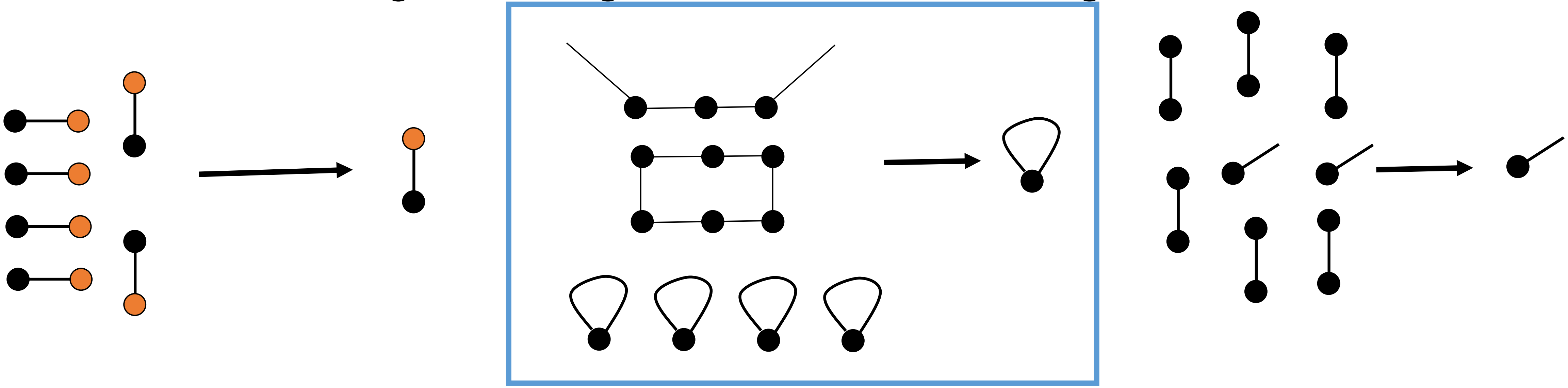
JK, Nedela (EUROCOMB 2023): True for B =  and B = 
with arbitrary A.

> relation on connected graphs

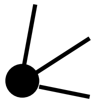
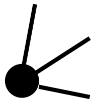
Thm 1 (JK,RN): For any graph *A*, *A* >  iff *A* →  .

Thm 2 (JK,RN): For any graph *A*, *A* >  iff *A semi-covers*  .

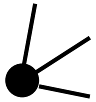
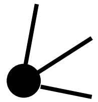
Definition: Preimages of edges in a semi-covering

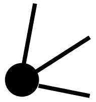
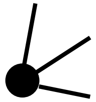


Sketch of proof of Thm 1

Thm 1 (JK,RN): For any graph A, $A >$  iff $A \to$  .

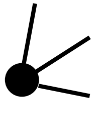
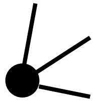
Proof: "$\Leftarrow$" is obvious.

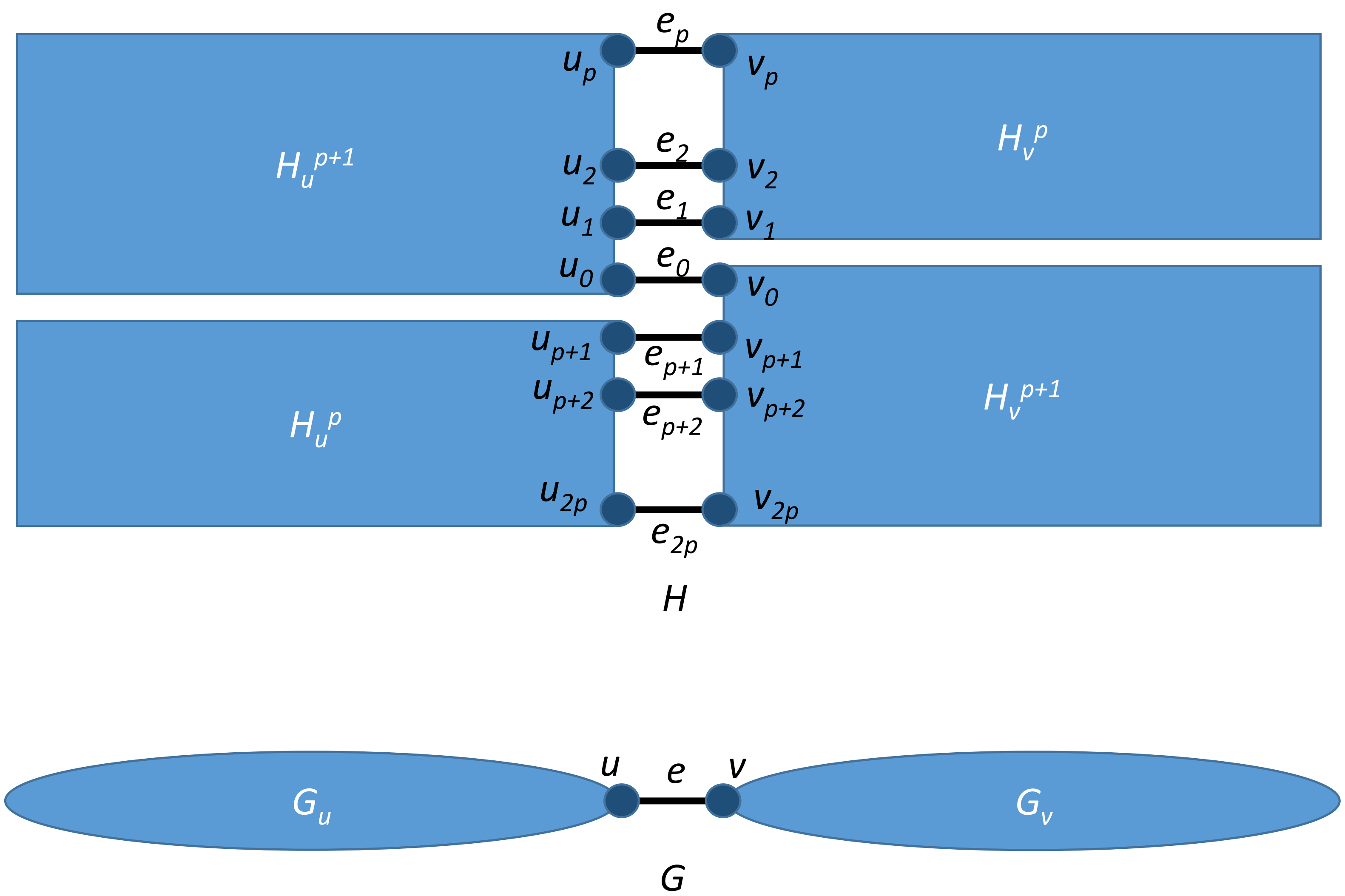
"$\Rightarrow$" We prove "$A \not\to$  $\Rightarrow \exists$ simple $H \to A$ s.t. $H \not\to$  ."

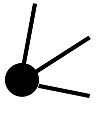
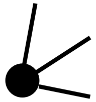
$A \not\to$  $\Rightarrow \exists$ simple $H \to A$ s.t. $H \not\to$ 

Case 1: *A* has no semi-edges

Case 1.1: If *A* has a bridge, then *A* has a simple cover which has a bridge.

$A \not\to$  $\Rightarrow \exists$ simple $H \to A$ s.t. $H \not\to$ 



$A \not\to$  $\Rightarrow \exists$ simple $H \to A$ s.t. $H \not\to$ 

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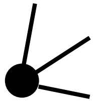
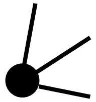
$A \not\to$  $\Rightarrow \exists$ simple $H \to A$ s.t. $H \not\to$ 

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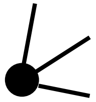
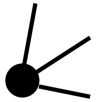
$A \not\to$  $\Rightarrow \exists$ simple $H \to A$ s.t. $H \not\to$ 

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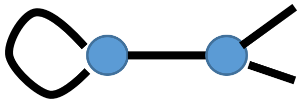
Case 2: *A* has semi-edges

Consider A^o, show $\chi'(A^o) = \chi'(A) > 3$, and by Case 1, A^o (and hence also *A*) has a simple cover *H* with $\chi'(H)>3$, the witness.



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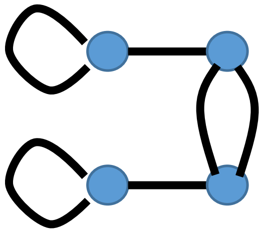

A^o

FACE

Covering directed graphs

Thm (JK, Proskurowski, Telle + Fiala 1997): If *H* is simple undirected *k*-regular graph, *k*>2, then *H*-COVER is NP-complete.

Covering directed graphs

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Thm (Bok, Fiala, Hlineny, Jedlickova, JK 2021): If H is semi-simple undirected k-regular graph, $k>2$, then H-COVER is NP-complete.

Conjecture: If H is simple connected directed k-in-k-out-regular graph with $k>2$, then H-COVER is NP-complete.

Covering directed graphs

Observation: If *H* is connected undirected 2-regular graph, then *H*-COVER is polynomial time solvable.

Question: What about connected directed 2-in-2-out-regular graphs?

Covering directed graphs

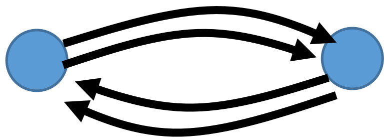
Observation: If *H* is connected undirected 2-regular graph, then H-COVER is polynomial time solvable.

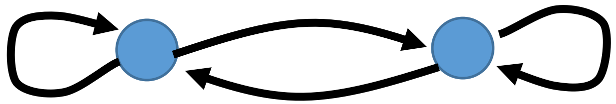
Question: What about connected directed 2-in-2-out-regular graphs?

Answer: A complete jungle.

Covering directed 2-in-2-out regular graphs

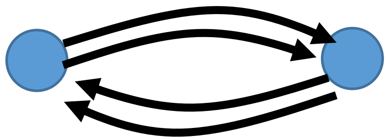
2-vertex graphs



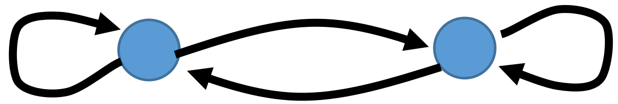


Covering directed 2-in-2-out regular graphs

2-vertex graphs



Polynomial time



Covering directed 2-in-2-out regular graphs

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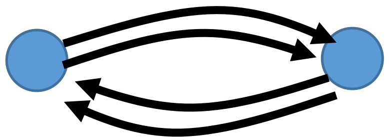

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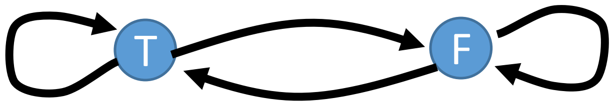
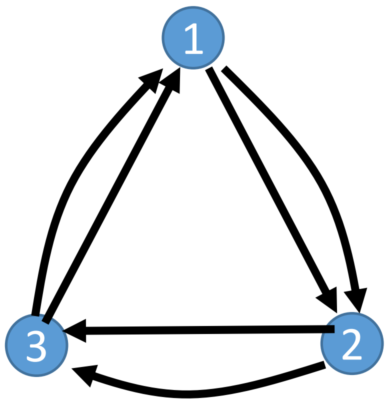


Polynomial time
via 2-SAT

Covering directed 2-in-2-out regular graphs

3-vertex graphs



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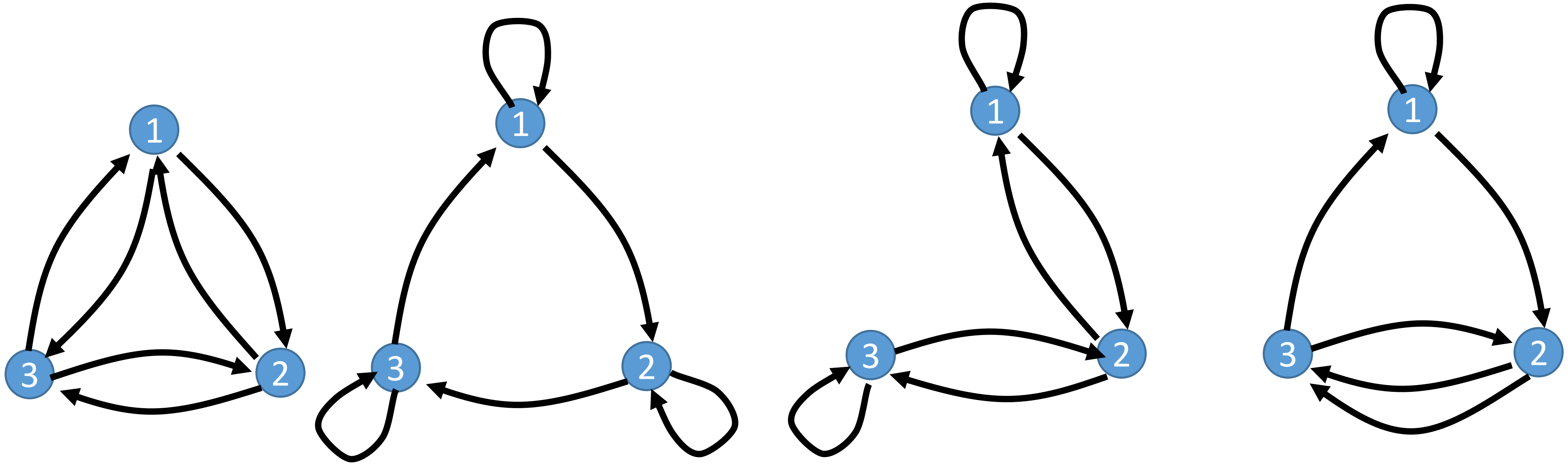
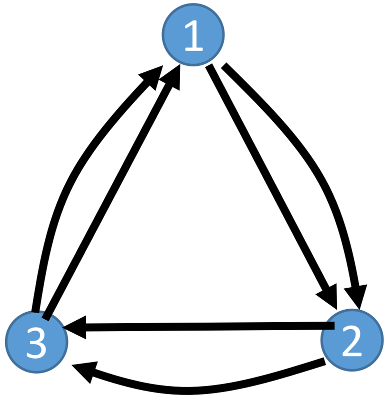
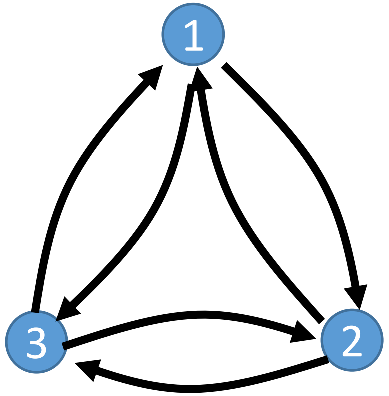


Covering directed 2-in-2-out regular graphs

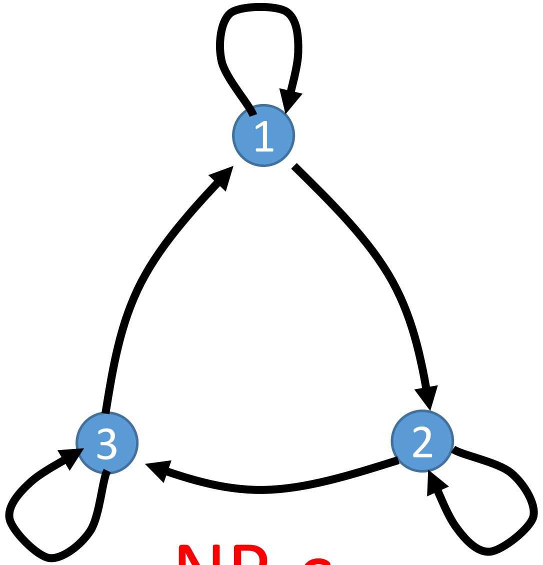
3-vertex graphs



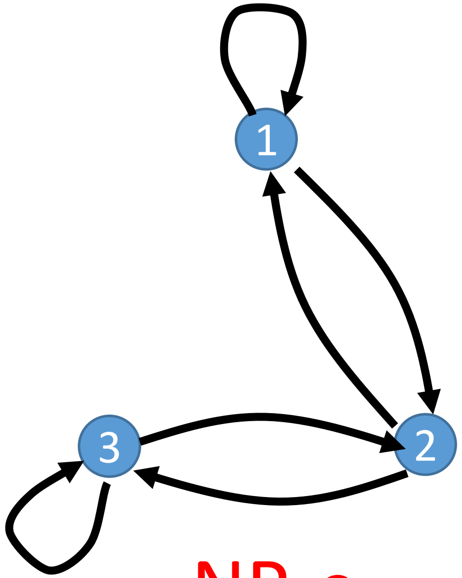
Polynomial time



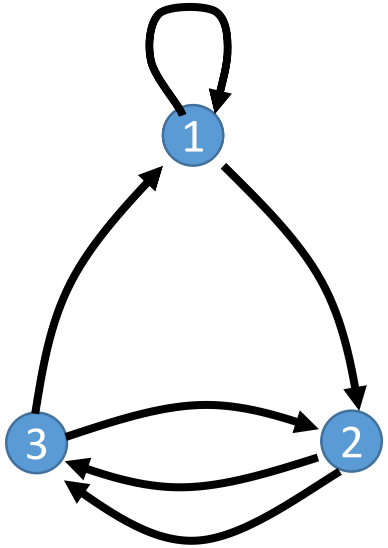
NP-c



NP-c



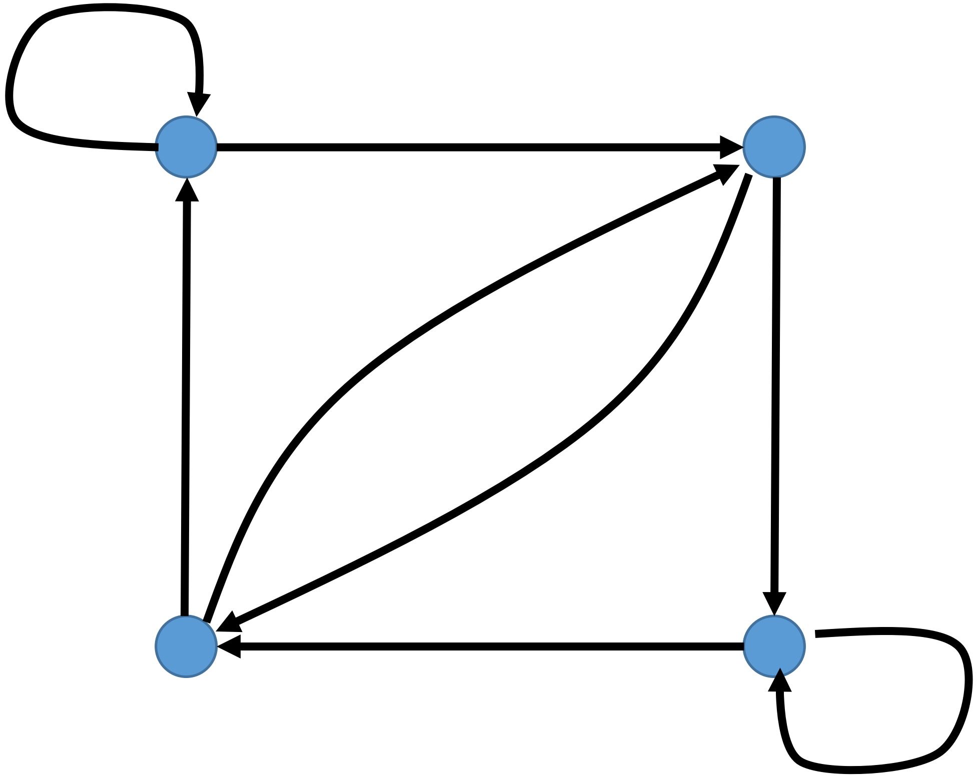
NP-c



NP-c

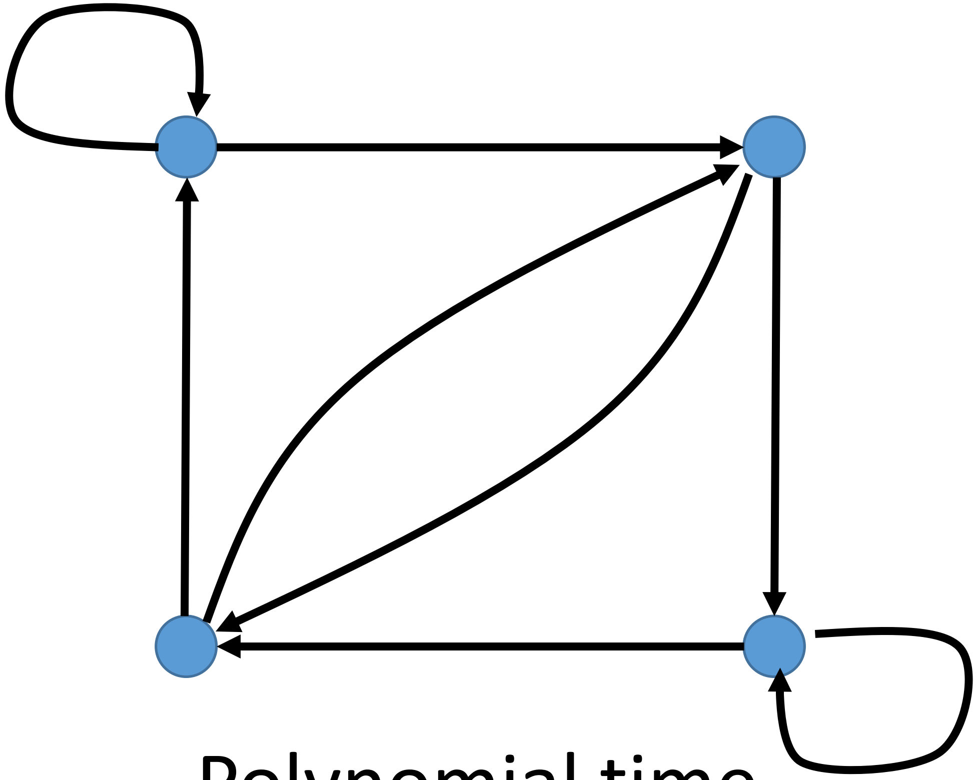
Covering directed 2-in-2-out regular graphs

4-vertex graphs



Covering directed 2-in-2-out regular graphs

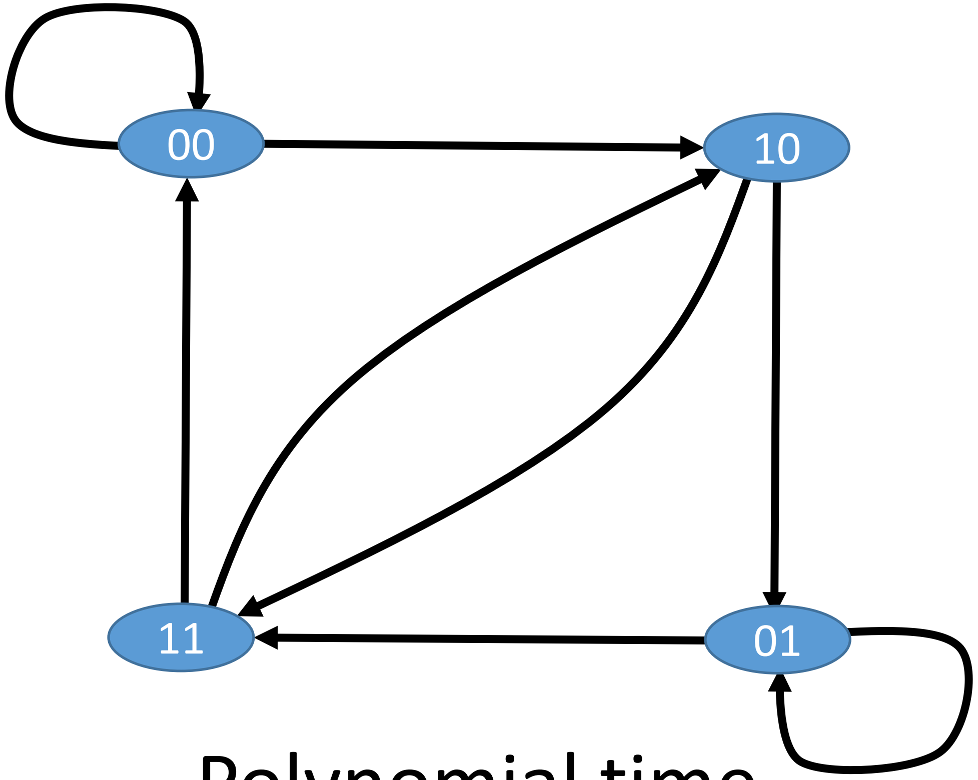
4-vertex graphs



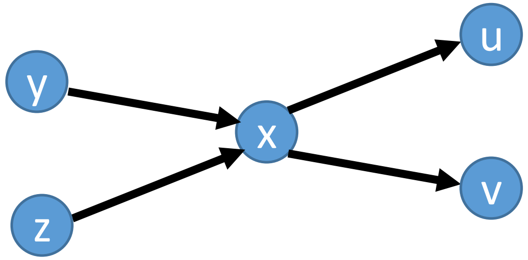
Polynomial time

Covering directed 2-in-2-out regular graphs

4-vertex graphs



Polynomial time



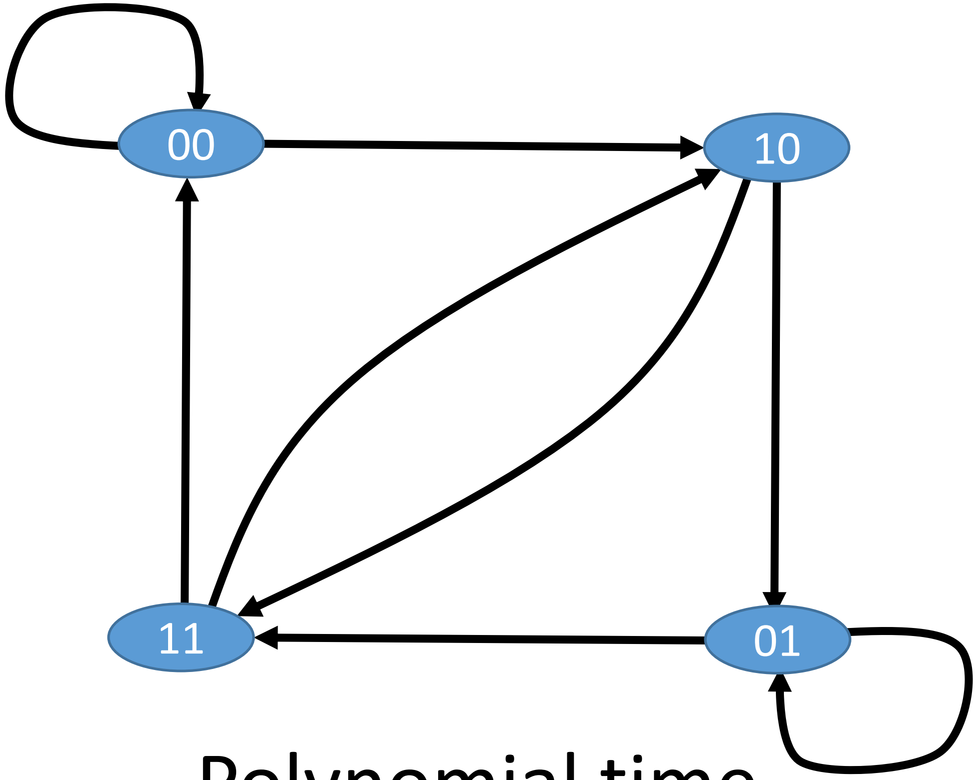
x : $(x(1),x(2)) \in GF(2)^2$

$u(2)+v(2)=0$

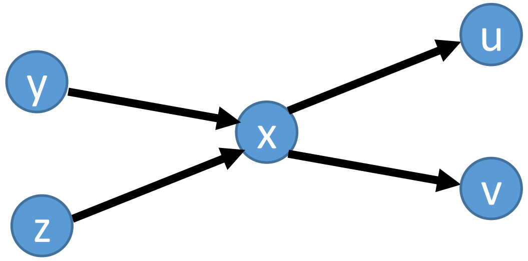
$u(1)+v(1)=1$

Covering directed 2-in-2-out regular graphs

4-vertex graphs



Polynomial time



x : $(x(1),x(2)) \in GF(2)^2$

$u(2)+v(2) = 0$

$u(1)+v(1) = 1$

$x(1)+x(2)+u(2)=0$

$y(2)+z(2) = 1$

$y(1)+z(1) = 1$

$y(1)+y(2)+x(2) = 0$

Thank you

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