Hamiltonicity of Dense Graphs

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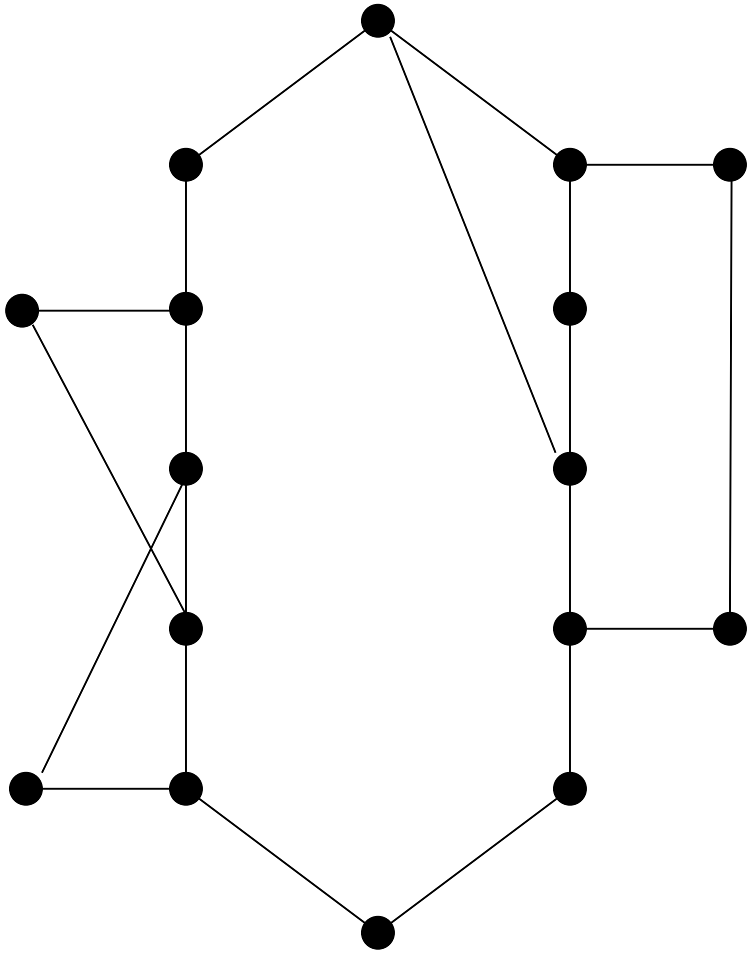
Hamiltonian path – a path in a graph passing through every vertex exactly once



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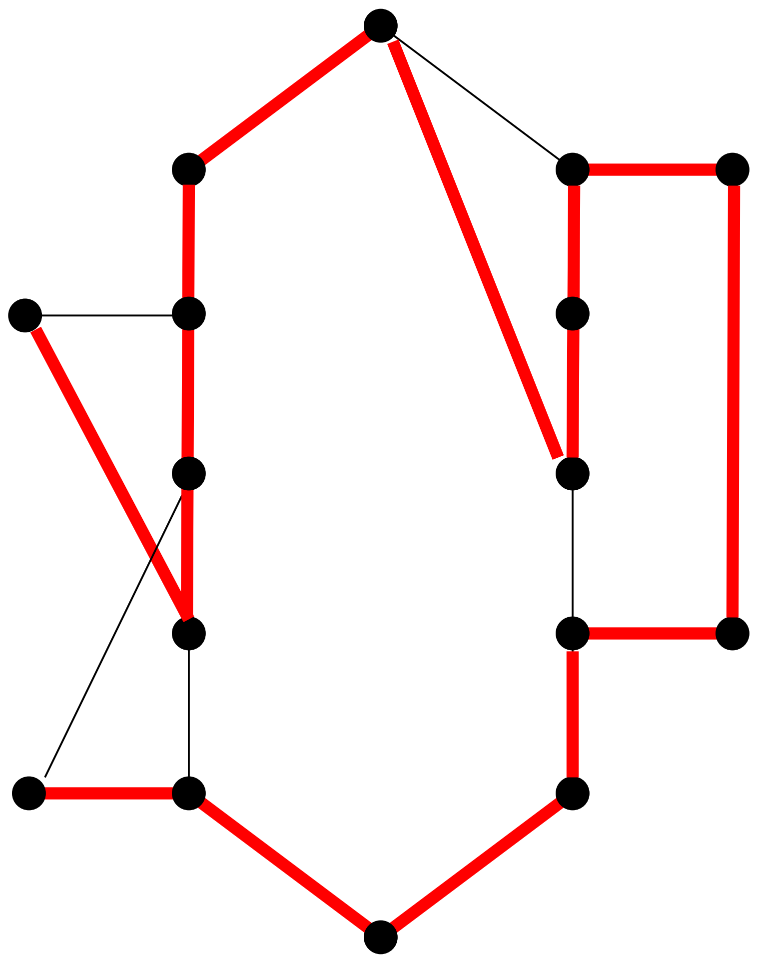

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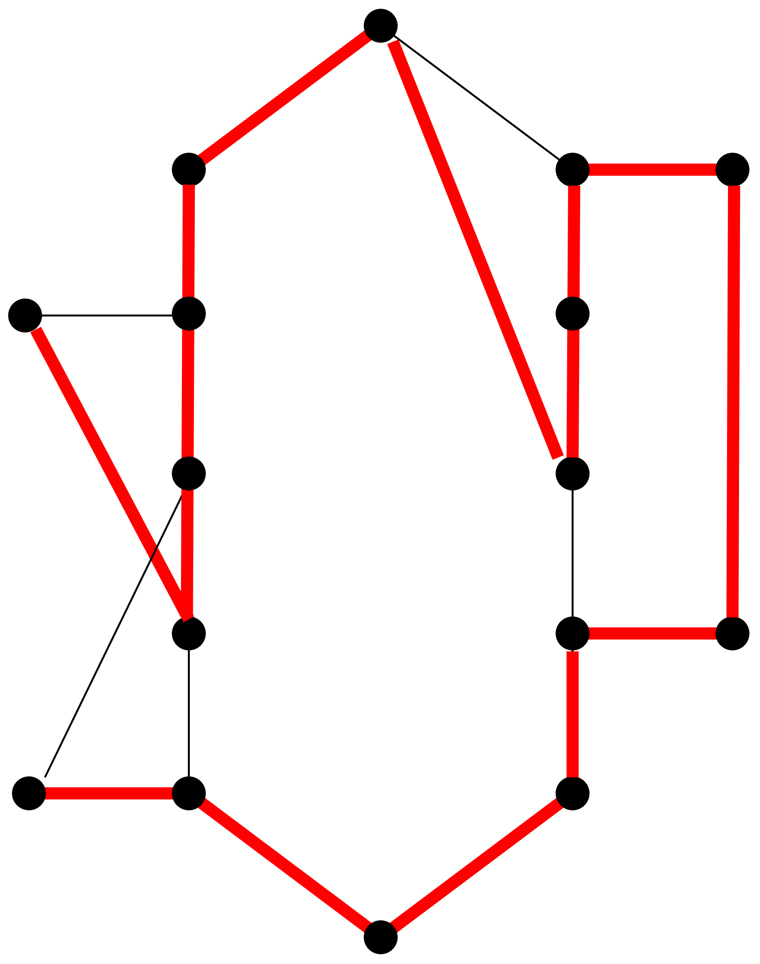

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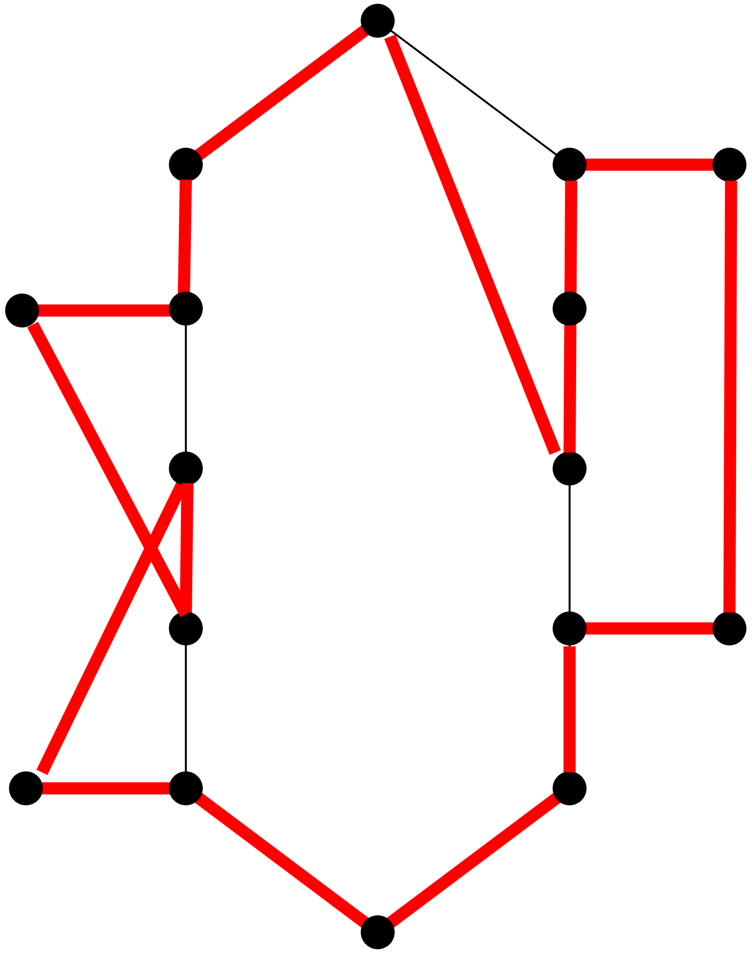

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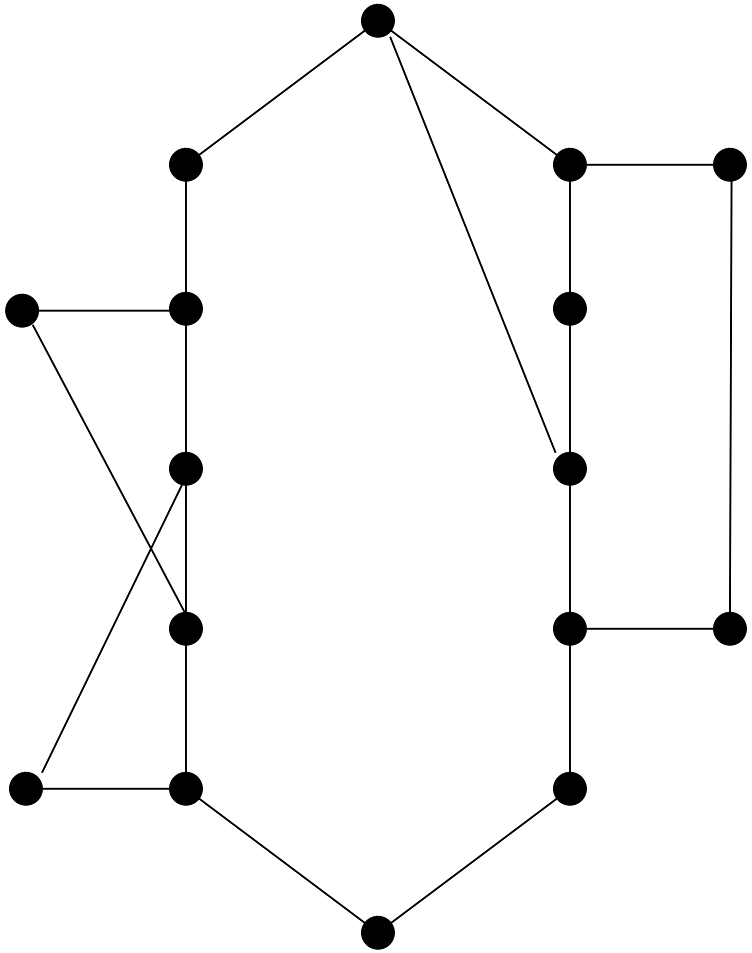

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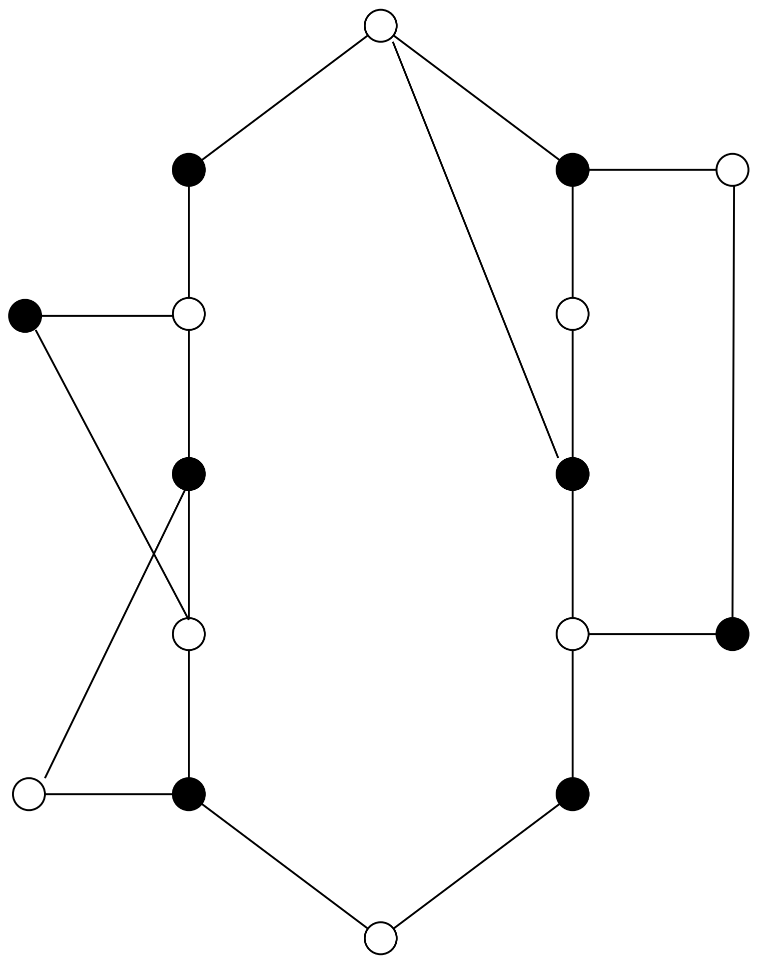
The graph is **Hamilton connected** if every pair of vertices is connected by a Hamiltonian path



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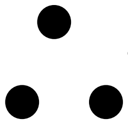
Theorem (Chvátal, Erdös 1972): Let G be a vertex-s-connected graph. Then

- If $\alpha(G)<s+2$, then G has a Hamiltonian path
- If $\alpha(G)<s+1$, then G has a Hamiltonian cycle
- If $\alpha(G)<s$, then G is Hamilton connected.

Dense graphs = bounded $\alpha(G)$

Observation: $\alpha(G) < k$ iff G is kK_1-free

 -free = $3K_1$-free = complement of a triangle-free graph

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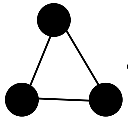 -free NP-complete

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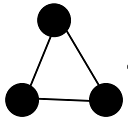
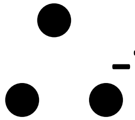
 -free NP-complete

 -free polynomial time

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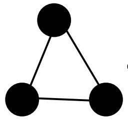
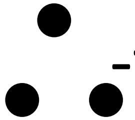
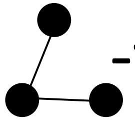
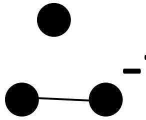
-free   NP-complete

-free   polynomial time

-free and -free   polynomial time, since subclasses of co-graphs

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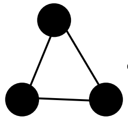
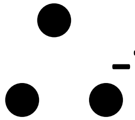
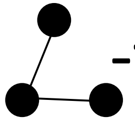
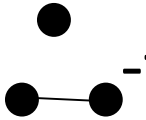
-free   NP-complete

-free   polynomial time

-free and -free   polynomial time, since subclasses of co-graphs

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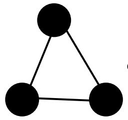
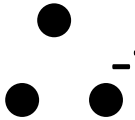
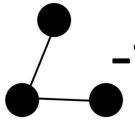
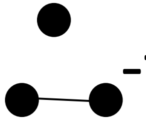
 -free NP-complete

 -free polynomial time

 -free and  -free polynomial time, since subclasses of co-graphs

Even $4K_1$-free was open

Our result [arXiv:2309.09228]: For every k, HamPath and HamCycle are polynomial time solvable on kK_1-free graphs.

Path Cover and Hamiltonian Linkage

Def: The *path cover number* of a graph is the minimum number of paths that cover all vertices.

Observation: $pc(G)=1$ iff G has a Hamiltonian path

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Def: HamLinkage = PathCover with specified end-vertices of the paths

A graph is *Ham-ℓ-linkable* if it has a Ham linkage for every choice of 2ℓ end-vertices of the paths

Observation: G is Ham-1-linkable iff it is Hamiltonian connected

Theorem [arXiv:2309.09228]: For every k and ℓ, Ham-ℓ-Linkage and Ham-ℓ-Linkability are polynomial time solvable on kK_1-free graphs.

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Theorem [arXiv:2309.09228]: For every k and ℓ, if a graph G satisfies $\alpha(G) < k$ and $c_v(G) \geq \max\{k\ell, 10\ell\}$, then G is Hamiltonian-ℓ-linkable.

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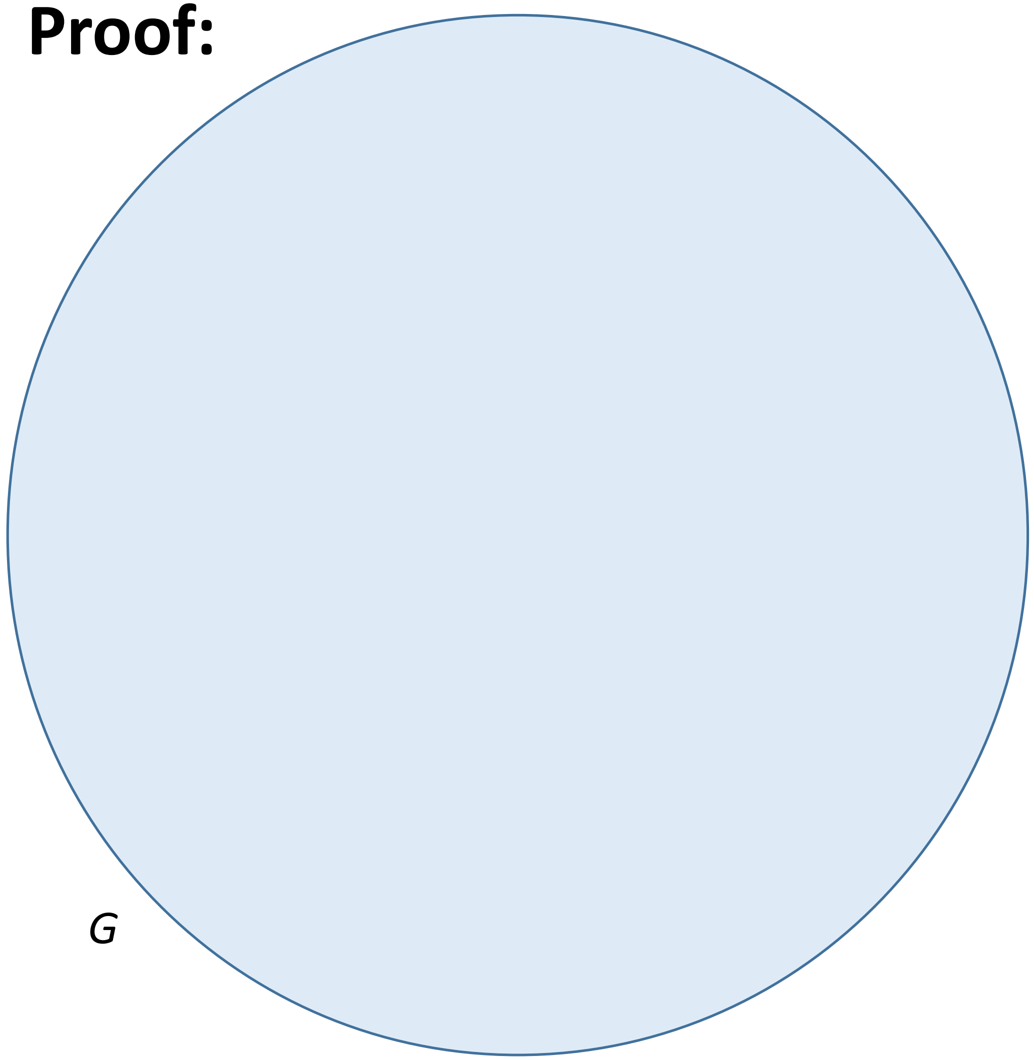

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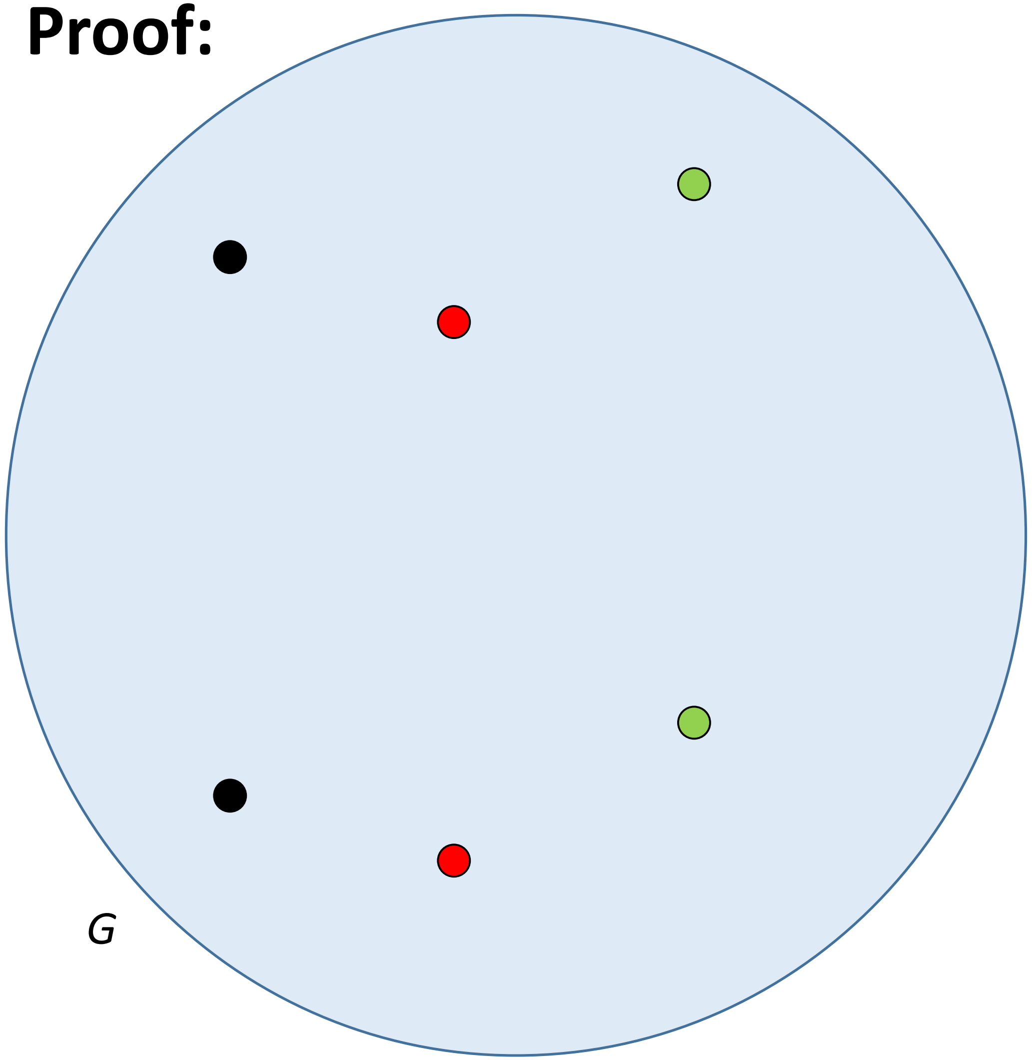

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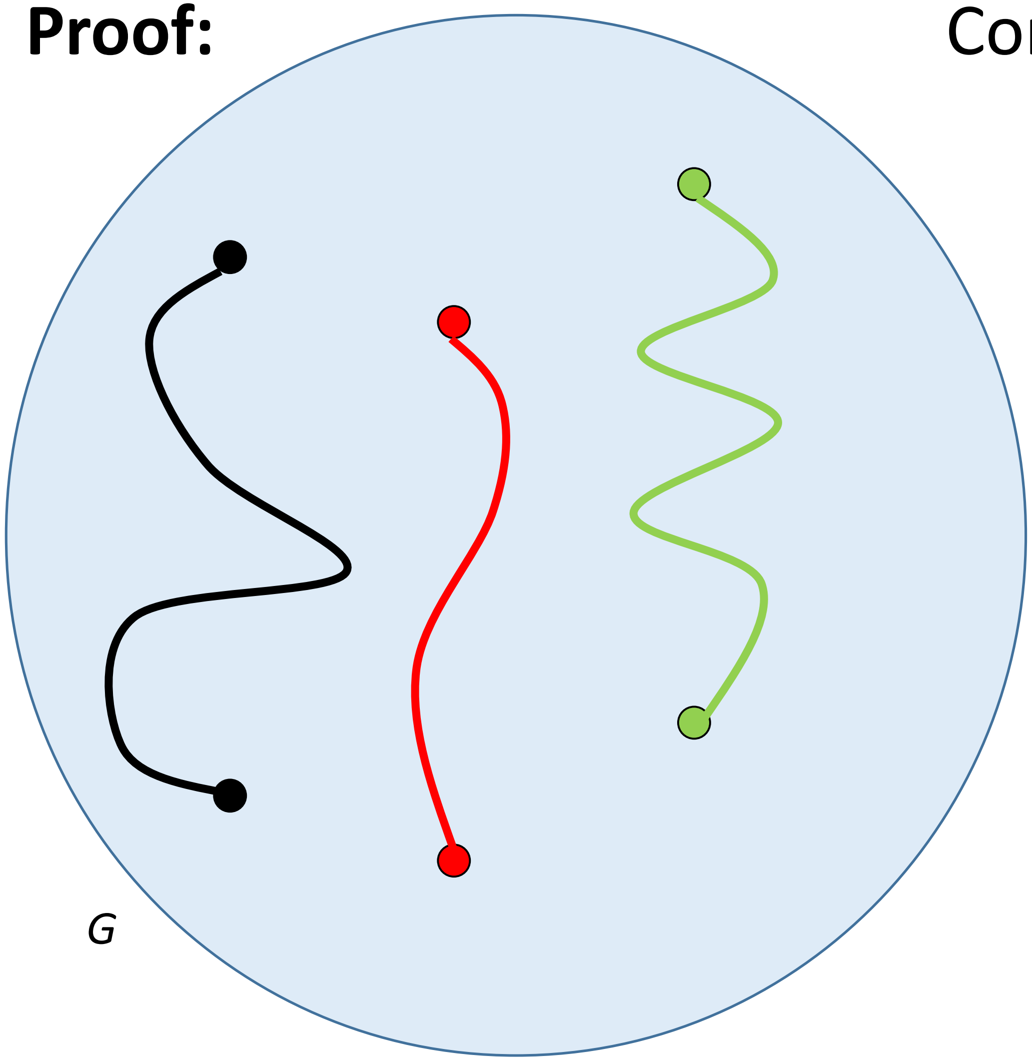
Consider a linkage of maximum size



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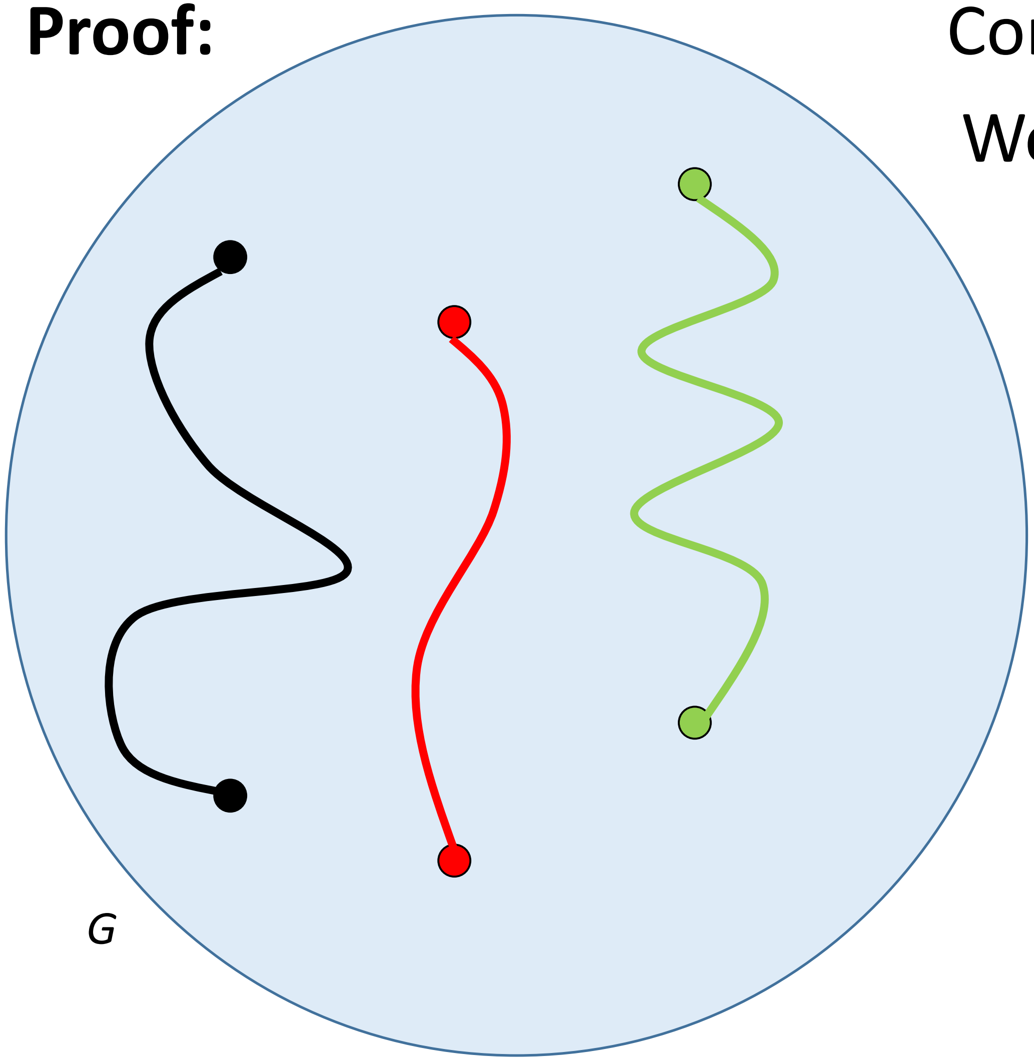

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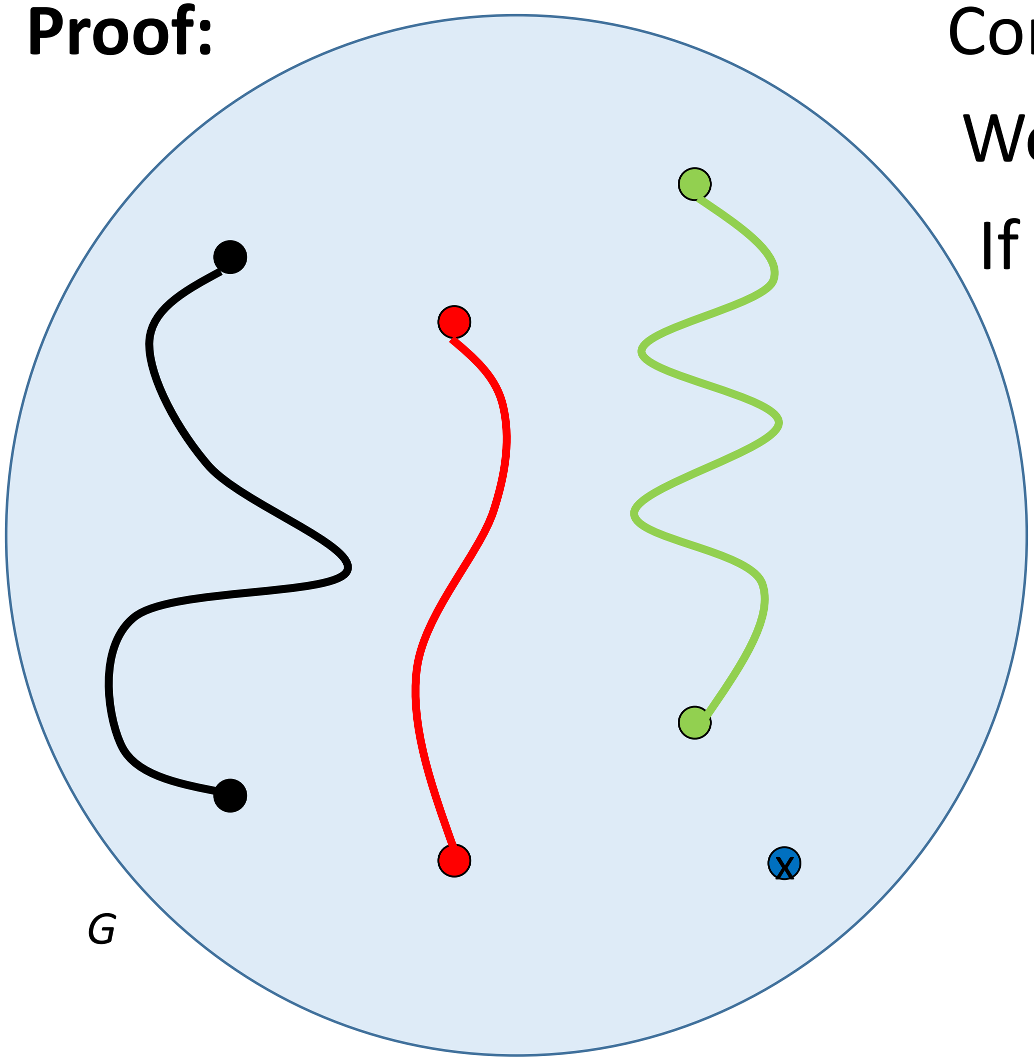

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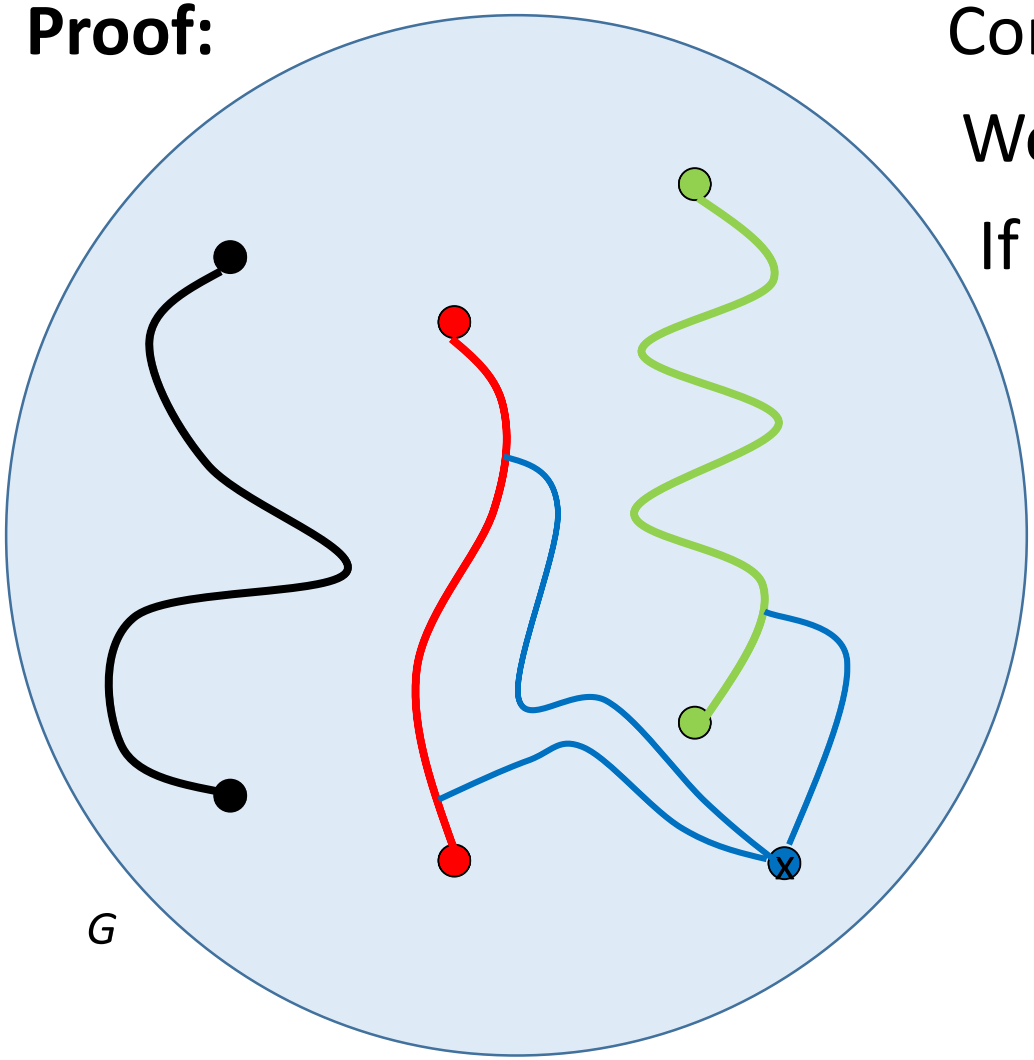

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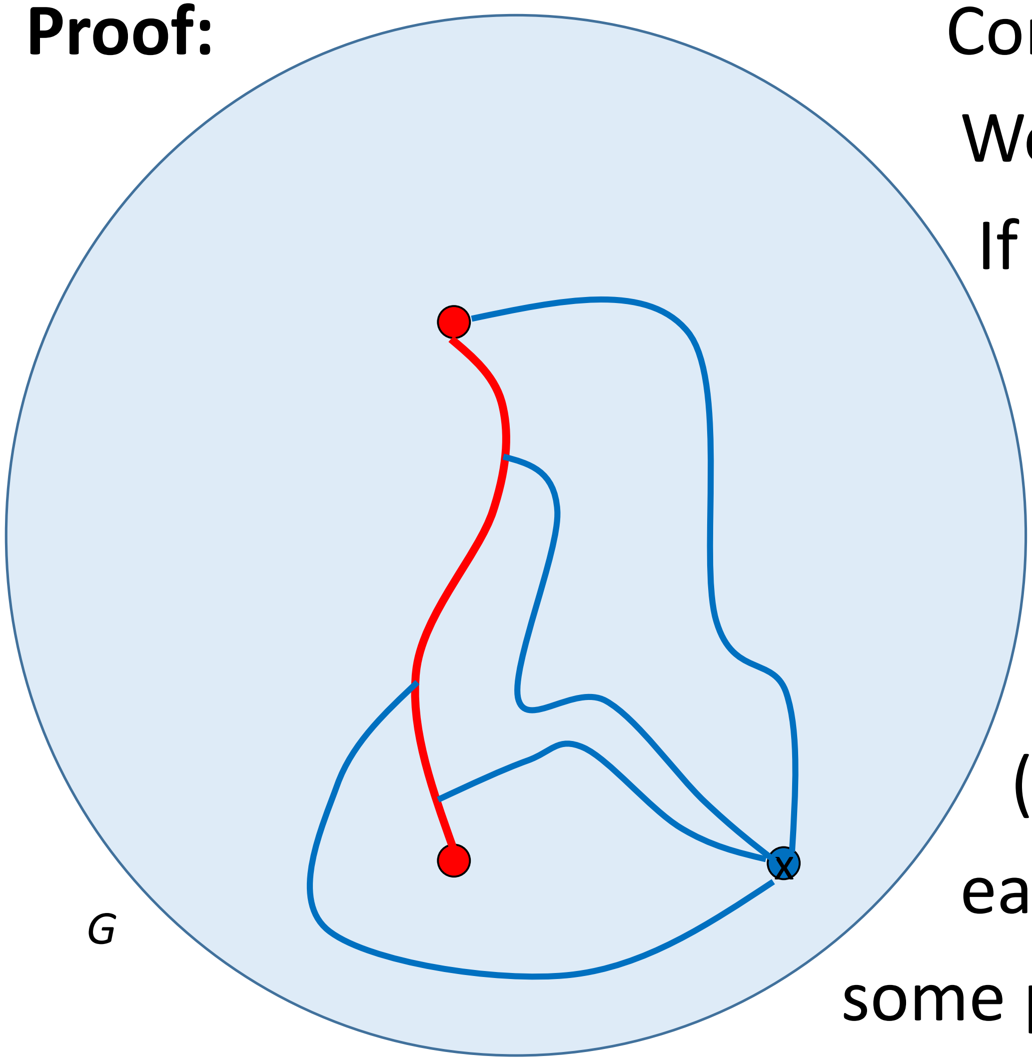

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If not, there is a vertex x that does not belong to the linkage.

Since G is highly connected, there are many disjoint paths from x to the linkage. At least one link is saturated (at least k paths end on this link, or each vertex of this link is end-vertex of some path).

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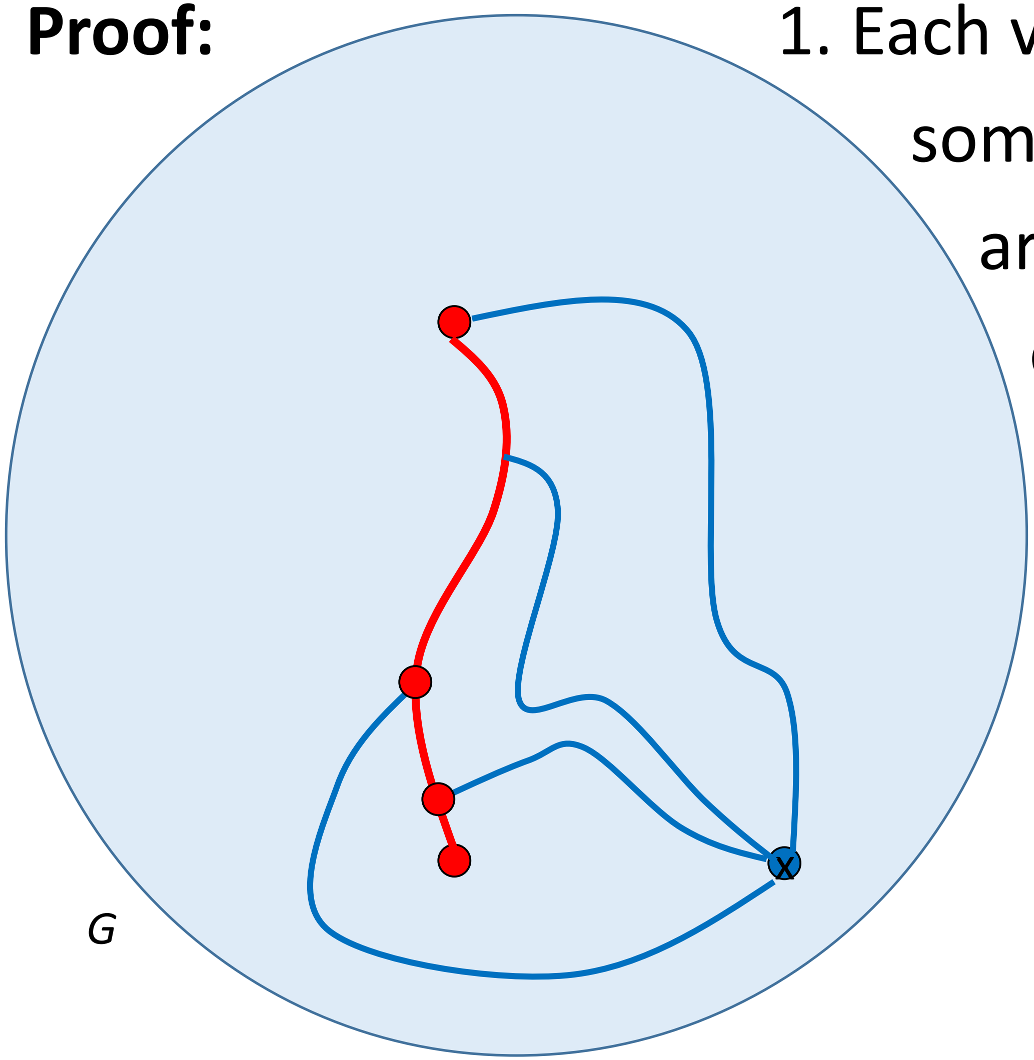

1. Each vertex of this link is end-vertex of some path, hence 2 consecutive vertices are end-vertices and the linkage can be extended. Contradicting its assumed maximality.

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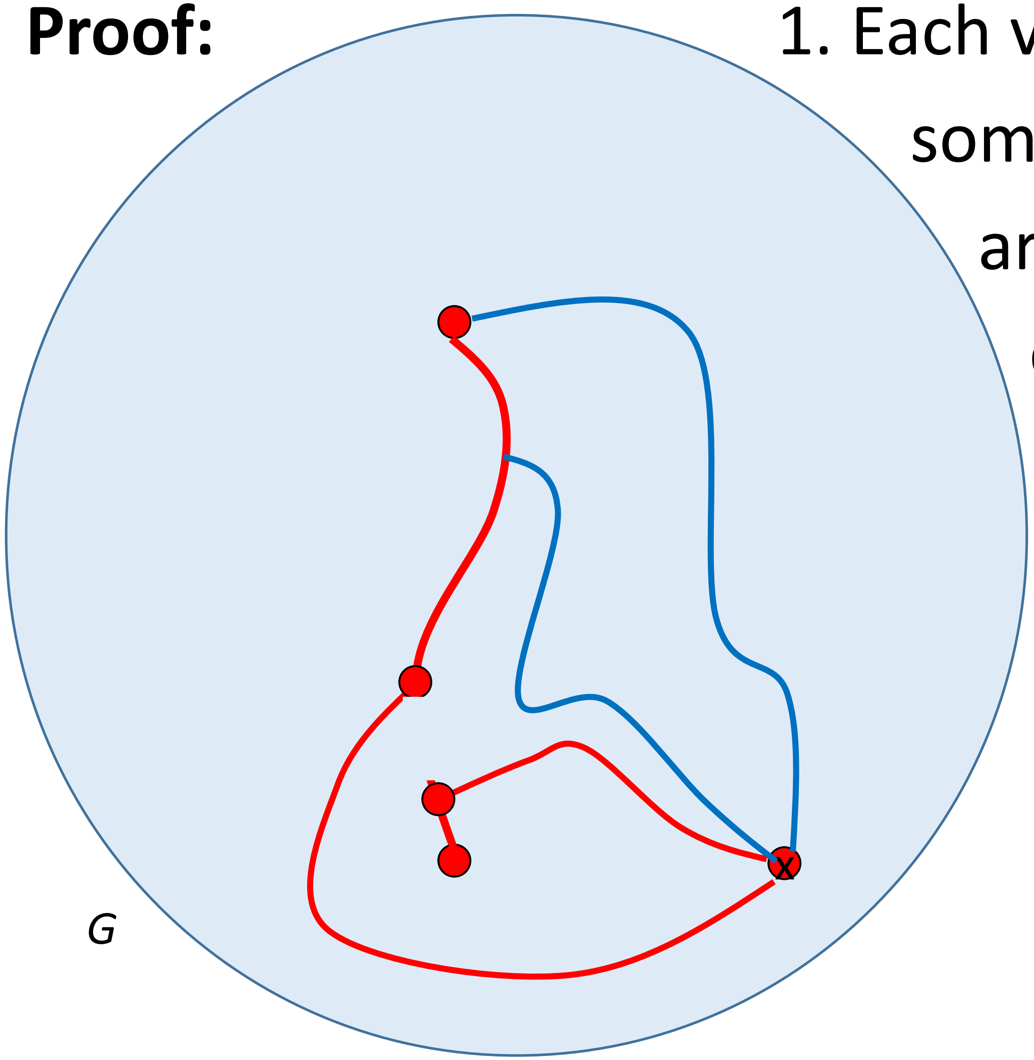

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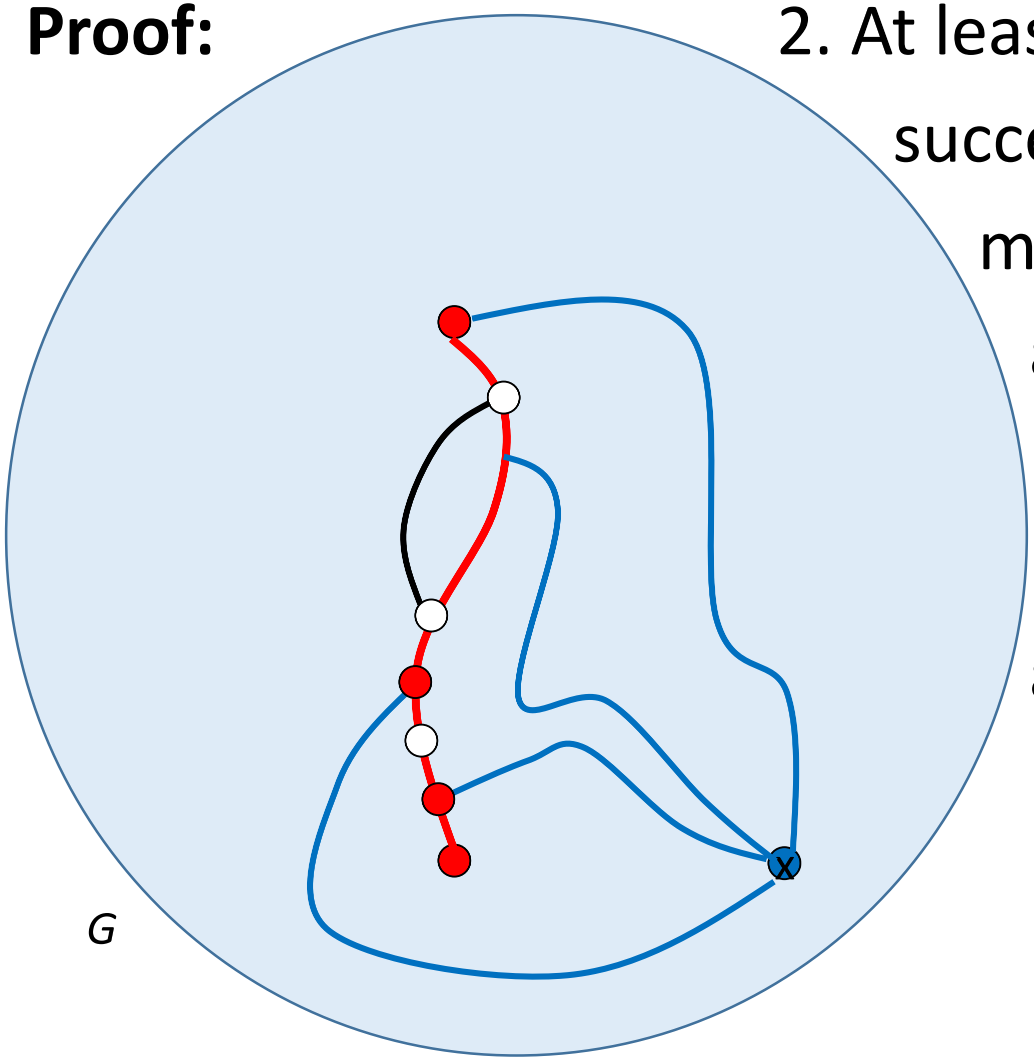

2. At least k paths end on this link. Consider successors of their end-points. Two of them must be adjacent (otherwise we have at least k independent vertices, contradicting $\alpha(G) < k$). Extend the link using this edge, contradicting the assumed maximality of the linkage.

Theorem [arXiv:2309.09228]: For every k and ℓ, if a graph G satisfies $\alpha(G) < k$ and $c_v(G) \geq \max\{k\ell, 10\ell\}$, then G is Hamiltonian-ℓ-linkable.

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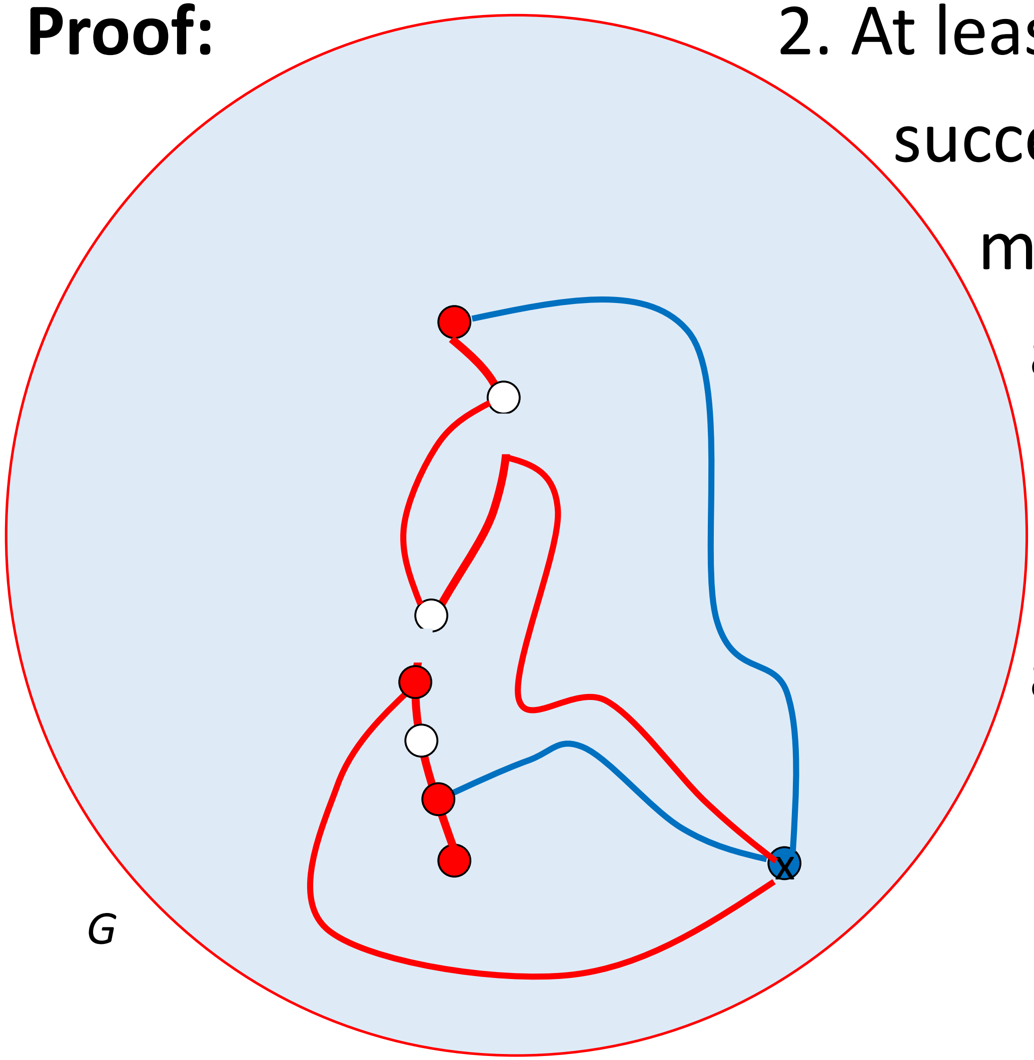

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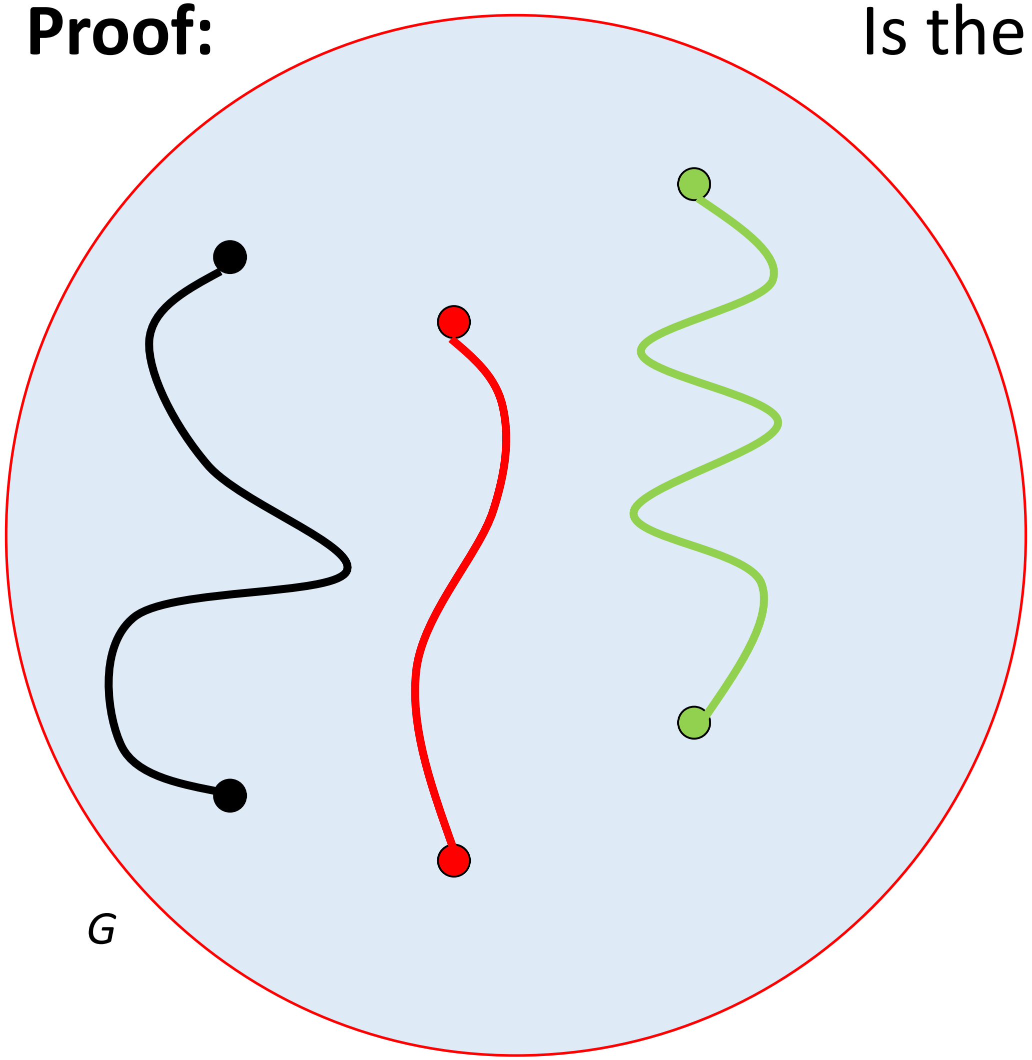
Proof: Is there any linkage to start with?



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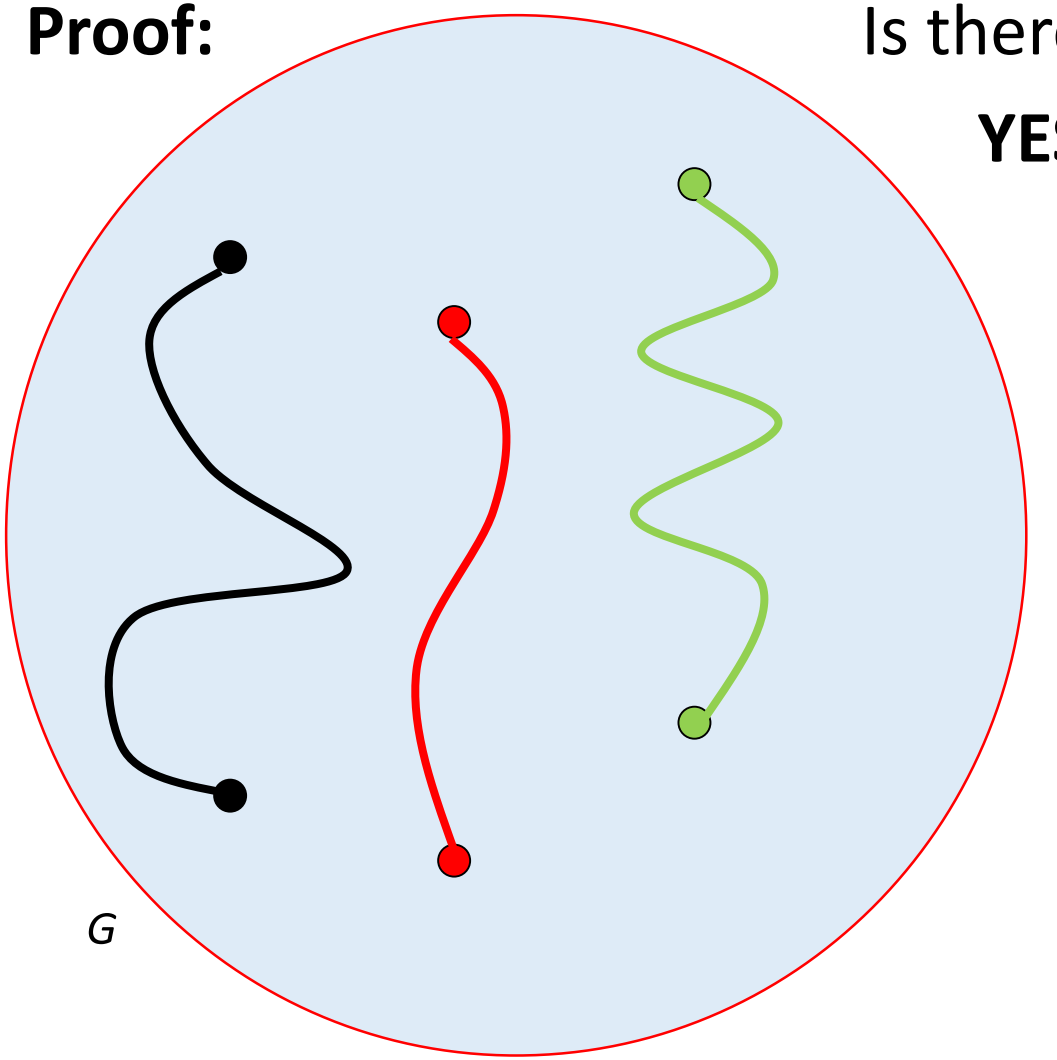
YES Thomas, Wollan [EJC 2005]



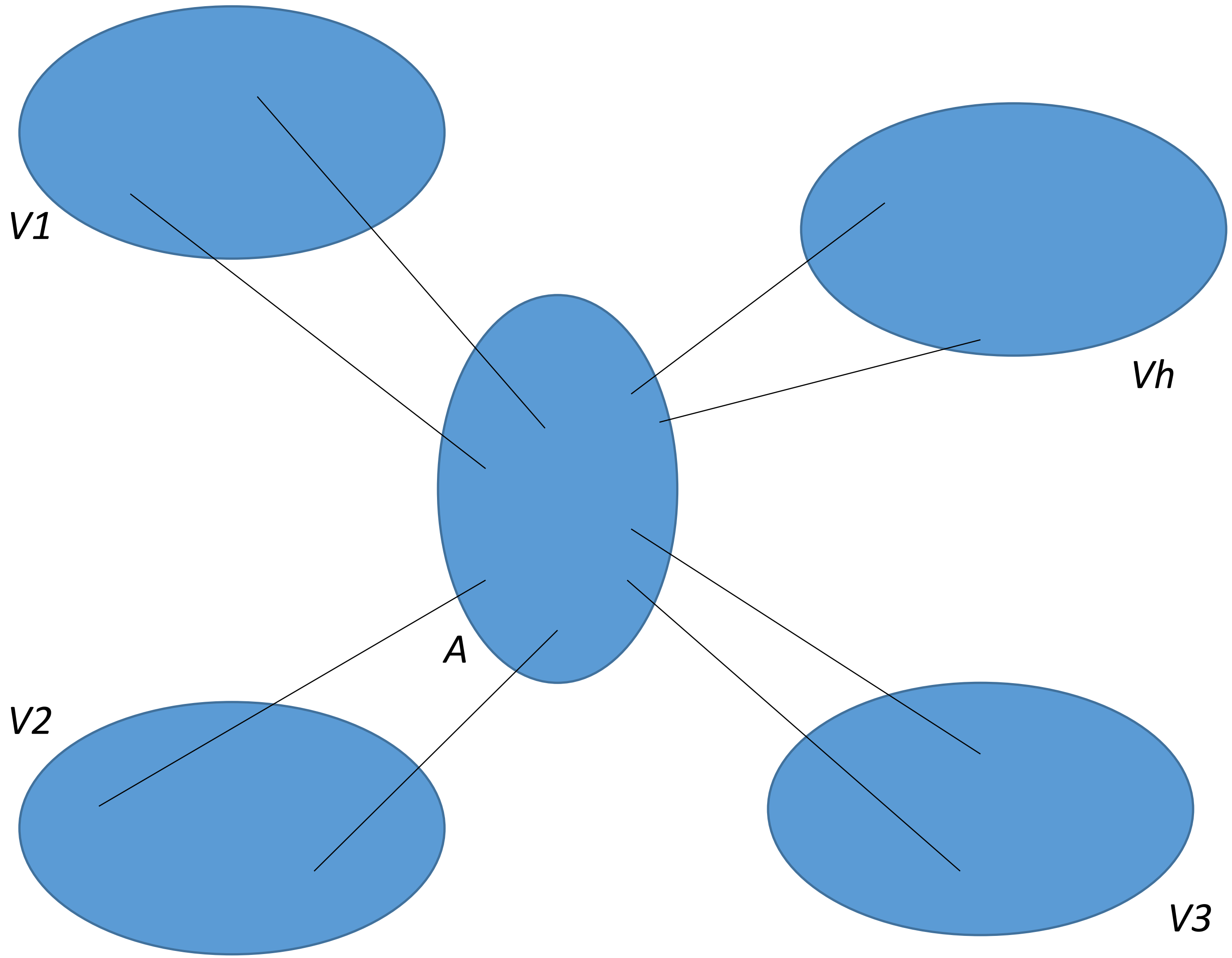
Theorem [arXiv:2309.09228]: For every k and ℓ, Ham-ℓ-Linkage and Ham-ℓ-Linkability are polynomial time solvable on kK_1-free graphs.

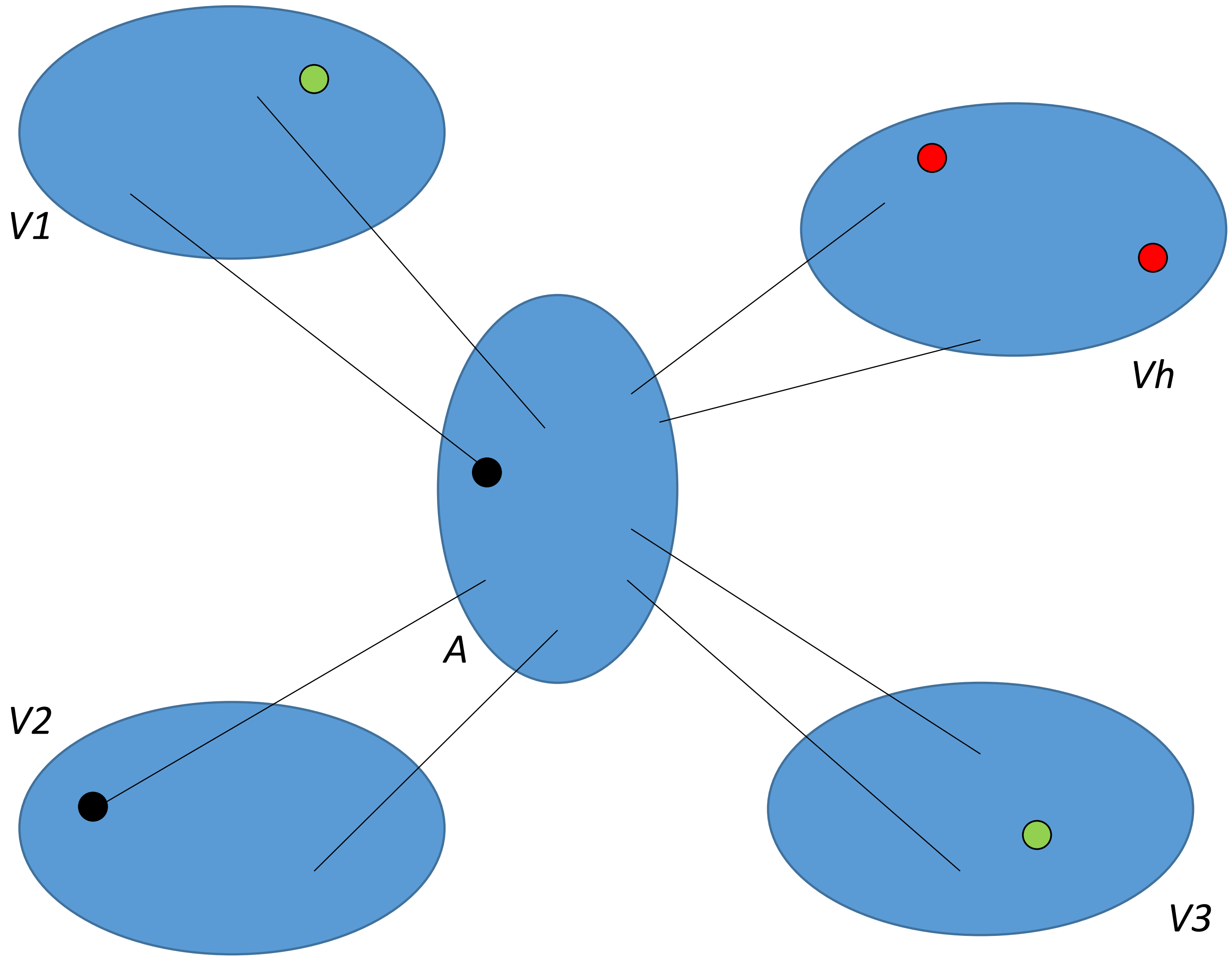
Algorithm:

1. Determine the vertex connectivity of G.
2. **If** $c_v(G) \geq g(k, \ell) = \max\{k\ell, 10\ell\}$, **then** answer **yes**
   else determine a small cut and iterate

$|A| < g(k, \ell)$

Each $G[Vi]$ is $(k\text{-}1)K_1$-free





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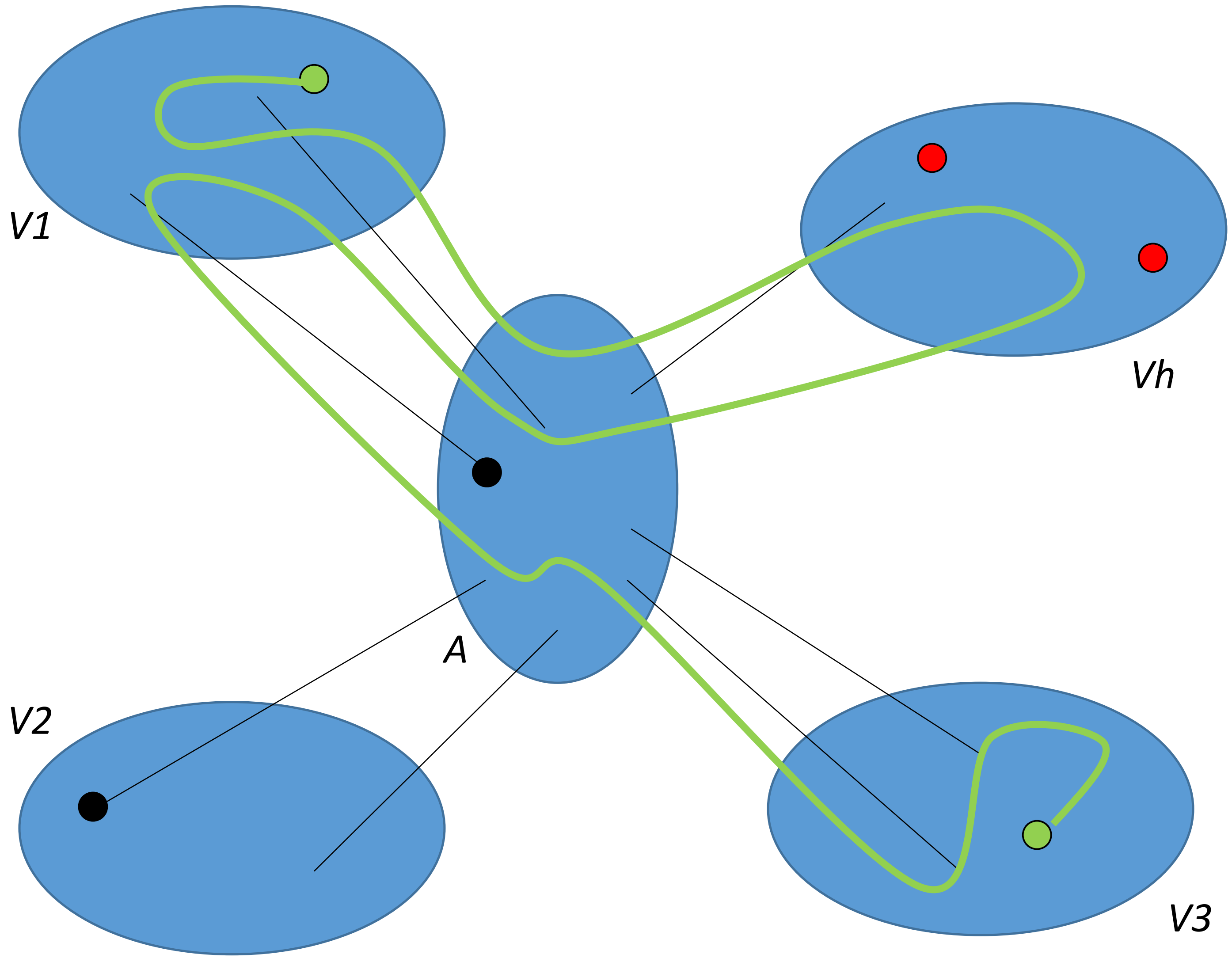

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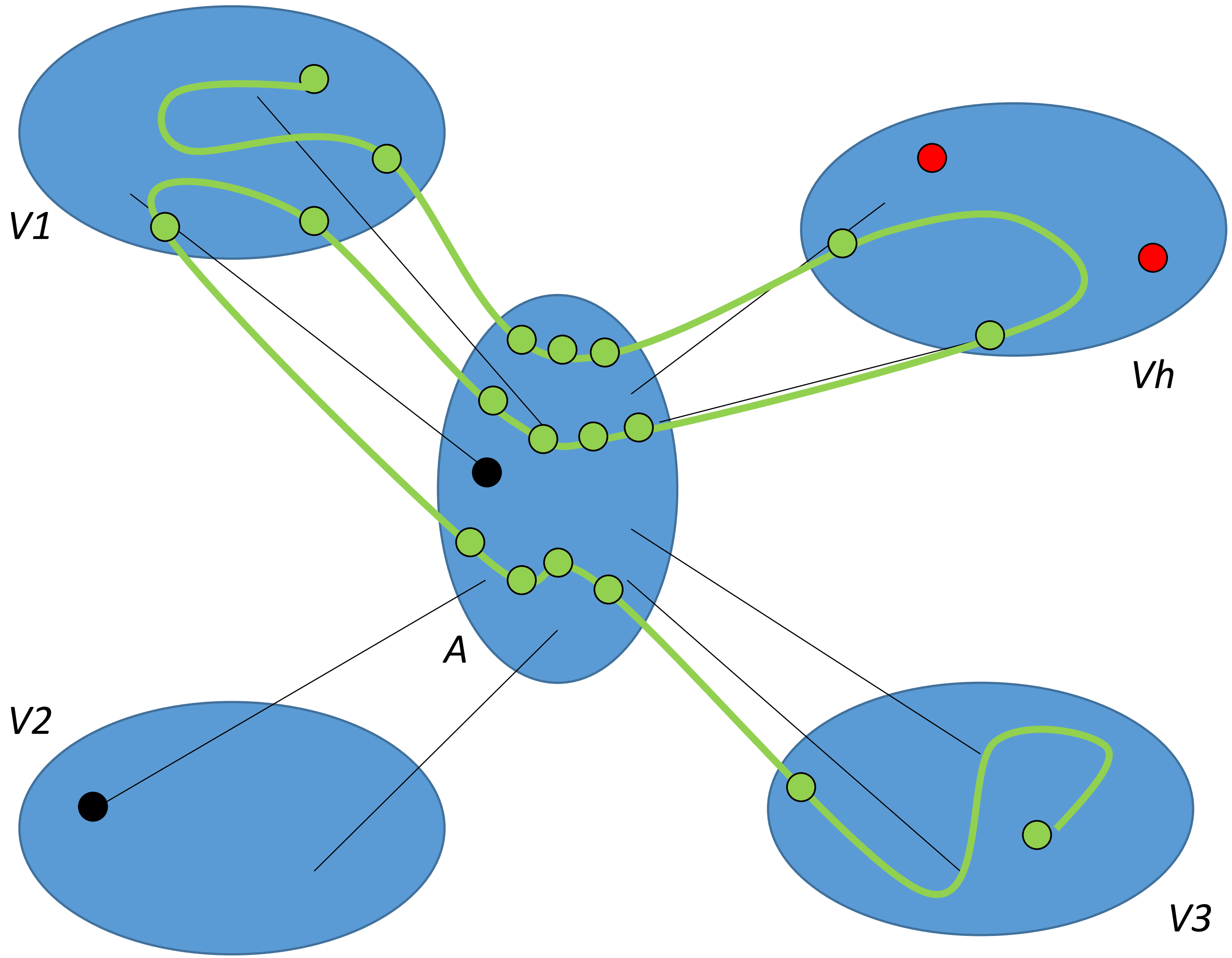

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Consider all possible **scenarios** for the links

And check their feasibility

The number of scenarios is $O(n^{f(k,\ell)})$

Recursive calls are for $(k\text{-}1)K_1$-free components

Induction on k

Theorem [arXiv:2309.09228]: For every k and ℓ, Ham-ℓ-Linkage and Ham-ℓ-Linkability are polynomial time solvable on kK_1-free graphs.

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The running time can be bounded by $O(n2^{\ell k^{k+1}})$.

For small values of k, HamPath can be solved in time

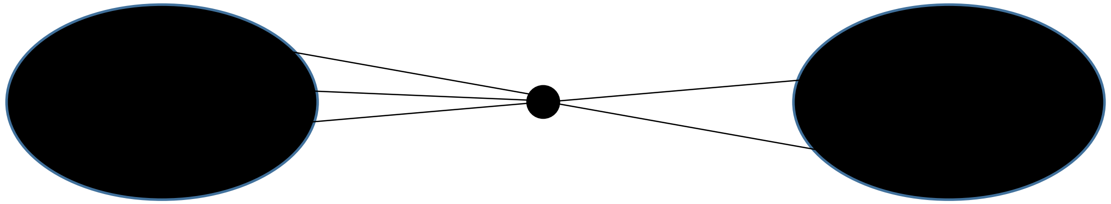
$O(n^{22})$ for $3K_1$-free graphs

$O(n^{242})$ for $4K_1$-free graphs

$O(n^{2662})$ for $5K_1$-free graphs

Forbidden configurations for Hamilton Path

Theorem 3 [IWOCA 2024]: A connected $3K_1$-free graph always has a Hamiltonian path. It has a Hamiltonian path starting in a vertex u if and only if u is not an articulation point of the graph.



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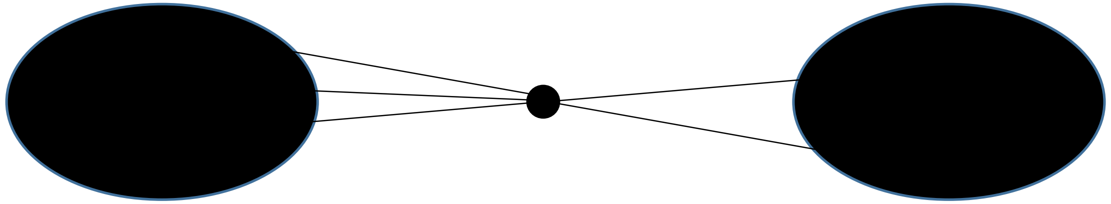


Proof: If *G* is 2-connected, then it has a Hamiltonian cycle [Chvátal, Erdös], and hence a Hamiltonian path starting in any vertex.

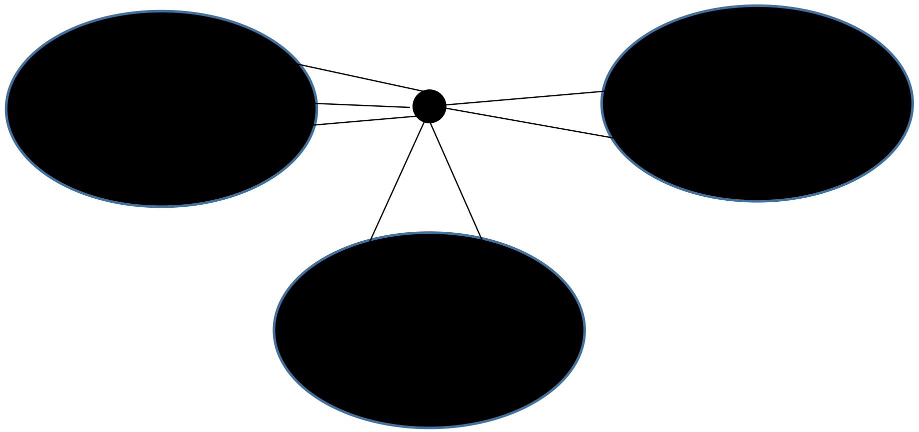
If it is not 2-connected and *v* is an articulation point, then *G*-*v* has 2 connected components and both are cliques.

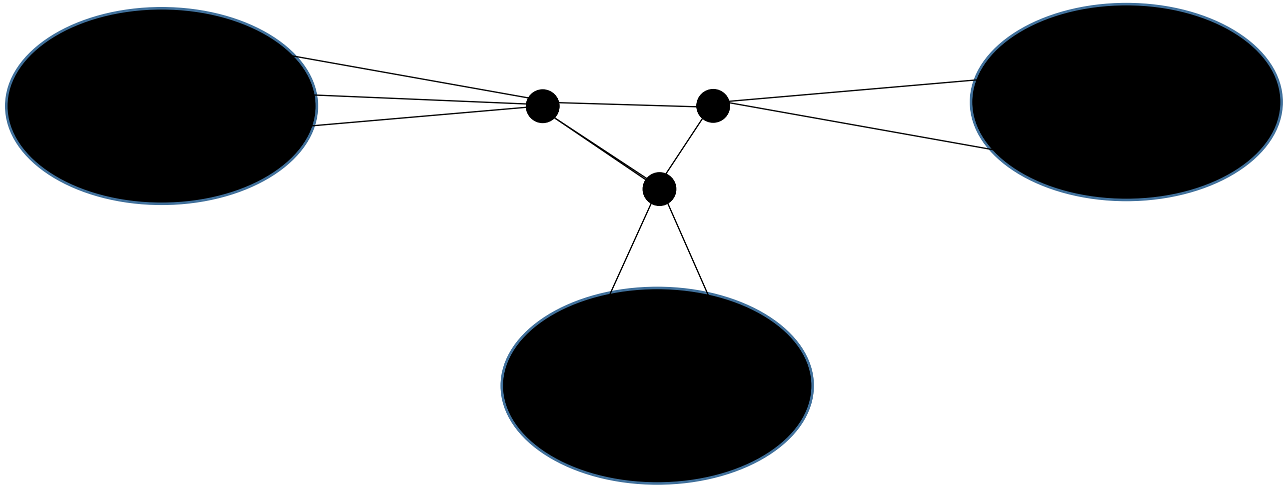
Forbidden configurations for Hamilton Path

Theorem 4 [IWOCA 2024]: A connected $4K_1$-free graph has a Hamiltonian path if and only if

(a) For every articulation point *u,* the graph *G-u* has 2 connected components, and

(b) There are no 3 articulation points inducing a triangle.





Forbidden configurations for Hamilton Path

Theorem 5 [IWOCA 2024]: A connected $5K_1$-free graph G has a Hamiltonian path if and only if

(a) For every articulation point u, the graph G-u has 2 connected components, and

(b) There are no 3 articulation points inducing a triangle.

(c) If G is not 2-connected, and x is an articulation point such that one of the components of G-x, denoted by Q_1, is a clique and the other one, denoted by Q_2, induces a $4K_1$-free graph, then there exists a vertex u in Q_2 adjacent to x such that $G[Q_2]$ has a Hamiltonian path starting in u.

(d) If G is 2-connected, but not 3-connected, then for every minimum vertex cut $\{x,y\}$ in G, G-$\{x,y\}$ has at most 3 connected components.

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Theorem [Fomin, Golovach, Sagunov, Simonov arXiv:2403.05943]: Ham-ℓ-Linkage and Ham-ℓ-Linkability are FPT when parameterized by ℓ and $\alpha(G)$.

Theorem [arXiv:2309.09228]: For every k and ℓ, Ham-ℓ-Linkage and Ham-ℓ-Linkability are polynomial time solvable on kK_1-free graphs.

i.e., Ham-ℓ-Linkage and Ham-ℓ-Linkability are in XP when parameterized by ℓ and $\alpha(G)$.

Theorem [Fomin, Golovach, Sagunov, Simonov arXiv:2403.05943]: Ham-ℓ-Linkage and Ham-ℓ-Linkability are FPT when parameterized by ℓ and $\alpha(G)$.

Our obstacles-for-small-independence-graphs paper won the **Best Paper Award at IWOCA 2024**.


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