

Perfect (1,2)-Domination and Regular Induced Subgraphs

or

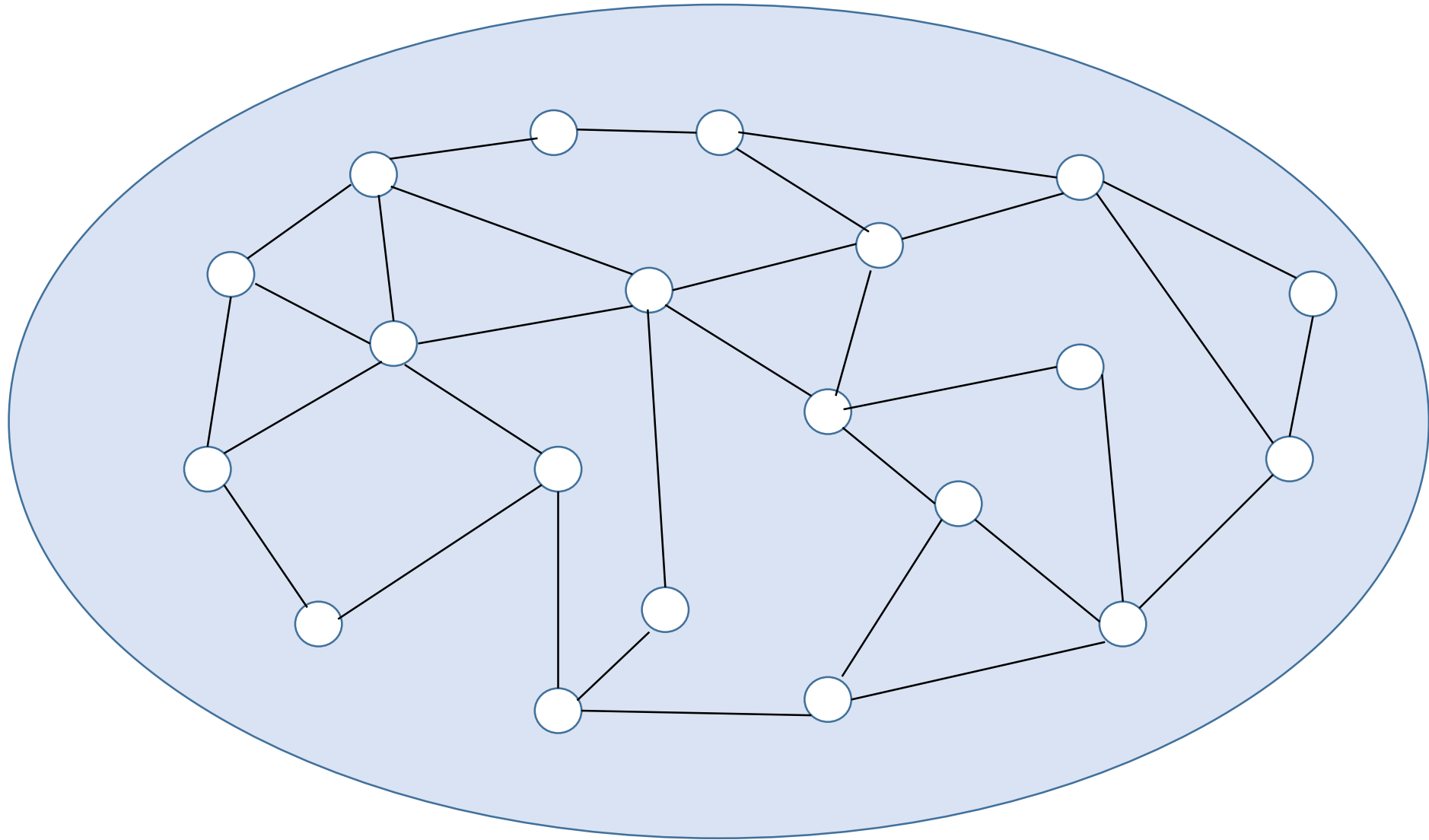
My First Encounters with Generative AI a Year before it Started Proving Theorems

Jan Kratochvil, Charles University, Prague

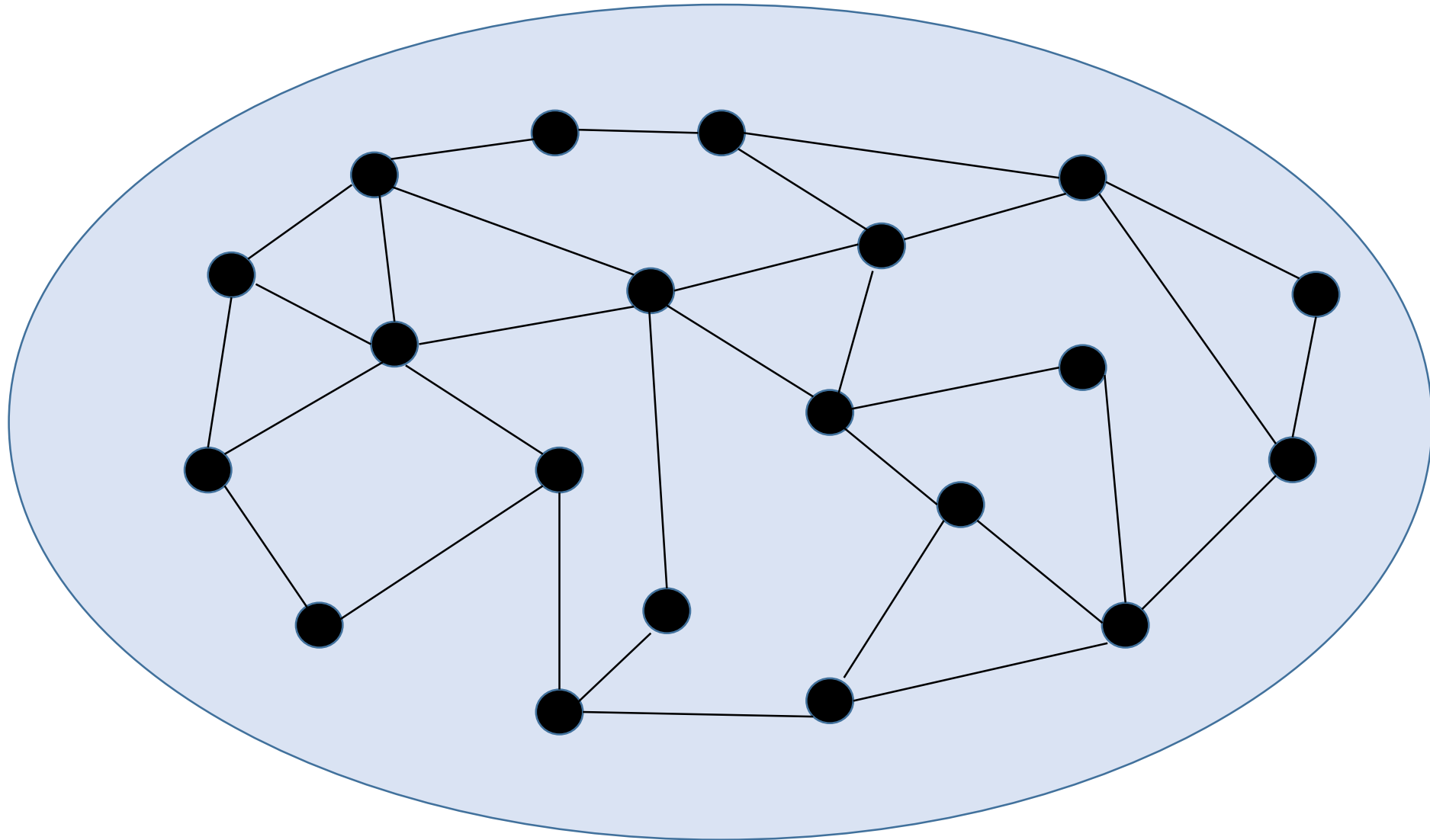
Based on joint work with Urszula Bednarz, Filip Filipi, Adrian Michalski,
Marta Pavelka and Jan Volec



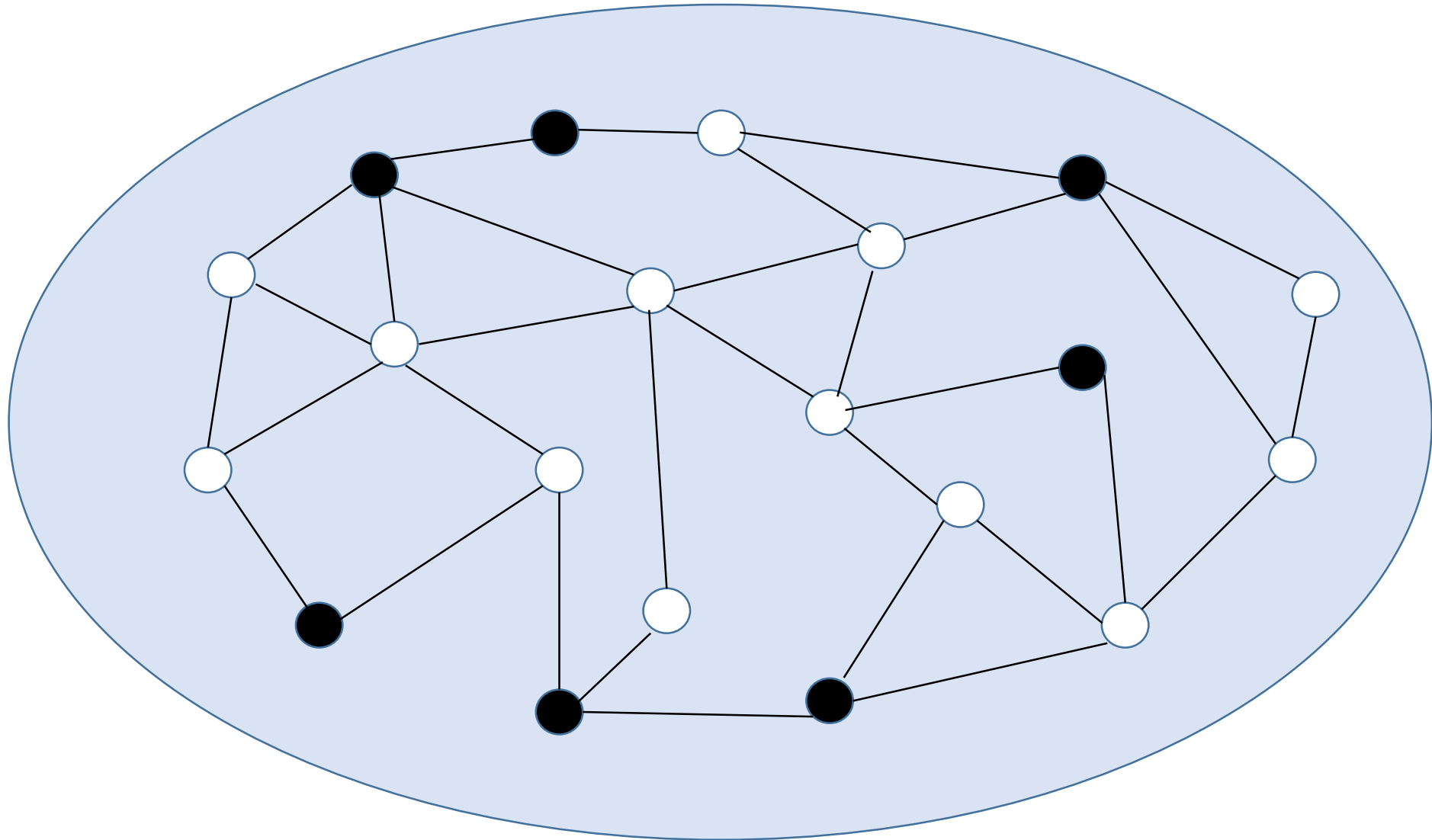
1. Dominating set



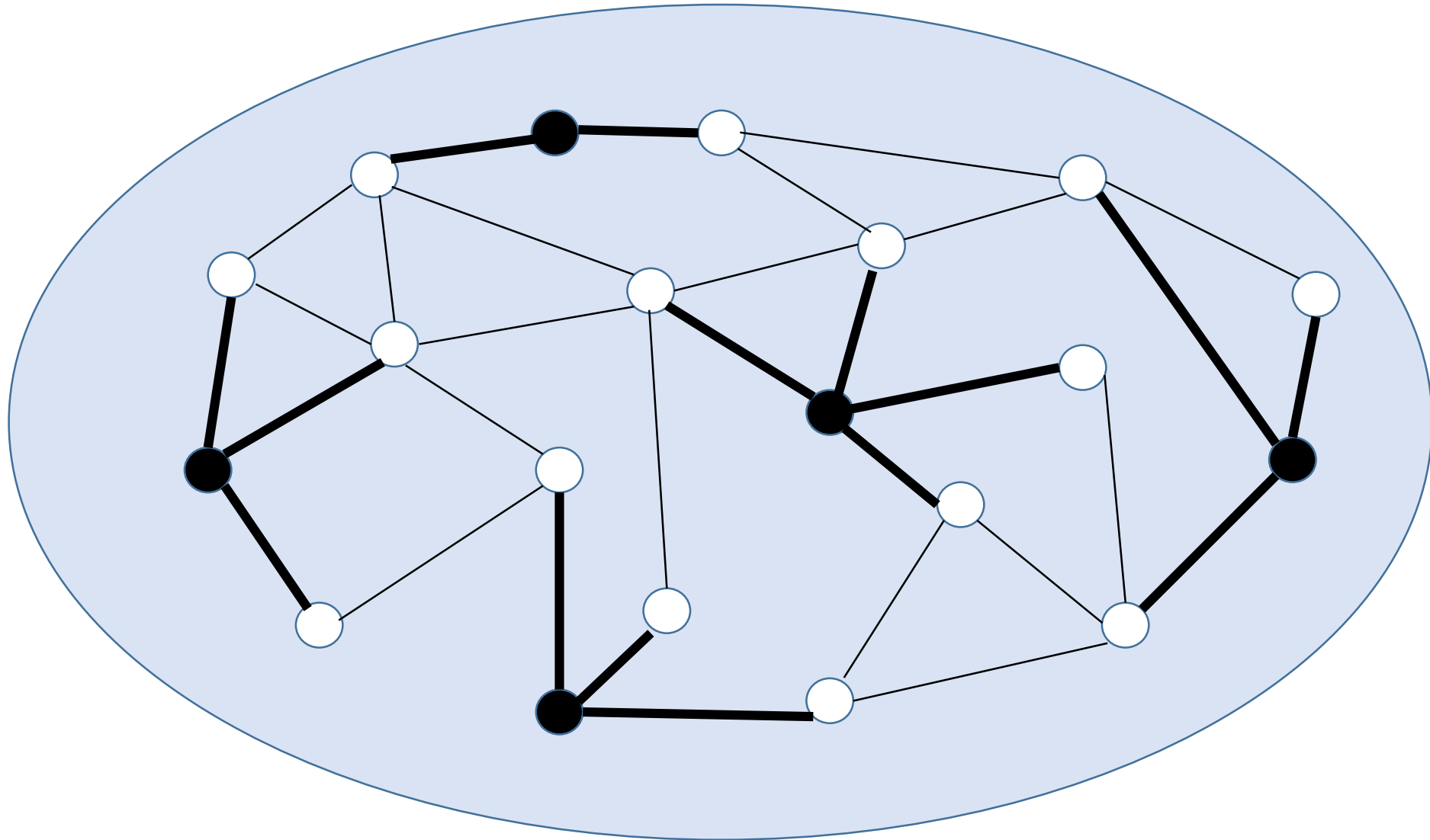
1. Dominating set



1. Dominating set



1. Efficient independent dominating set = 1-perfect code



1. Efficient independent dominating set = 1-perfect code

Dominating set always exists, minimizing the size is NP-hard

Independent dominating set always exists (any maximal independent set is dominating), minimizing the size is NP-hard

1-perfect code does not always exist, deciding its existence is NP-hard

1. Dominating sets

Many variants – independent dominating set, 2-fold dominating set, distance 2 dominating set

[Haynes, Hedetniemi, Slater: Fundamentals of Domination in Graphs (1998)]

Secondary domination – $(1,k)$ -domination

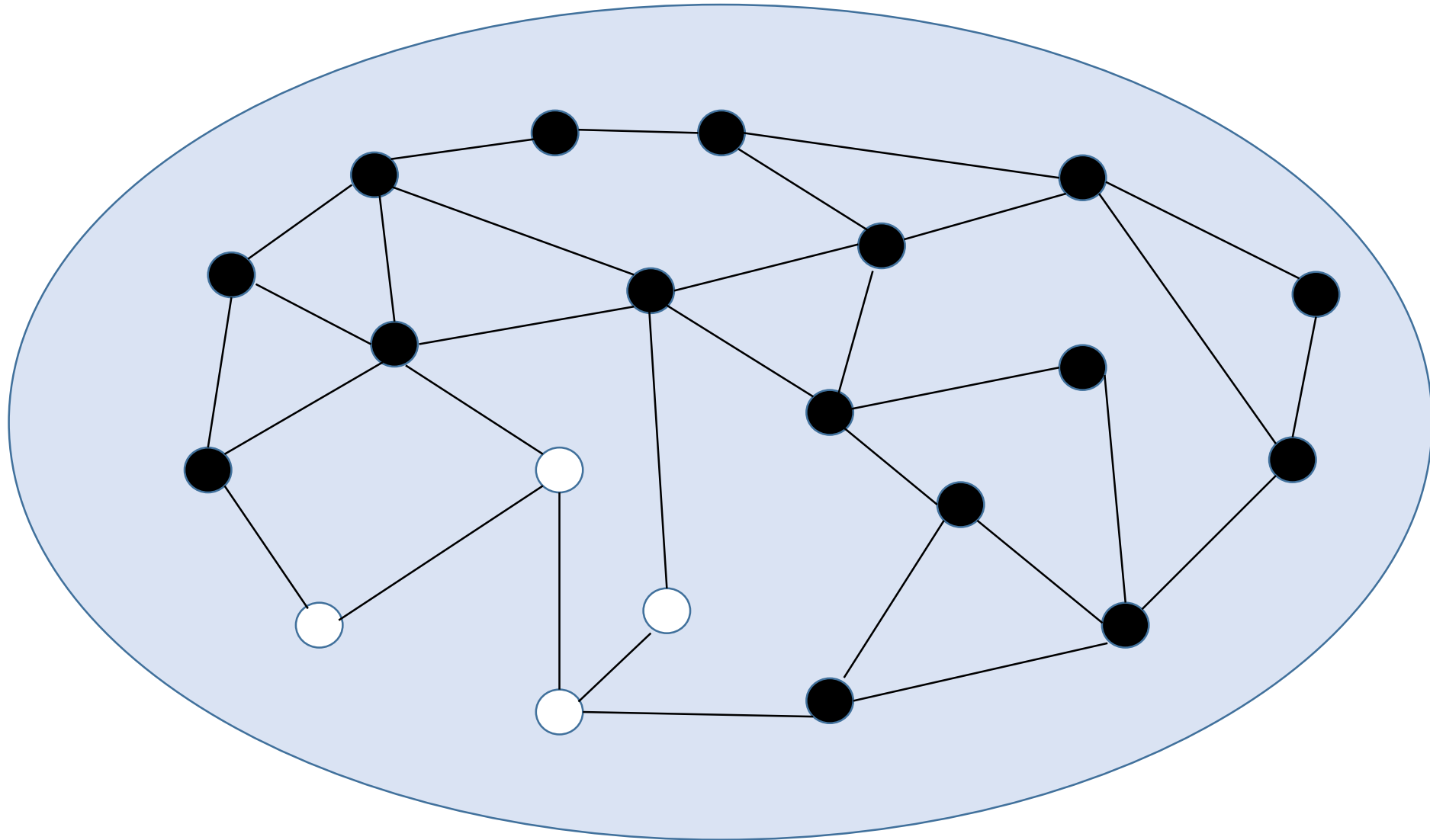
[Hedetniemi, Hedetniemi, Knisely, Rall: Secondary domination in graphs (2008).]

2. Perfect (1,2)-dominating sets

Introduced in [Bednarz, Pirga: (1, 2)-PDS in graphs with the small number of vertices of large degrees (2025)]

- D is nontrivial ($D \neq V(G)$)
- Every vertex outside D has exactly one neighbor in D
- Every vertex outside D has at least one vertex at distance 2 in D

2. Perfect (1,2)-dominating set



2. Perfect (1,2)-dominating sets

Perfect (1,2)-dominating set does not always exist

[Bednarz, Pirga (2025)] classified graphs with perfect (1,2)-dominating sets which have large maximum degree.

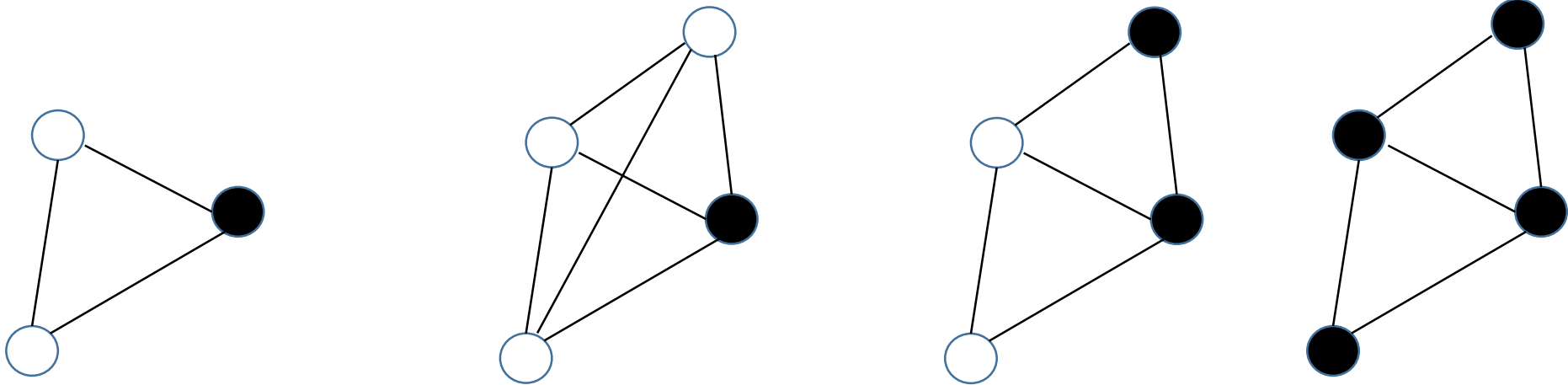
[Bednarz, JK, Michalski, WALCOM 2026]:

Theorem 1: Every connected subcubic graph with at least 5 vertices contains a perfect (1,2)-dominating set.

Theorem 2: Deciding the existence of a perfect (1,2)-dominating set in a 4-regular graph is NP-hard.

3. Subcubic graphs

$K_1, K_2, K_3, K_4, K_4^-$ do not contain perfect (1,2)-dominating sets, every other connected graph of maximum degree 3 does.



4. Max-degree 4 graphs

Plausible conjecture: Again a finite number of “bad graphs”, every other has a perfect $(1,2)$ -dominating set.

4. Four-regular graphs

Plausible conjecture: Again a finite number of “bad graphs”, every other has a perfect $(1,2)$ -dominating set.

NO! Infinitely many minimal 4-regular graphs with no perfect $(1,2)$ -dominating set.

4. Four-regular graphs

Plausible conjecture: Again a finite number of “bad graphs”, every other has a perfect $(1,2)$ -dominating set.

NO! Infinitely many minimal 4-regular graphs with no perfect $(1,2)$ -dominating set.

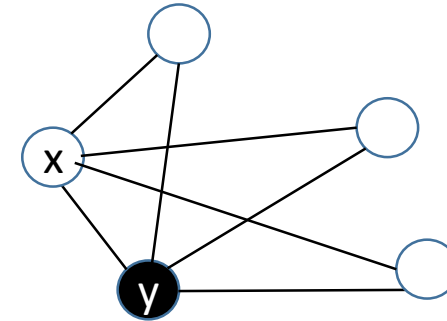
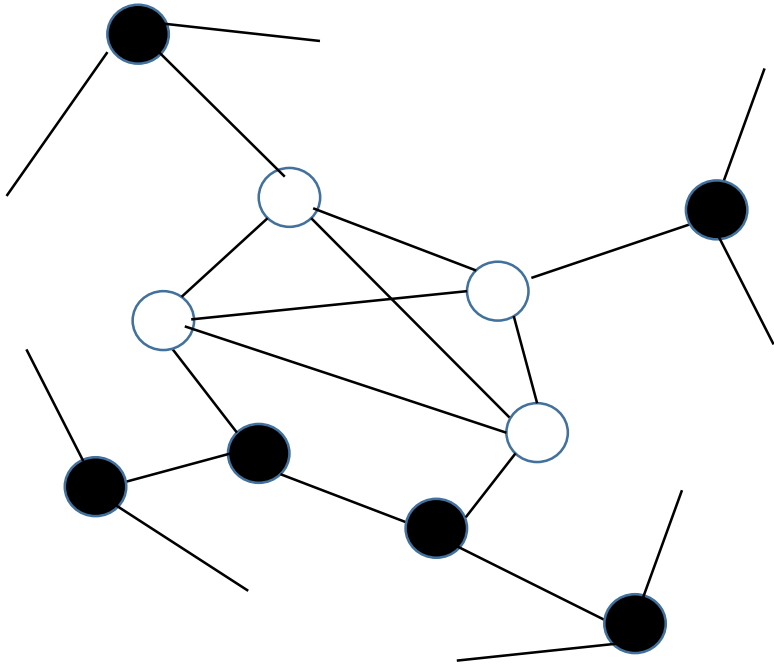
Observation: If a 4-regular graph has the property that no edge belongs to more than 2 triangles, then it contains a perfect $(1,2)$ -dominating set if and only if it contains a non-empty induced 3-regular subgraph.

Proof: $V(G) - D$ induces a 3-regular subgraph.

4. Four-regular graphs

Proof: $V(G) - D$ induces a 3-regular subgraph.

An induced 3-regular subgraph gives a perfect (1,2)-DS.



4. Four-regular graphs

Is it NP-hard to decide if a 4-regular graph contains a nontrivial induced 3-regular subgraph?

AI entertainment

July 2, 2025: Google AI said that the problem is NP-complete because deciding existence of 3-regular subgraph in a graph of maximum degree 4 is NP-complete.

AI entertainment

July 2, 2025: Google AI said that the problem is NP-complete because deciding existence of 3-regular subgraph in a graph of maximum degree 4 is NP-complete.

NOT induced subgraph

NOT 4-regular input graph

And it provided a reference: [Plesník: A note on the complexity of finding regular subgraphs. Discrete Mathematics (1984).]

A NOTE ON THE COMPLEXITY OF FINDING REGULAR SUBGRAPHS

J. PLESNÍK

*Katedra numerických a optimalizačných metód MFF UK, Mlynská dolina, 842 15 Bratislava,
Czechoslovakia*

Received 8 September 1981

Revised 2 June 1982

Given a graph G and an integer r , does there exist a regular subgraph of G with degree r ? In this note we establish NP-completeness for the r -regular subgraph problem for each $r \geq 3$ and certain restrictions on G . In particular, the cubic subgraph problem is NP-complete even for the simple case where G is a bipartite planar graph with maximum degree 4.

this result is attributed to Chvátal, 1976). Here we extend this result to each r and strengthen by adding certain restrictions on G .

Our graph-theoretic terminology is based on [1] or [6]. As to the NP-completeness, the reader is referred to [5].

2. Bipartite graphs

Theorem 1. *For every $r \geq 3$ the r -regular subgraph problem is NP-complete, even for the class of bipartite graphs with maximum degree $r + 1$.*

Proof. Since the problem is obviously in the class NP [5], it suffices to prove the completeness. We give a polynomial transformation of the following NP-complete problem (see [7] or [5, p. 221]) to our problem.

Exact cover by 3-sets: Given a set X of $3k$ elements and a collection C of 3-element subsets of X . Does C contain an exact cover for X , that is a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C' ?

this result is attributed to Chvátal, 1976). Here we extend this result to each r and strengthen by adding certain restrictions on G .

Our graph-theoretic terminology is based on [1] or [6]. As to the NP-completeness, the reader is referred to [5].

2. Bipartite graphs

Theorem 1. *For every $r \geq 3$ the r -regular subgraph problem is NP-complete, even for the class of bipartite graphs with maximum degree $r + 1$.*

Proof. Since the problem is obviously in the class NP [5], it suffices to prove the completeness. We give a polynomial transformation of the following NP-complete problem (see [7] or [5, p. 221]) to our problem.

Exact cover by 3-sets: Given a set X of $3k$ elements and a collection C of 3-element subsets of X . Does C contain an exact cover for X , that is a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C' ?

AI entertainment

Asked Petr Golovach, July 2, 2025:

Hello Petr,

I noticed you have done some work on finding regular subgraphs. So you may know an answer to the following question:

Is existence of a (nonempty) induced 3-regular subgraph in a 4-regular graph known to be NP-complete?

ChatGPT tells me that yes, but the references it gives either make no sense, or relate to looking for non-induced cubic subgraphs.

Jan

AI entertainment

Asked Petr Golovach, July 2, 2025:

Hello Petr,

I noticed you have done some work on finding regular subgraphs. So you may know an answer to the following question:

Is existence of a (nonempty) induced 3-regular subgraph in a 4-regular graph known to be NP-complete?

ChatGPT tells me that yes, but the references it gives either make no sense, or relate to looking for non-induced cubic subgraphs.

Jan

Answer from Petr Golovach, July 2, 2025:

Dear Honza,

No, I don't know whether it was proved. I also tried ChatGPT, and it claims that you proved it in 1994 :)

Petr

AI entertainment

March 4, 2026:

It is not NP-hard to decide if a 4-regular graph contains a nontrivial induced 3-regular subgraph. In fact, such a subgraph **always exists** in any simple 4-regular graph, meaning the decision problem is trivial (always "yes").

Here are the details:

- **The Structure:** According to the **Berge–Sauer conjecture** (proven by Tâskinov in 1982, with a simplified proof by Alon, Friedland, and Kalai), every simple 4-regular graph contains a 3-regular subgraph.
- **Induced Subgraph:** While the conjecture traditionally refers to a 3-regular subgraph (not necessarily induced), research on r -regular induced subgraphs indicates that finding *maximum* r -regular subgraphs is generally NP-hard. However, for a 4-regular graph, a 3-regular induced subgraph is guaranteed by the structure proven by Tâskinov.
- **Reference:** A seminal paper on this is **Alon, N., Friedland, S., & Kalai, G. (1984). "Every 4-regular graph plus an edge contains a 3-regular subgraph," Journal of Combinatorial Theory, Series B, 37(1), 92-93.** [Mathematics Stack Exchange +4](#)

Note: The problem of finding a 3-regular subgraph is NP-complete for general graphs, but not when the input is specifically restricted to a 4-regular graph.

AI entertainment

July 30, 2026:

is existence of nontrivial induced 3-regular subgraph in a 4-regular graph NP-comple

◆ Přehled od AI

Yes, finding an r -regular induced subgraph is NP-complete for any fixed $r \geq 1$, including the existence of a 3-regular induced subgraph in a 4-regular graph.

Key Complexity Aspects

- **General Hardness:** The problem of deciding whether a graph contains an induced r -regular subgraph is **NP-complete** for any fixed integer $r \geq 1$.
- **Special Cases:** For $r = 0$, the problem is equivalent to finding an **Independent Set**, and for $r = 1$, it corresponds to finding an **Induced Matching**.
- **Bounded Degree:** Restricting the host graph to regular or bounded-degree graphs maintains the computational hardness for degree constraints where subgraphs are sought.

If you'd like, I can:

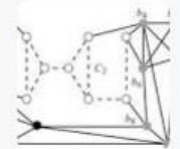
- Detail the **gadget reductions** used to prove this hardness.
- Explain the **parameterized complexity** (such as W-hardness) for finding an r -regular induced subgraph of size k .

4 weby

Parameterized complexity of finding regular induced subgraphs

15. 6. 2009 — Note that for $r = 0$, r -Regular Induced Subgraph is equivalent to...

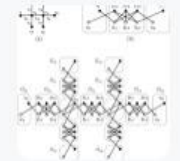
ScienceDirect.com



Complexity of finding maximum regular induced subgraphs ...

18. 9. 2014 — This problem has been also studied extensively: for example, it is know...

ScienceDirect.com



Zobrazit vše

AI entertainment

July 30, 2026:

Moser, Thilikos: Parameterized complexity of finding regular induced subgraphs, Journal of Discrete Algorithms 2009

Asahiro, Eto, Ito, Miyana: Complexity of finding maximum regular induced subgraphs with prescribed degree, Theoretical Computer Science 2014

AI entertainment

September 28, 2026:

Režim AI Vše Video Obrázky Krátká videa Více ▾

◆ Upgradovat



is deciding the existence of an induced 3-regular subgraph in a 4-regular graph NP-complete?

Yes, deciding the existence of an induced 3-regular subgraph in a 4-regular graph is NP-complete. ACM Digital Library +1

1. Equivalence to Perfect (1,2)-Domination

This fact follows directly from recent developments in low-degree graph complexity. The problem is closely tied to finding a **perfect (1,2)-dominating set** in a 4-regular graph: ACM Digital Library +1

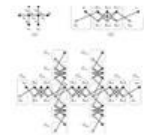
- By definition, a perfect (1,2)-dominating set S requires every vertex outside of S (i.e., in $V \setminus S$) to have **exactly one neighbor** inside S .
- Because the host graph is 4-regular, every vertex has a total of 4 neighbors. If a vertex in $V \setminus S$ has exactly 1 neighbor in S , it must have exactly $4 - 1 = 3$ neighbors within $V \setminus S$.

- Consequently, the set $V \setminus S$ forms a valid solution if and only if the induced subgraph

ScienceDirect.com

Complexity of finding maximum regular induced subgraphs ...

18. 9. 2014 — 2.1. 1. Main ideas of our reduction. We now explain our ideas of the...



Numdam

Polynomial time algorithms for two classes of subgraph problem

every 4-regular simple graph contains a 3-regular subgraph. every k -regular (≥ 3) it is an NP-Complete...

www.hmoser.info

Parameterized Complexity of Finding Regular Induced ...

k -Almost r -Regular Graph: Input: A graph $G = (V, E)$ and an integer k . Question: Is there a vertex subset $S \subseteq V$ of size...

Zobrazit vše

4. Four-regular graphs – the proof

[Plesník: A note on the complexity of finding regular subgraphs. Discrete Mathematics (1984).] does not claim anything about induced 3-regular subgraphs, but the proof therein **DOES** work for induced subgraphs as well (without any modification).

4. Four-regular graphs – the proof

[Plesník: A note on the complexity of finding regular subgraphs. Discrete Mathematics (1984).] does not claim anything about induced 3-regular subgraphs, but the proof therein **DOES** work for induced subgraphs as well (without any modification).

Exact cover $(X, C) \rightarrow G$

If (X, C) does not admit an exact cover, then G does not have a nontrivial 3-regular subgraph

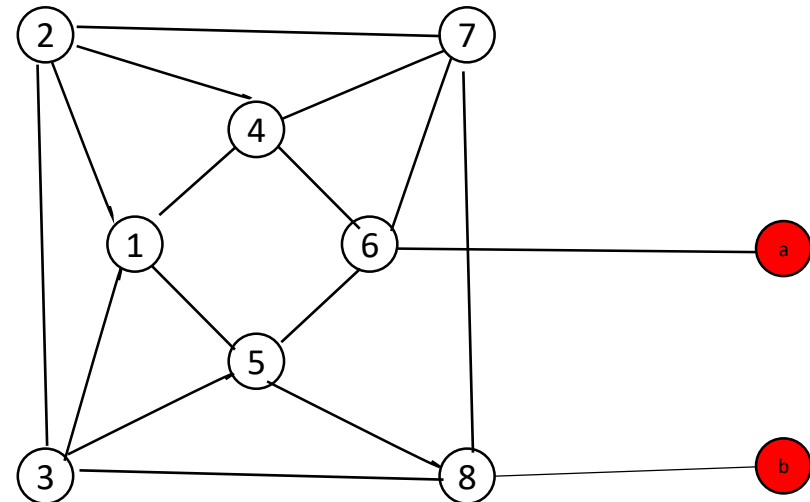
If (X, C) admits an exact cover, then G has a nontrivial induced 3-regular subgraph

4. Four-regular graphs – the proof

[Plesník: A note on the complexity of finding regular subgraphs. Discrete Mathematics (1984).] does not claim anything about induced 3-regular subgraphs, but the proof therein **DOES** work for induced subgraphs as well (without any modification).

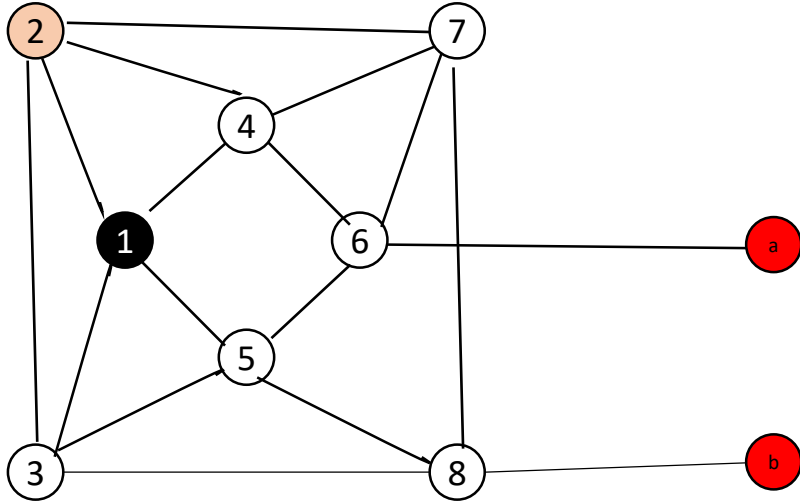
The graph constructed in the reduction has vertices of degrees 3 or 4 only. And the number of vertices of degree 3 is even. Pair these vertices and attach an auxiliary graph to each pair

Claim: Any induced 3-regular subgraph is disjoint with $\{1,2,3,4,5,6,7,8\}$



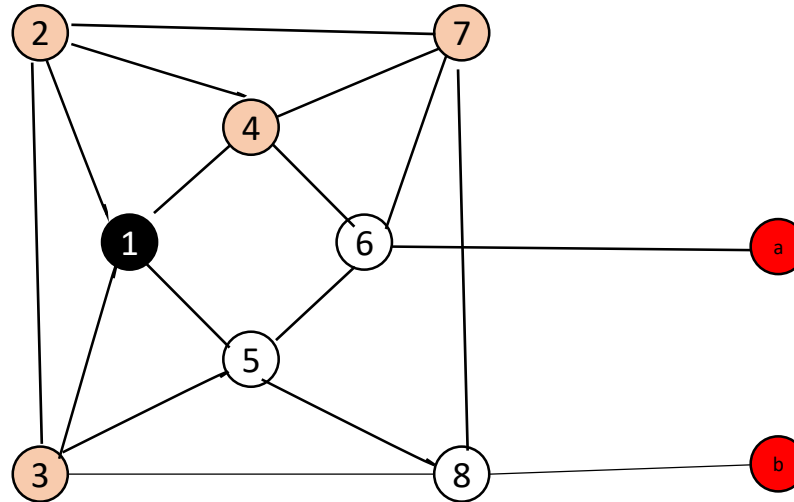
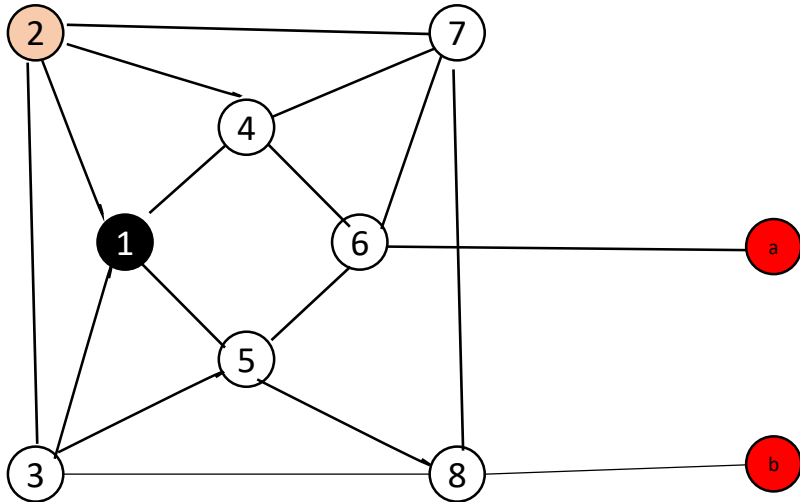
4. Four-regular graphs – the proof

Example of case analysis. Suppose $1 \in D$, $2 \notin D$.



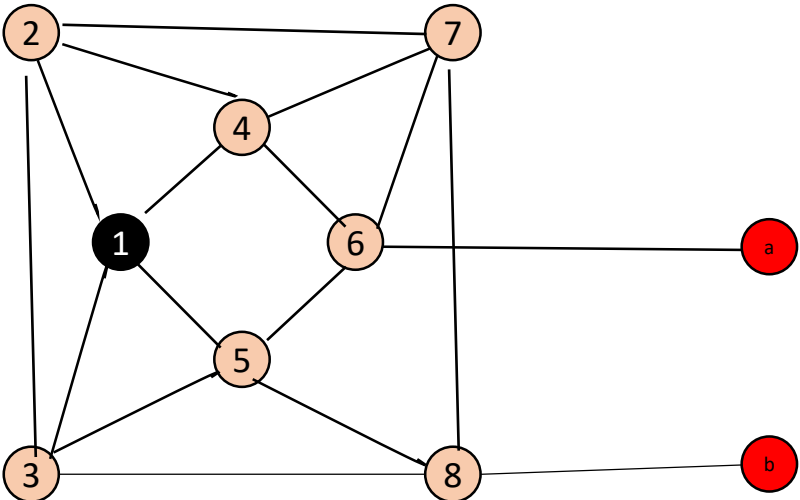
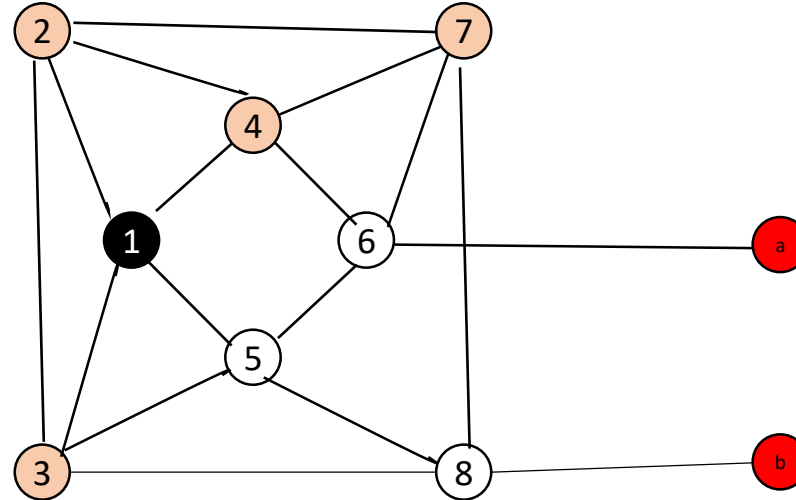
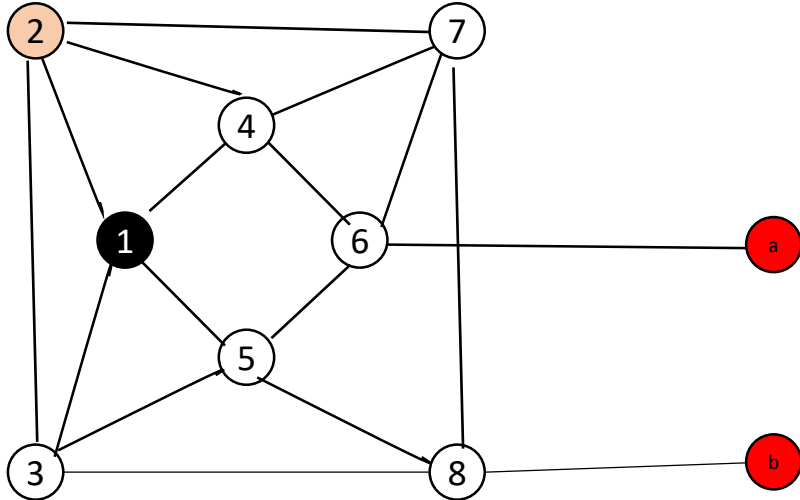
4. Four-regular graphs – the proof

Example of case analysis. Suppose $1 \in D$, $2 \notin D$. Then $3, 4, 7 \notin D$.



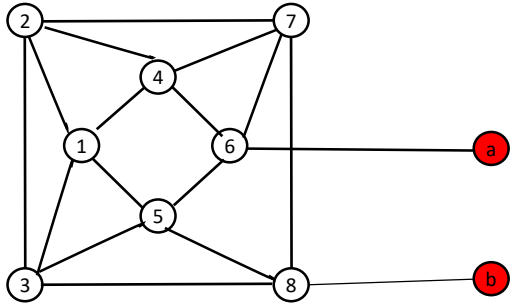
4. Four-regular graphs – the proof

Example of case analysis. Suppose $1 \in D$, $2 \notin D$. Then $3, 4, 7 \notin D$. Hence, $5, 6, 8 \notin D$.

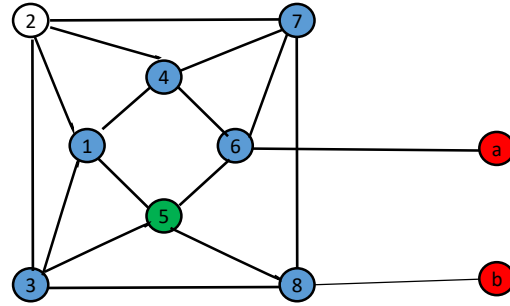


And 7 is not dominated at all.

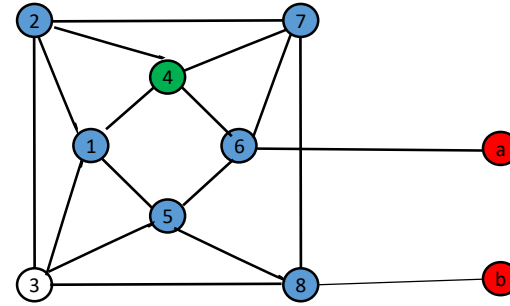
4. Four-regular graphs – the proof



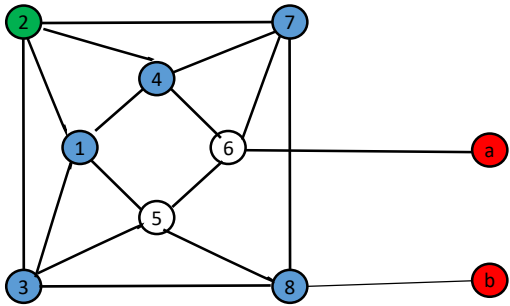
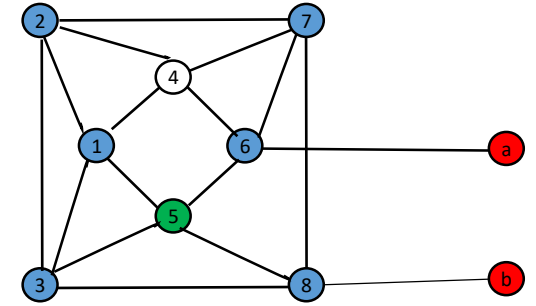
1 in T, 2 not



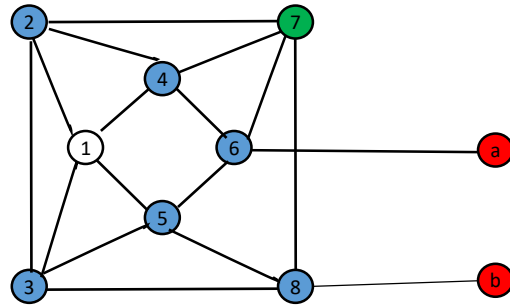
1 in T, 3 not



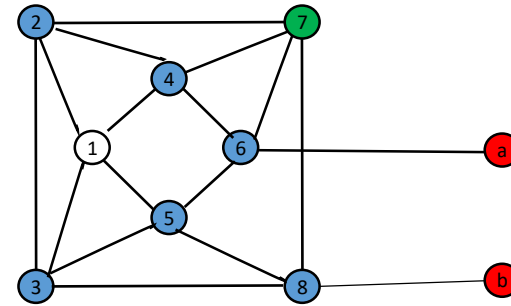
1 in T, 4 not



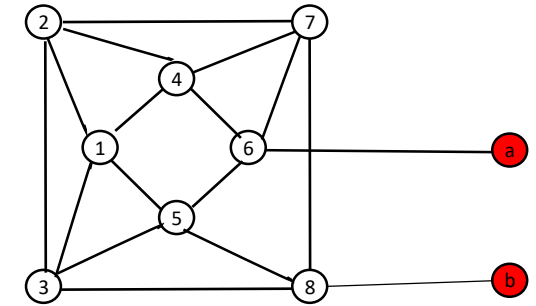
1 in T, 5 not



1 not in T, 2 in T



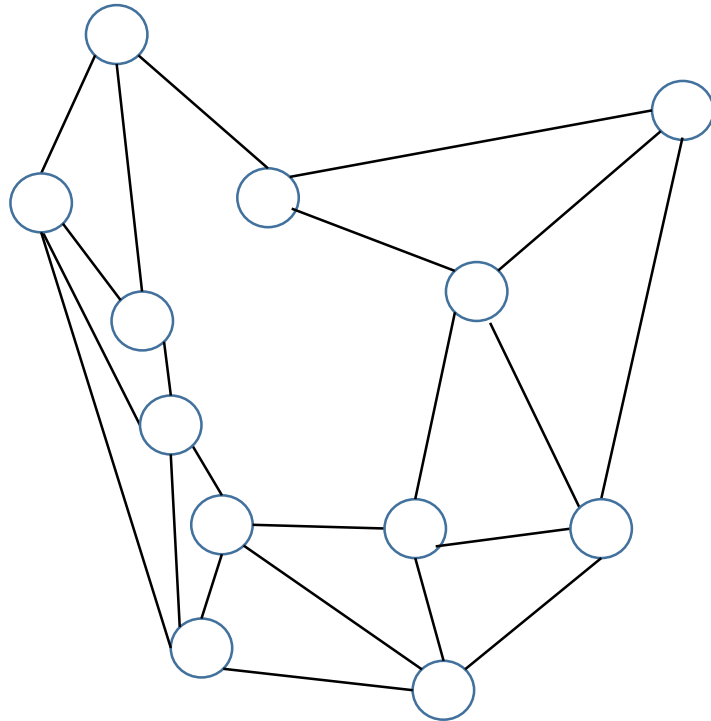
1 not in T, 3 in T



1, 2, 3 not in T

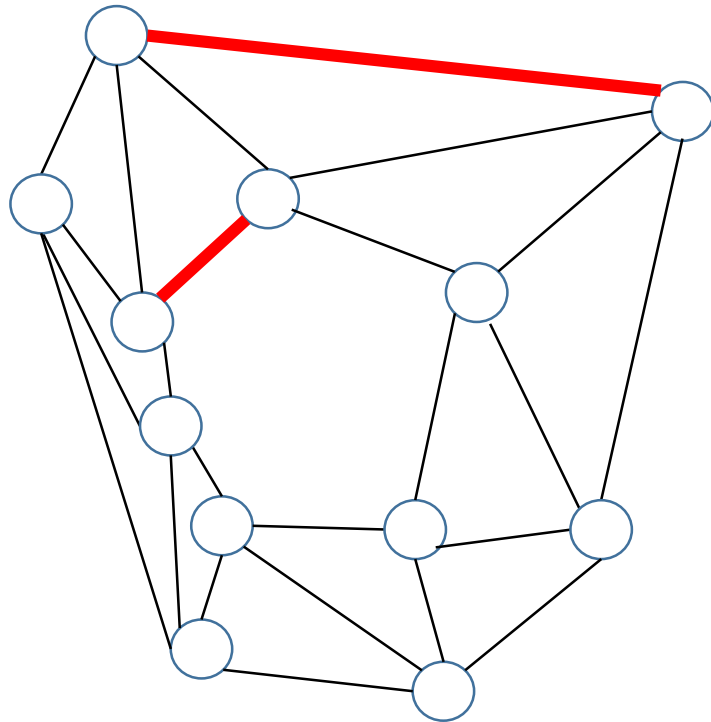
4. Four-regular graphs – the proof

Theorem 3: Deciding the existence of a non-trivial induced 3-regular subgraph in a 4-regular graph is NP-complete. Even for planar 4-regular graphs on the input.



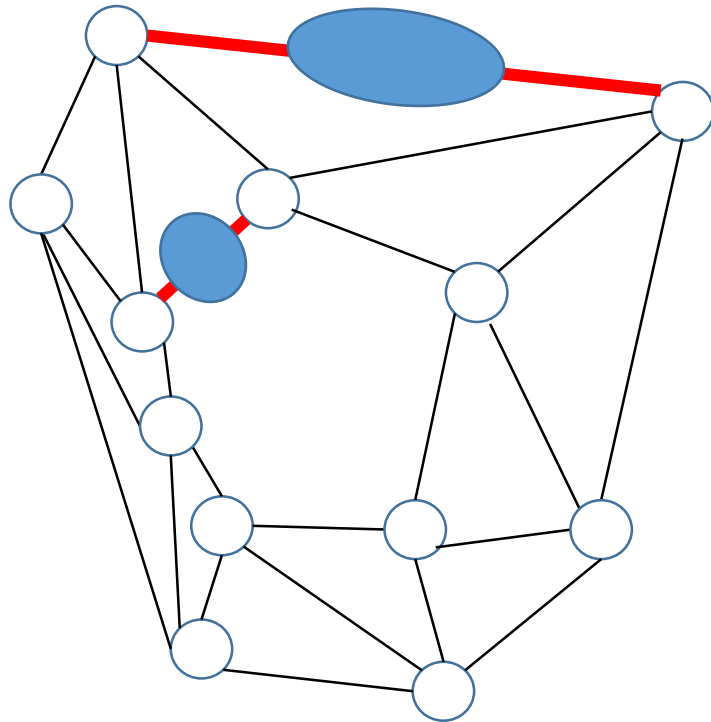
4. Four-regular graphs – the proof

Theorem 3: Deciding the existence of a non-trivial induced 3-regular subgraph in a 4-regular graph is NP-complete. Even for planar 4-regular graphs on the input.



4. Four-regular graphs – the proof

Theorem 3: Deciding the existence of a non-trivial induced 3-regular subgraph in a 4-regular graph is NP-complete. Even for planar 4-regular graphs on the input.



5. Induced k -regular subgraphs in d -regular graphs

Theorem 4: For every k and d such that $k \geq 2d/3$, deciding the existence of a nontrivial induced k -regular subgraph in a d -regular graph is NP-complete.

5. Induced 3-regular subgraphs in d -regular graphs

Question 1: For every $d > 3$, does there exist a d -regular graph which does not contain a non-trivial induced 3-regular subgraph?

Question 2: For which $d > 3$ is deciding the existence of a non-trivial induced 3-regular subgraph in a d -regular graph NP-complete?

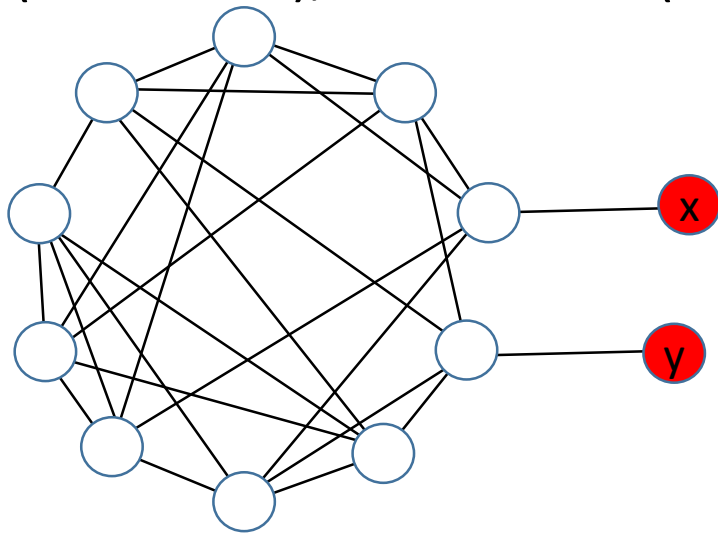
5. Induced 3-regular subgraphs in d -regular graphs

Question 1: For every $d > 3$, does there exist a d -regular graph which does not contain a non-trivial induced 3-regular subgraph?

Yes for $d=4$ (see above), 5, 6, 8 (found by generative AI), open for 7, 9, 10, ...

Question 2: For which $d > 3$ is deciding the existence of a non-trivial induced 3-regular subgraph in a d -regular graph NP-complete?

For $d=4$ (see above), and for $d=5$ (found by generative AI), open for all others



Dziękuję bardzo!

