

How Close is a Tree to a Euclidean Minimum Spanning Tree?

Todor Antić ¹, Jiří Fiala ¹, Jelena Glišić ¹, Grzegorz Gutowski ²,
Konstanty Junosza-Szaniawski ³, Jan Kratochvíl ¹, Giuseppe Liotta ⁴, Morteza Saghaian ⁵,
Maria Saumell ⁶, Krisztina Szilágyi ⁶, Pavel Valtr ¹

¹ Charles University, Prague, Czech Republic

² Jagiellonian University, Kraków, Poland

³ Warsaw University of Technology, Warsaw, Poland

⁴ Università degli Studi di Perugia, Perugia, Italy

⁵ Algorithm and Complexity Group, TU Wien, Austria

⁶ Czech Technical University in Prague, Czech Republic





HOMONOLO workshop 2025





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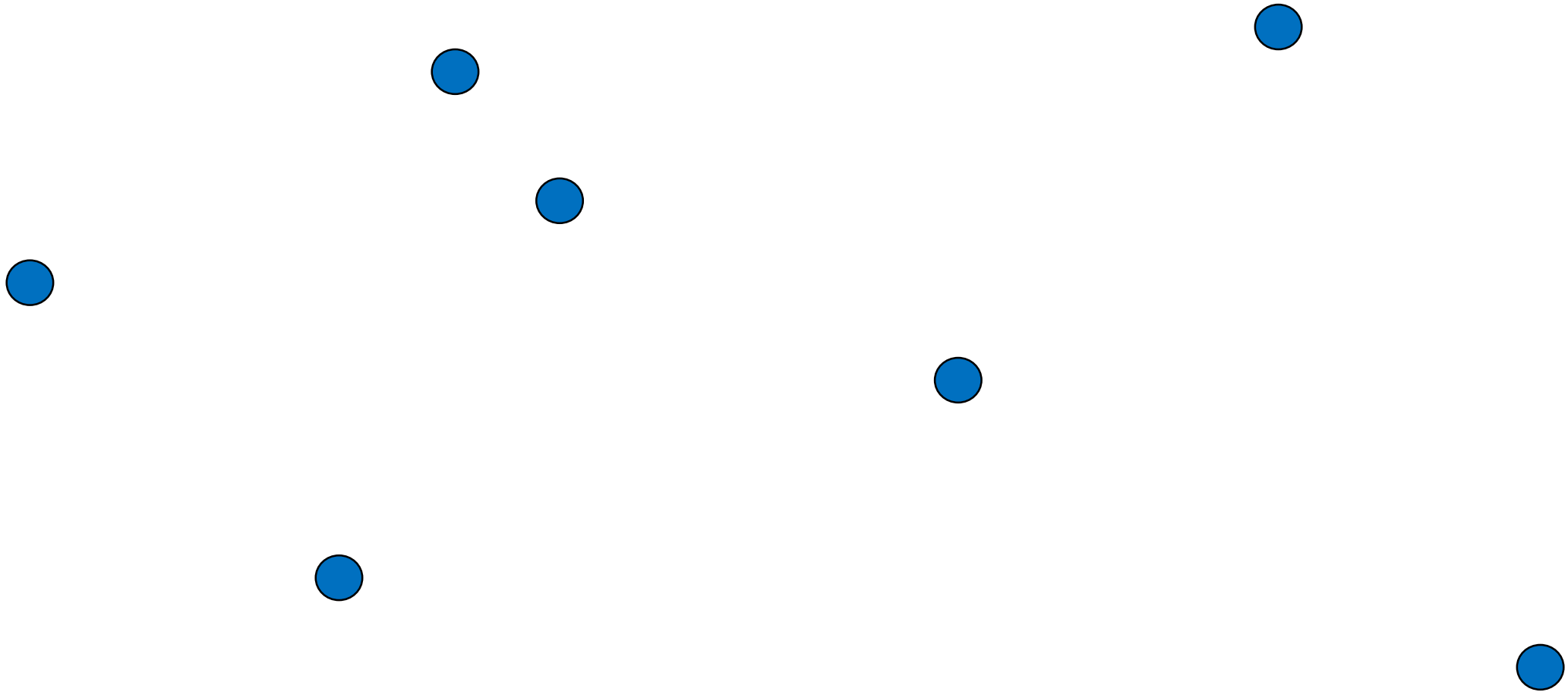
HOMONOLO workshop 2025



Photo taken by Jirka Fiala

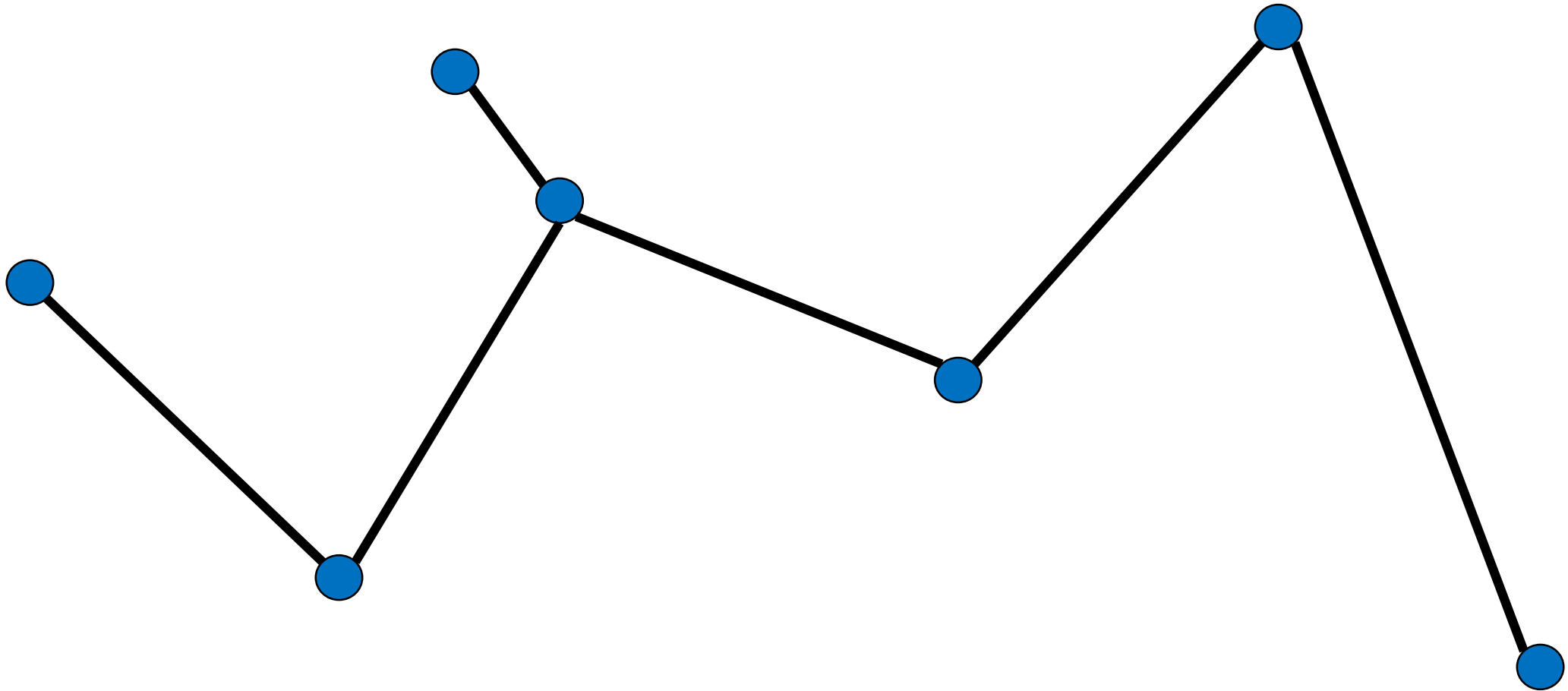


EMST – Euclidean Minimum Spanning Tree



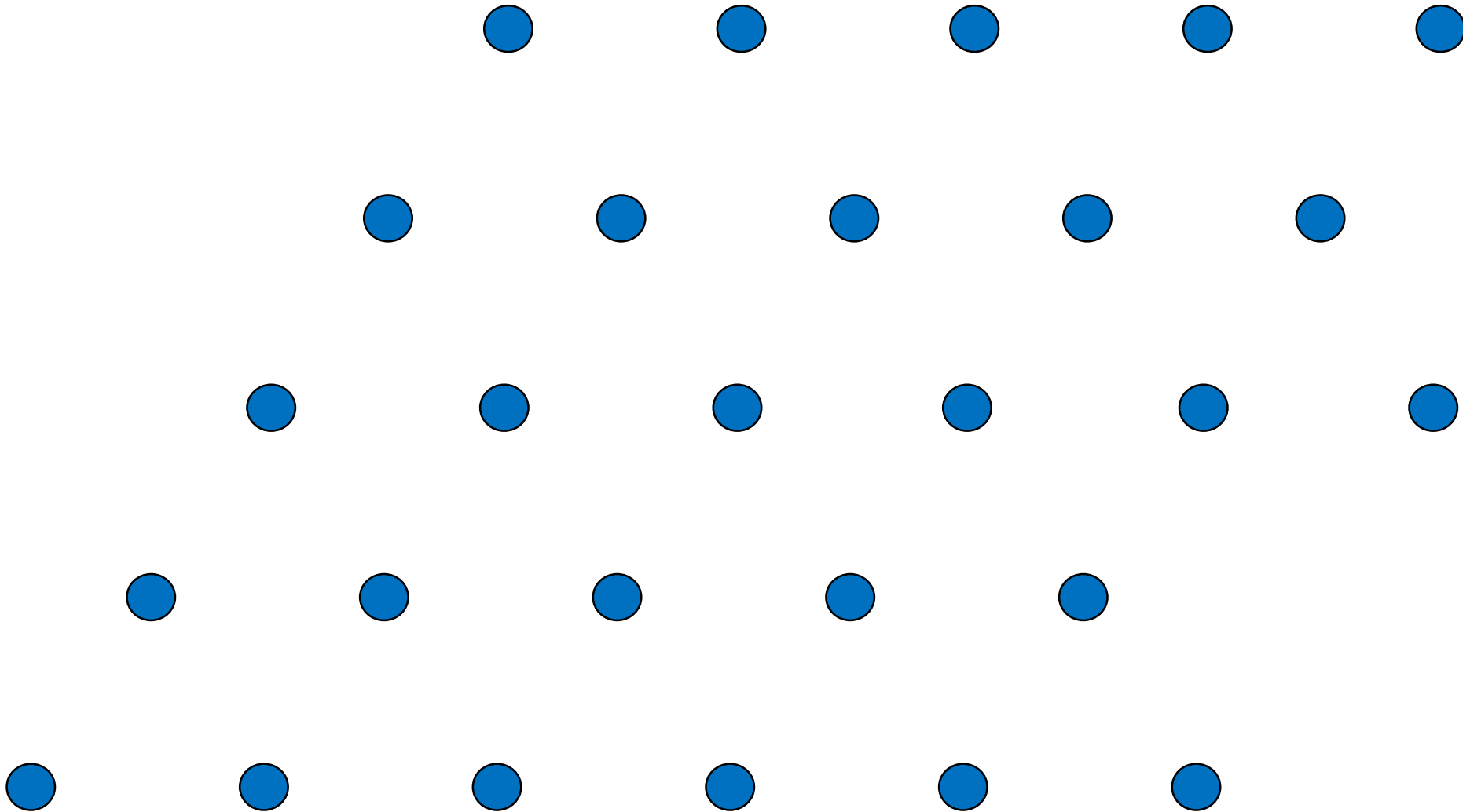


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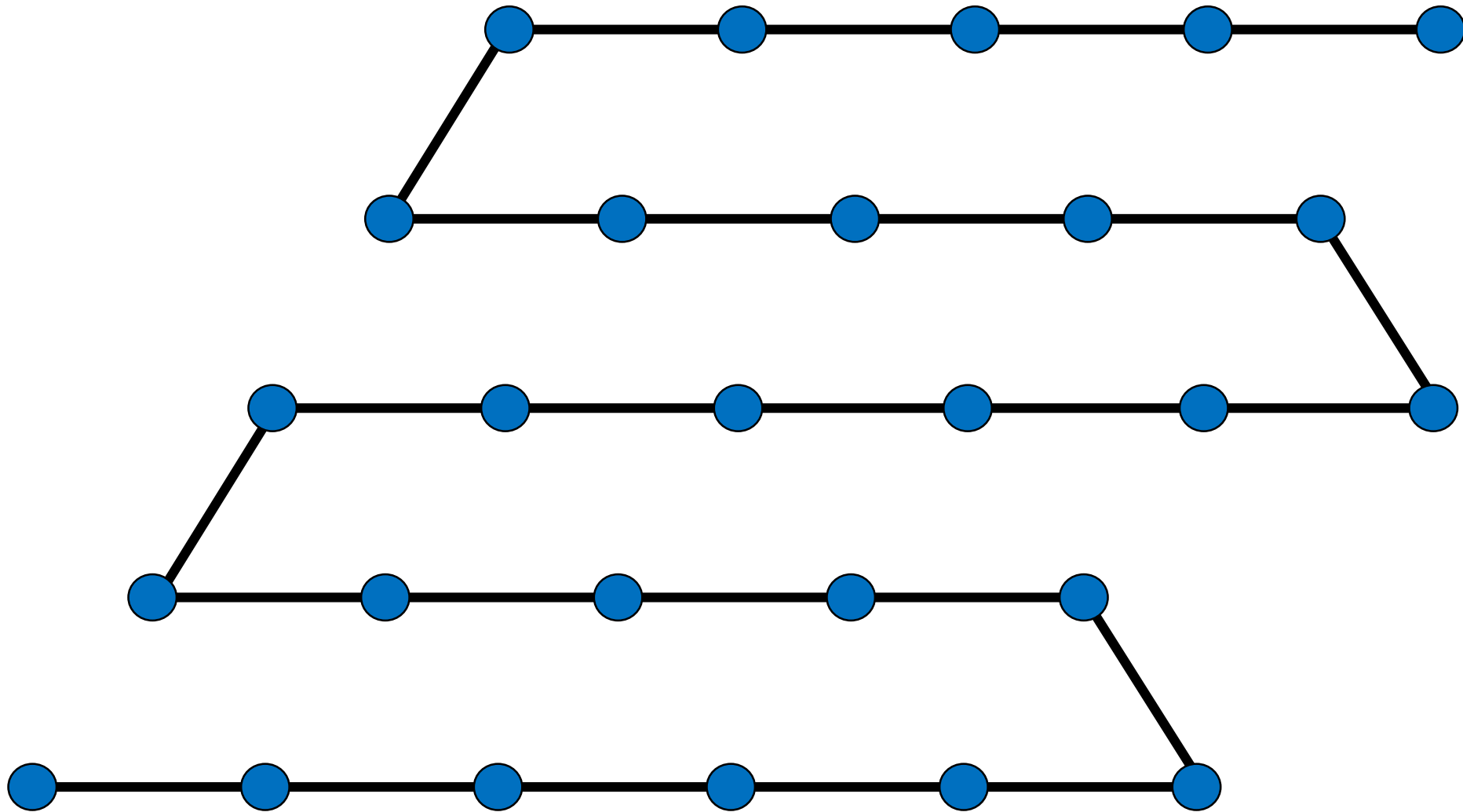


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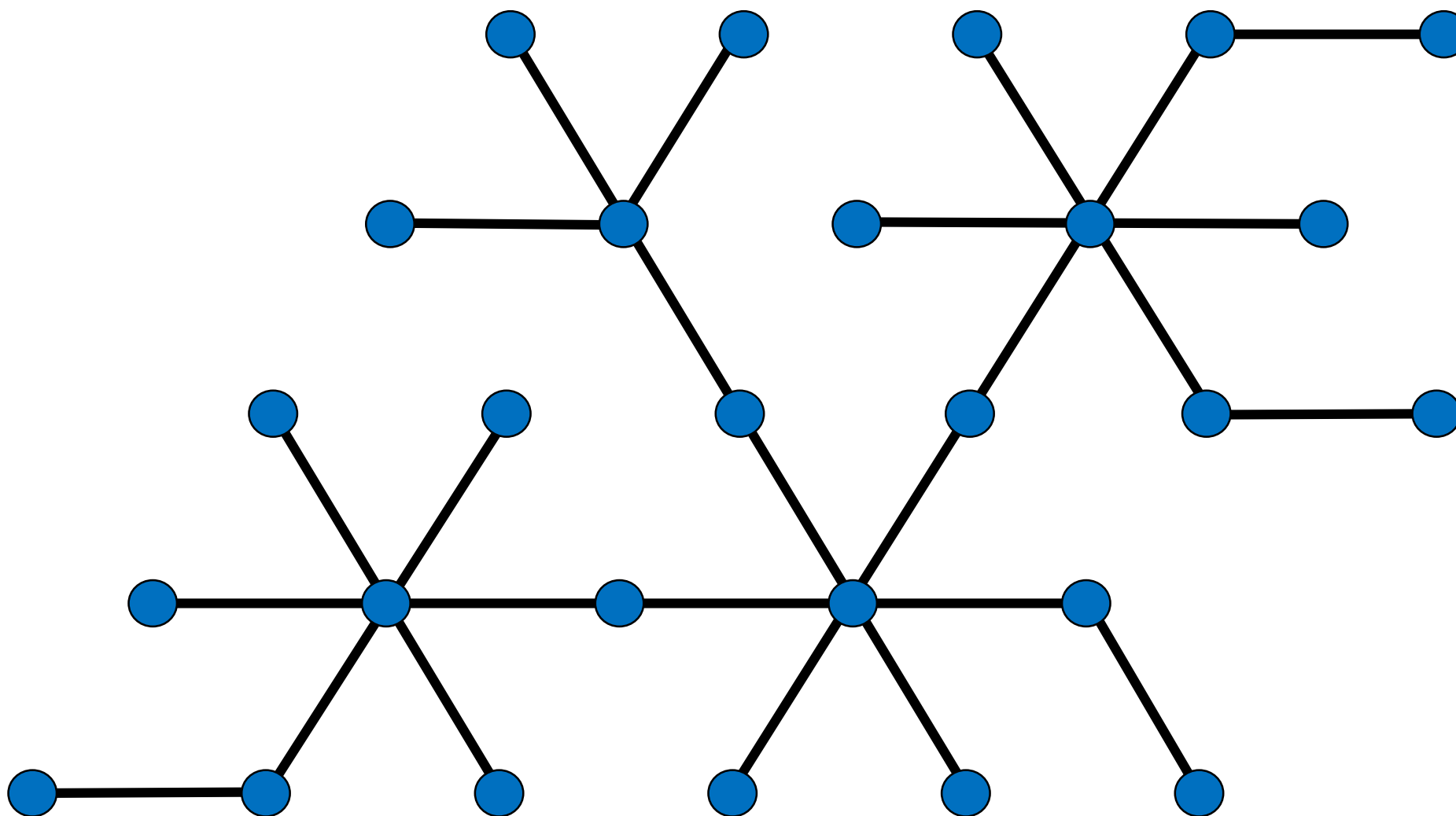
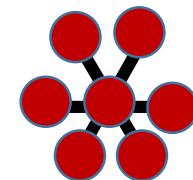


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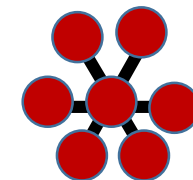


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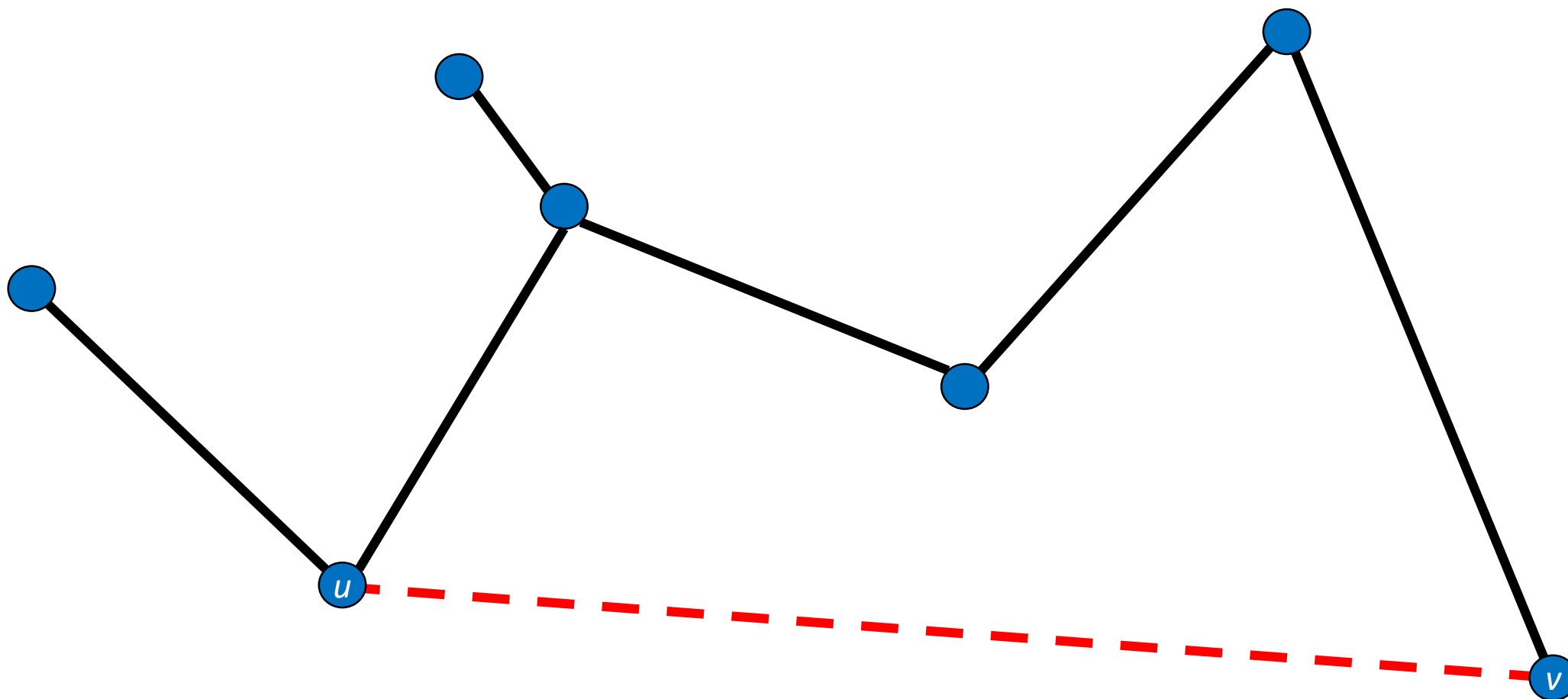


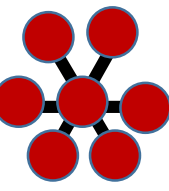


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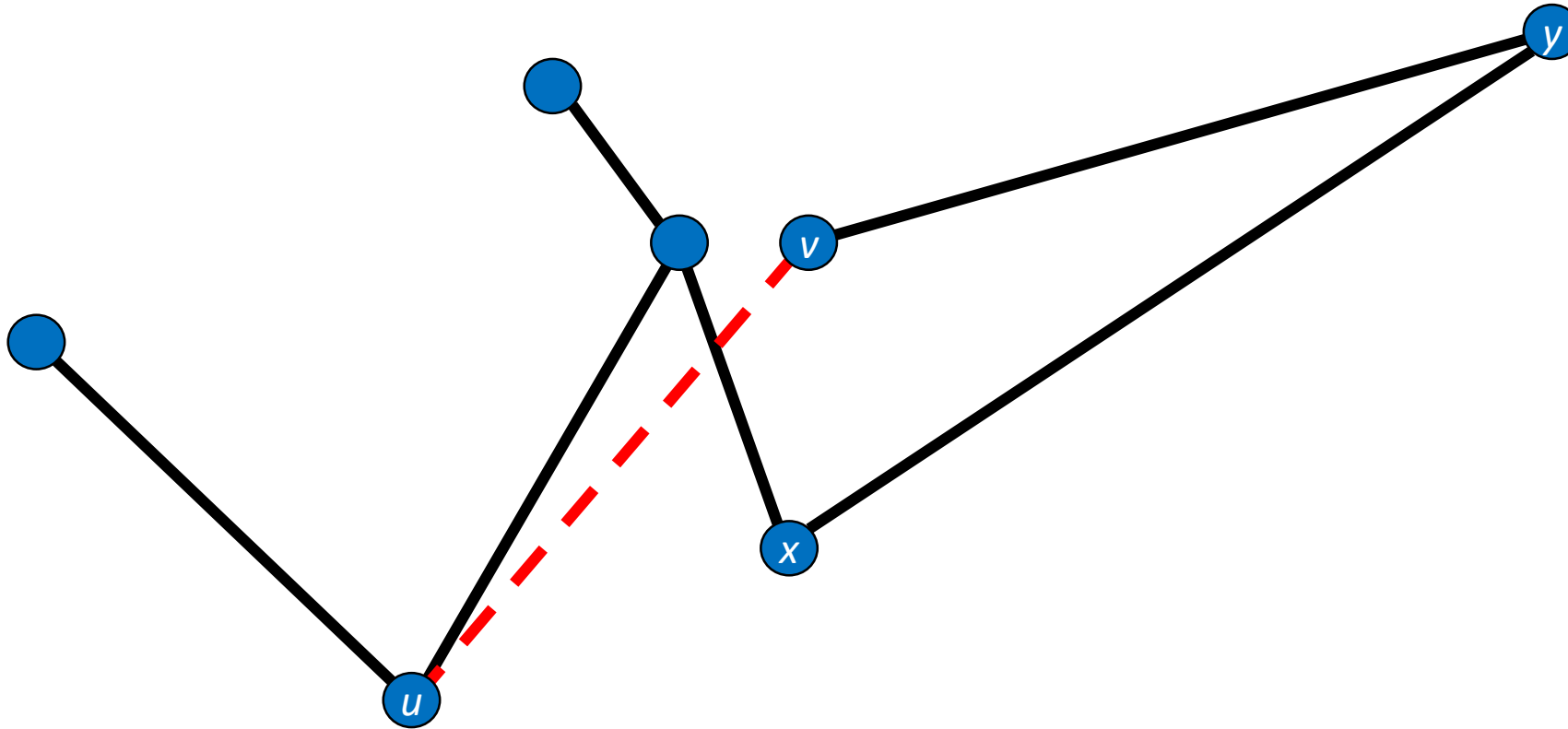
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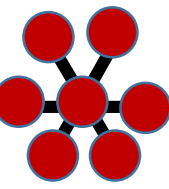




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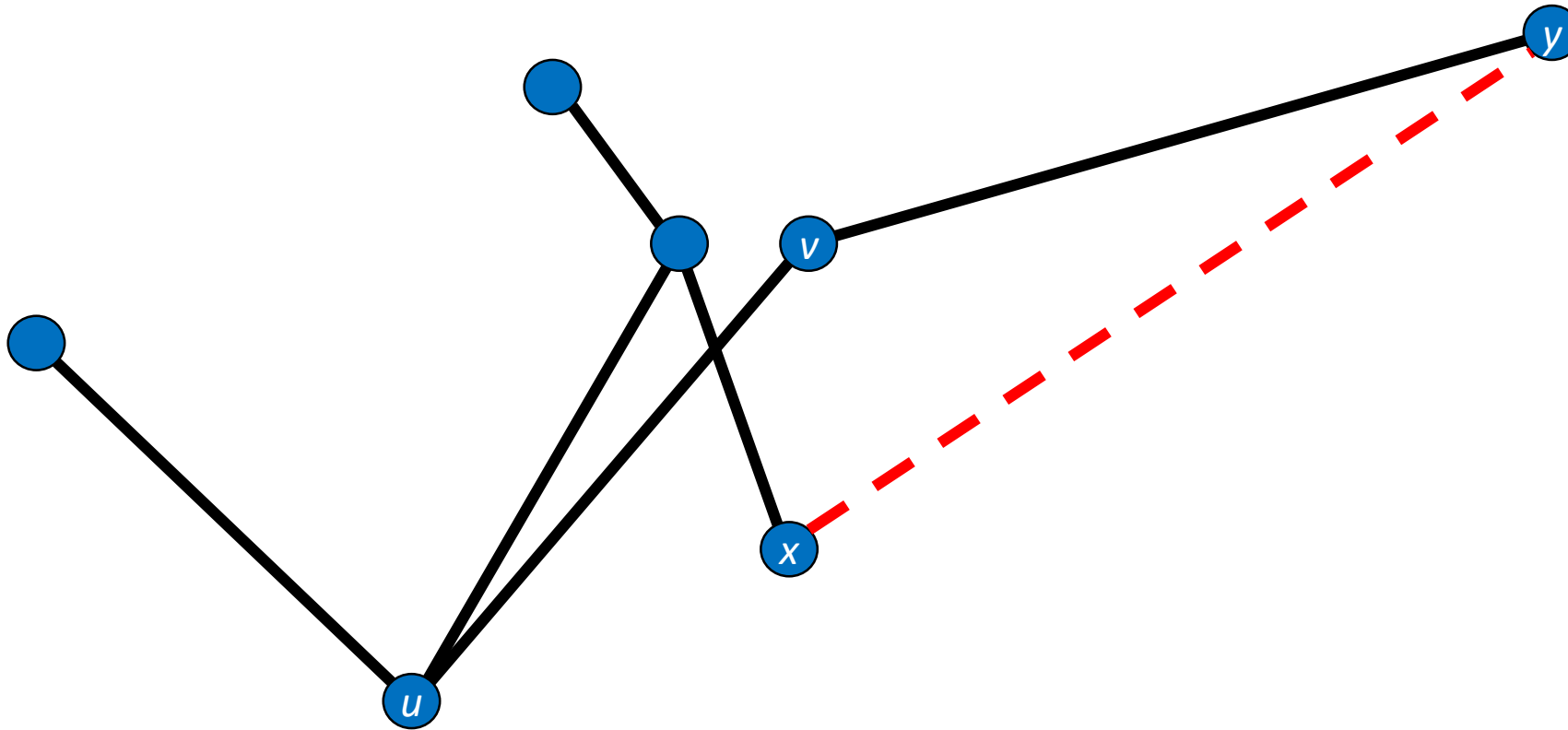
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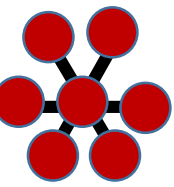
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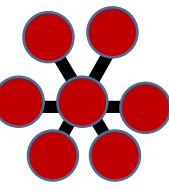


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An embedded tree Γ is an EMST for the point set formed by its vertices if and only if

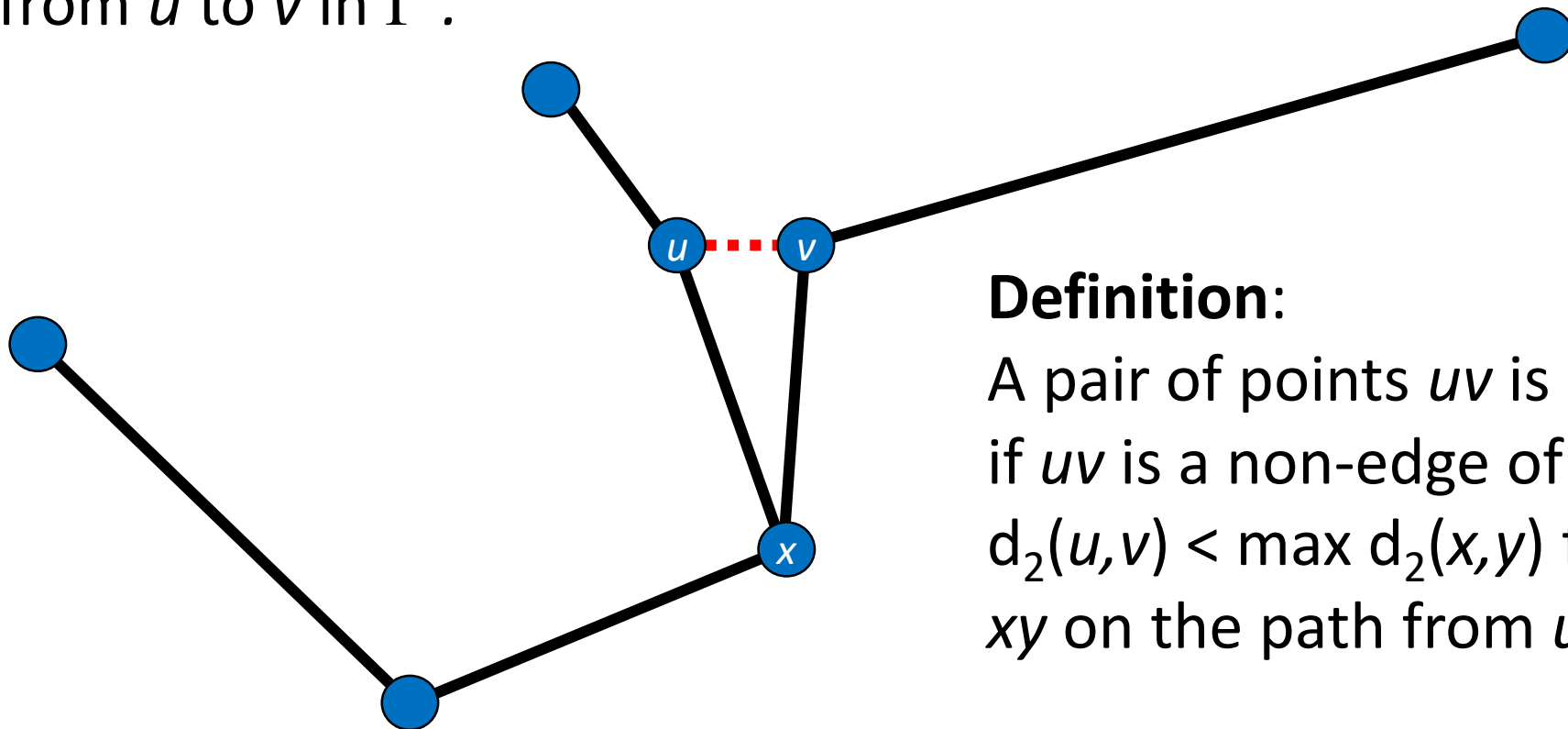
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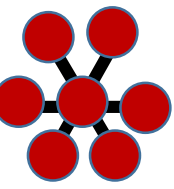


Definition:

A pair of points uv is called a **bad pair** if uv is a non-edge of Γ and $d_2(u,v) < \max d_2(x,y)$ for some edge xy on the path from u to v in Γ .



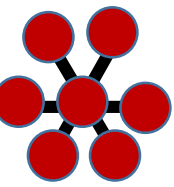
Which trees can be embedded as EMST?



Given an abstract tree T , does there exist a planar embedding Γ of T such that Γ is an EMST for the point set formed by its vertices?



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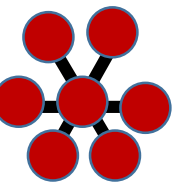
Definition:

A pair of points uv in an embedding Γ of a tree T is called a **bad pair** if uv is a non-edge of T and $d_2(u, v) < \max d_2(x, y)$ for some edge xy on the path from u to v in T .

Let $\text{bp}(\Gamma)$ be the number of bad pairs of the embedding Γ . Define $\text{bp}(T)$ as the minimum of $\text{bp}(\Gamma)$ over all planar embeddings Γ of T .



Which trees can be embedded as EMST?



Given an abstract tree T , does there exist a planar embedding Γ of T such that $\text{bp}(\Gamma) = 0$?

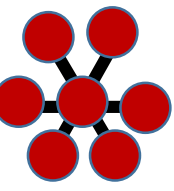
Monma and Suri [1992]: Every tree of maximum degree at most 5 can be embedded as an EMST.

Eades and Whitesides [1996]: It is NP-hard to decide if a tree of maximum degree 6 can be embedded as an EMST.

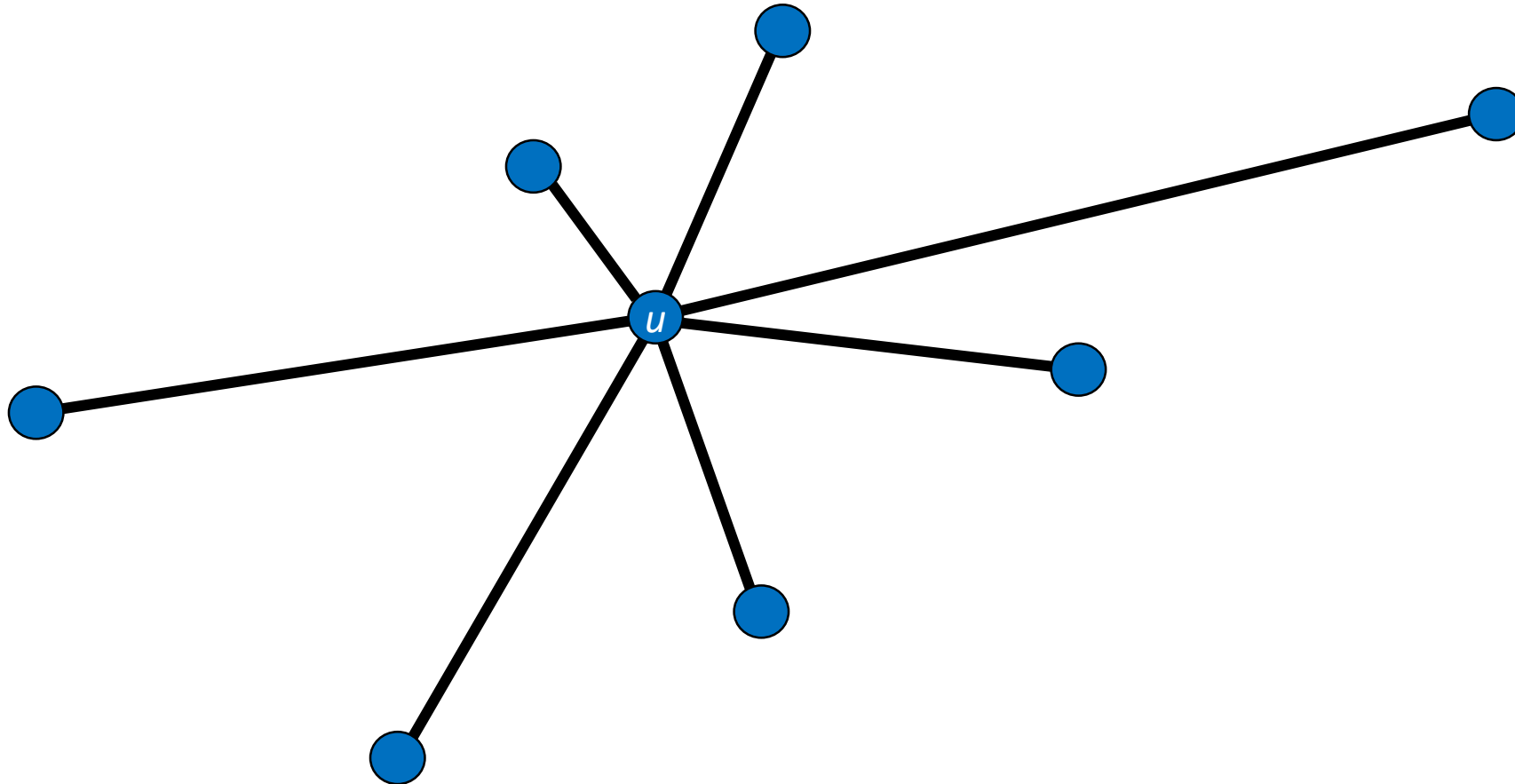
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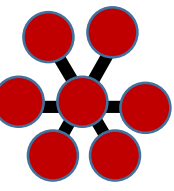


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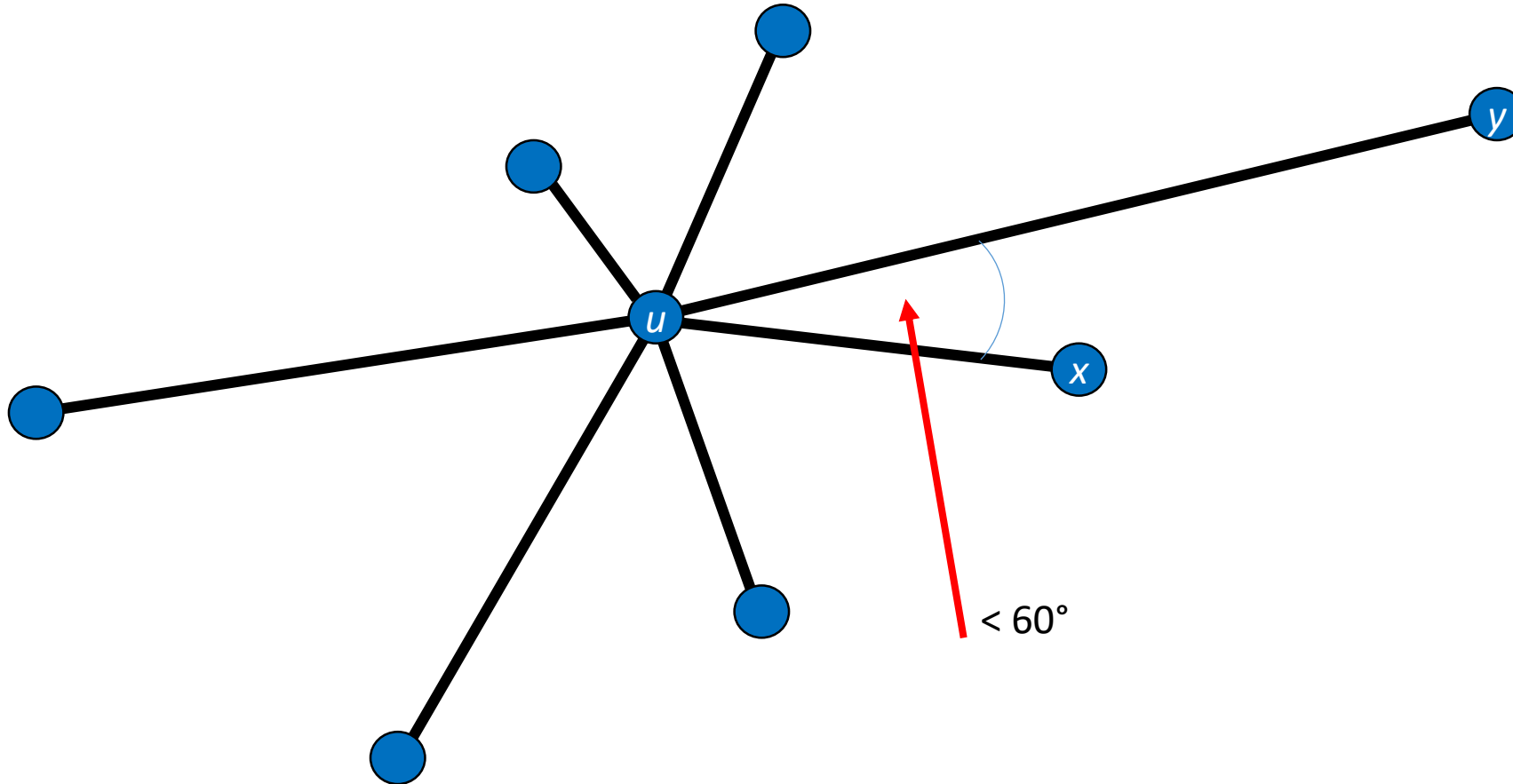




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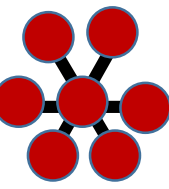


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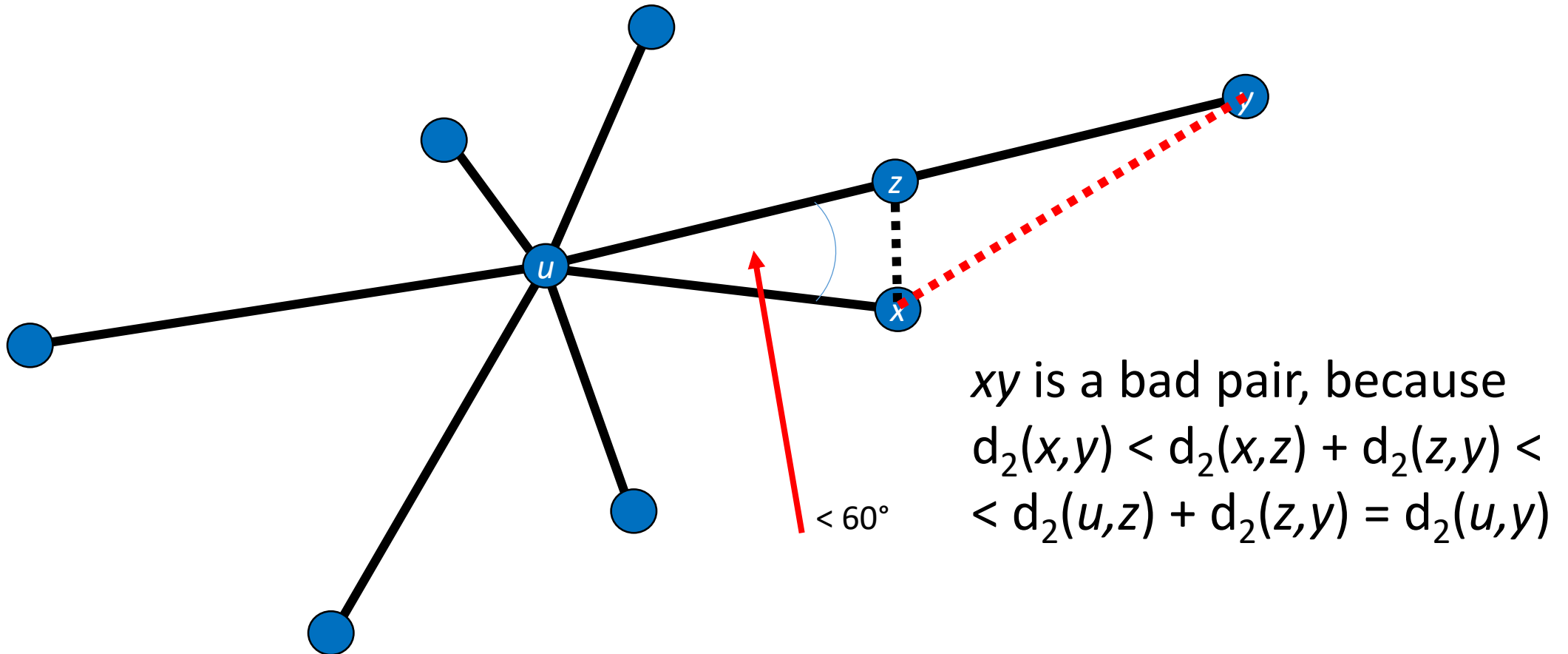




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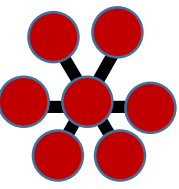


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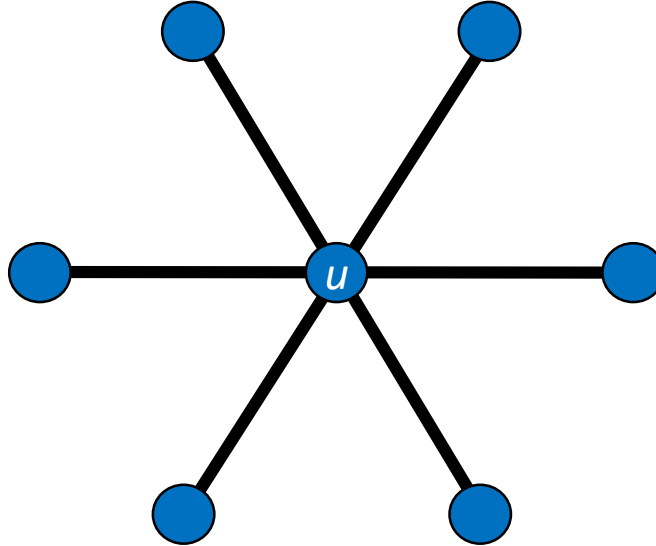


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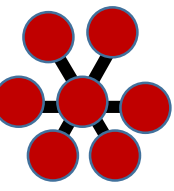
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Similarly, the neighbors of a degree 6 vertex in an EMST drawing must be placed in the corners of a regular hexagon centered in that vertex.





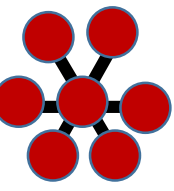
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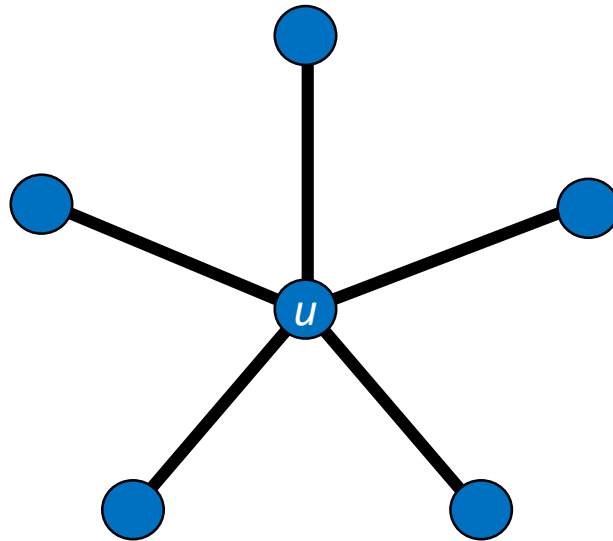
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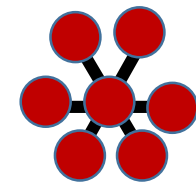


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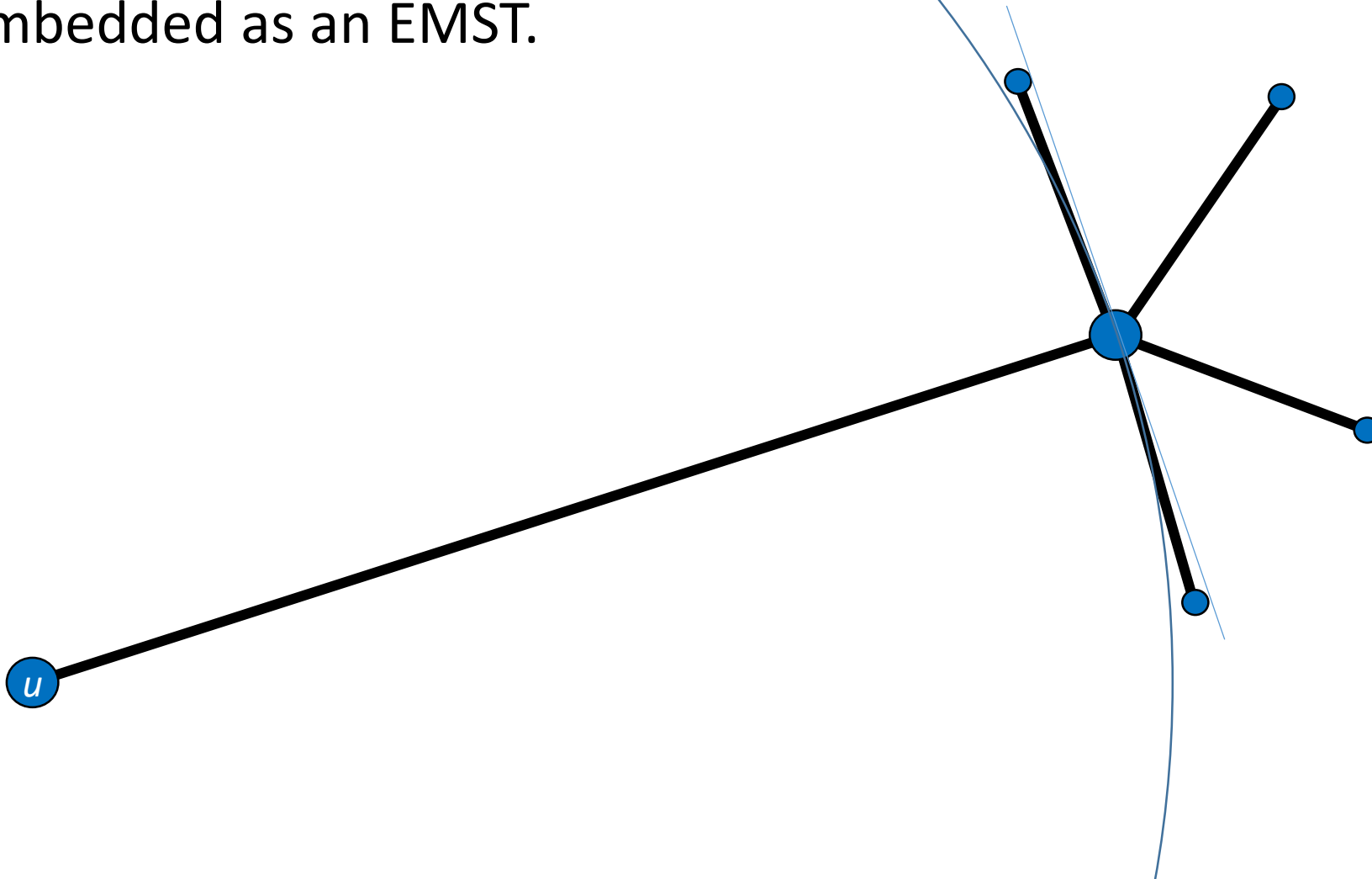
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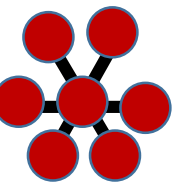




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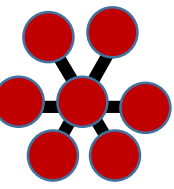
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Our results

- Characterization of caterpillars that can be embedded as EMST.
- Polynomial-time algorithm for determining $\text{bp}(T)$ for caterpillars T of maximum degree ≤ 6 .
- Exact formula for $\text{bp}(K_{1,s})$ for arbitrary s .
- Upper bound $\text{bp}(T) \leq \Delta^2 n \log n$ for trees T of maximum degree Δ .



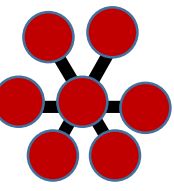
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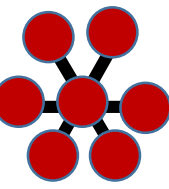
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All results are constructive, drawings matching the promised bounds can be constructed in polynomial time in the real RAM model.

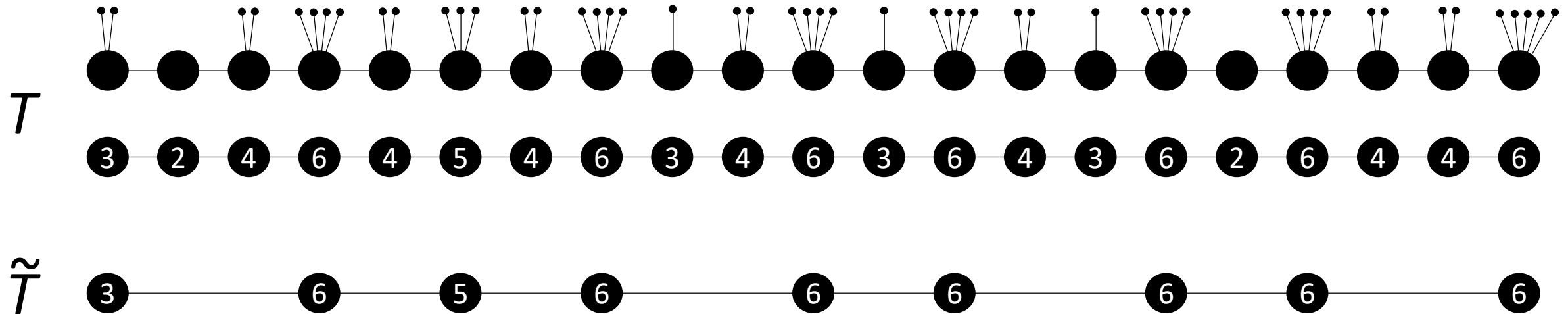


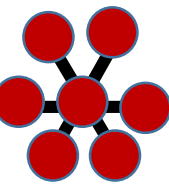
1. Caterpillars as EMST





Reduced model $\tilde{T}$ of a caterpillar T - the edges of $\tilde{T}$ correspond to maximal segments on the spine with no inner vertex of degree 5 or 6.





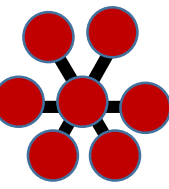
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Theorem 1. A caterpillar T is EMST-drawable if and only if

- the maximum degree is at most 6
- every segment of T corresponding to a 66 edge of $\tilde{T}$ contains a vertex of degree at most 2, or at least two vertices of degree 3
- every segment of T corresponding to a 65 edge of $\tilde{T}$ contains a vertex of degree at most 3.

$\tilde{T}$





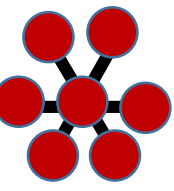
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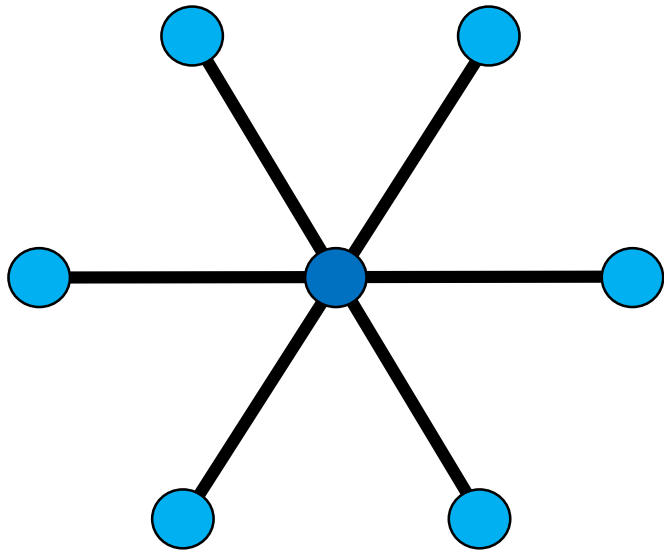
In other words, $64..46$, $64..434..46$ and $64..45$ are the only minimal non-EMST-drawable caterpillars of maximum degree at most 6.

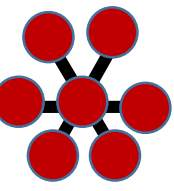




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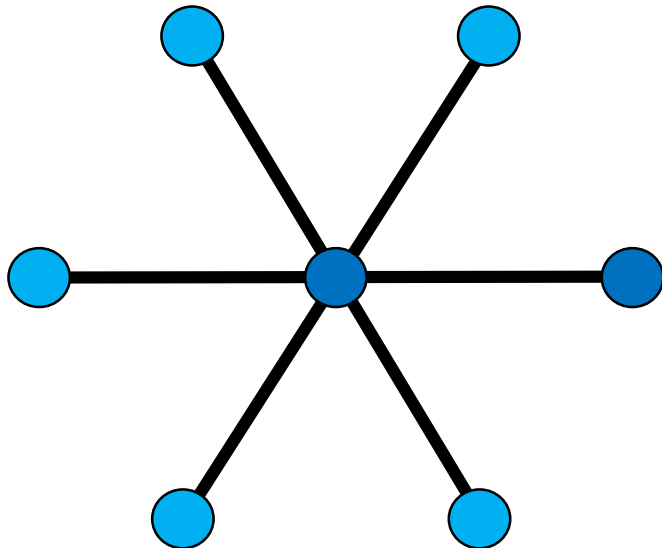
Lemma: Every EMST-drawing of $G_{64..4}$ has the vertices placed in a triangular grid and each leaf of the last spine vertex has only 3 available locations for possible neighbors.

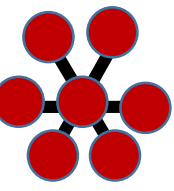




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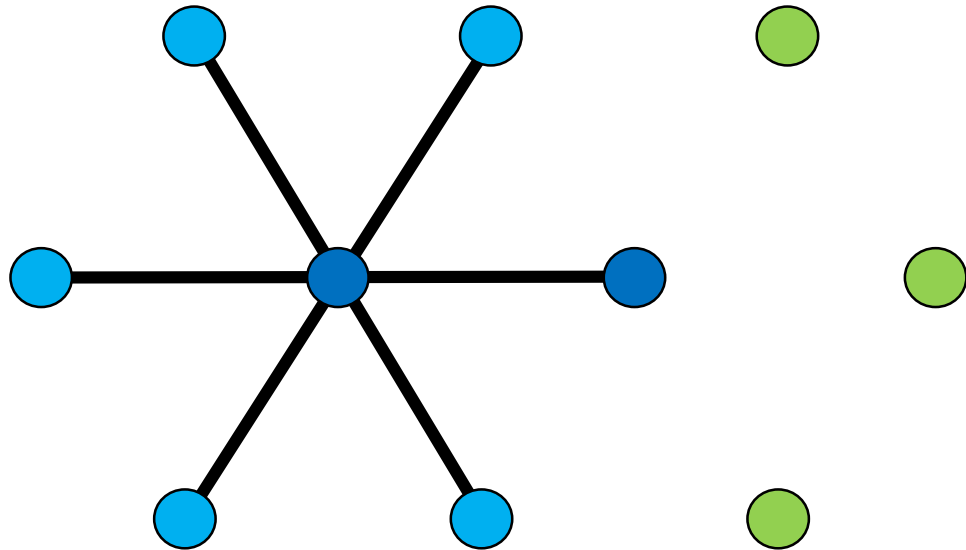
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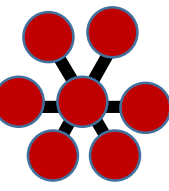




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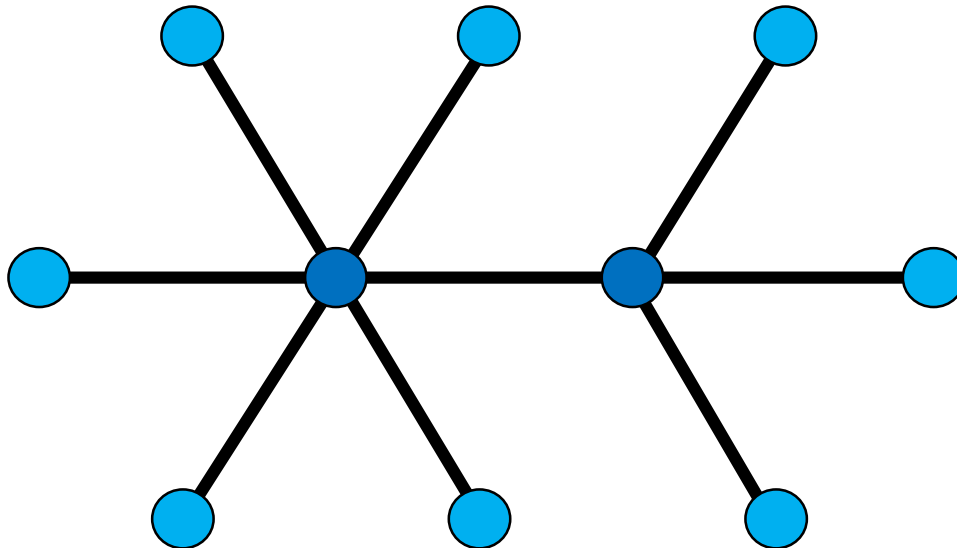
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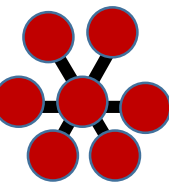




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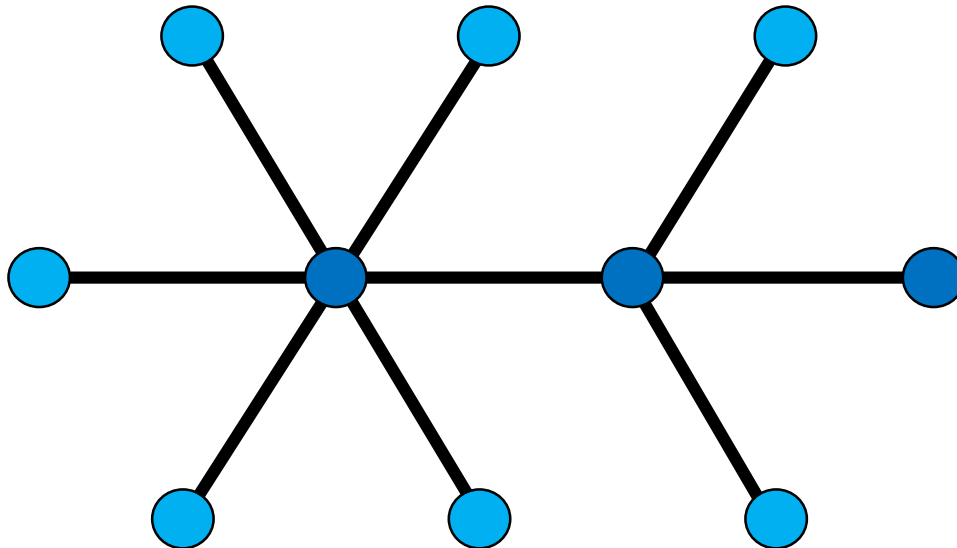
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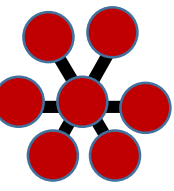




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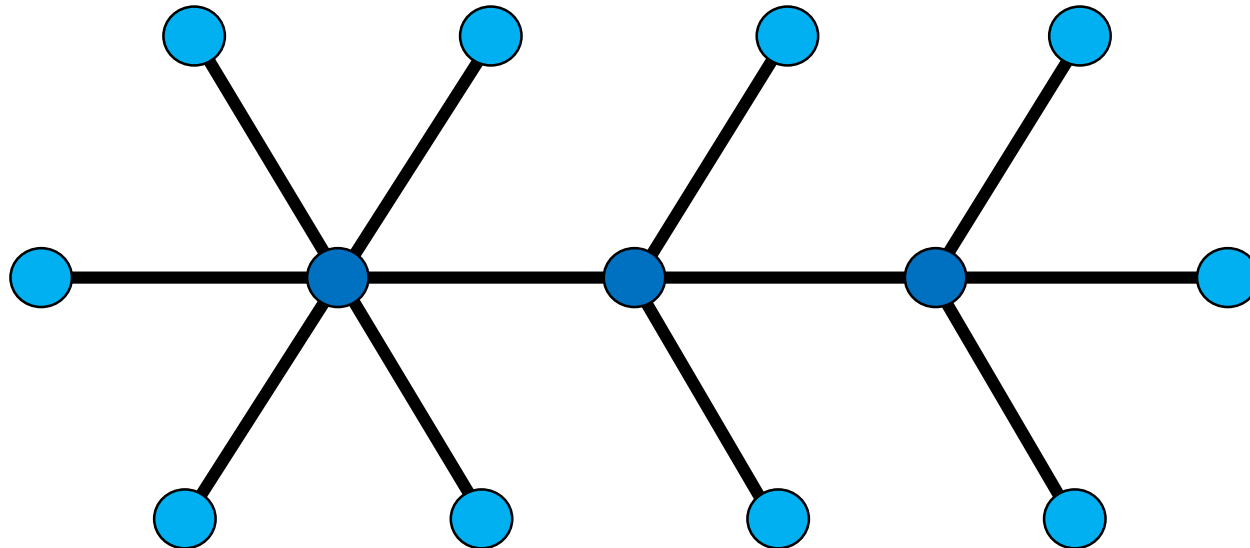
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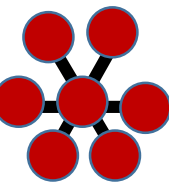




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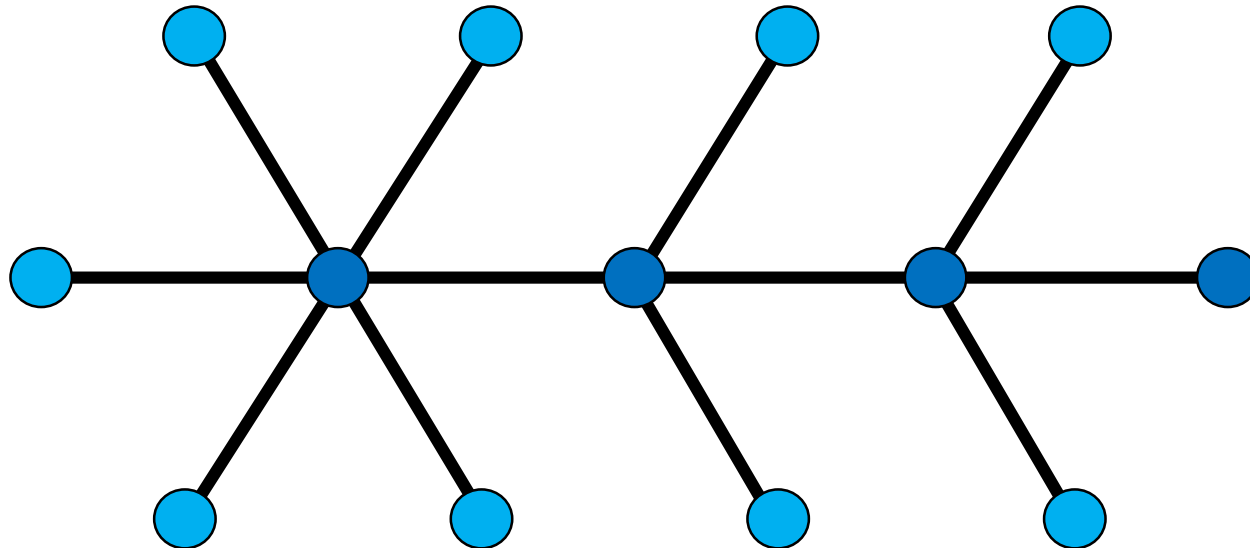
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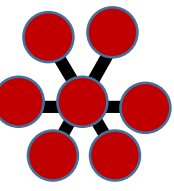




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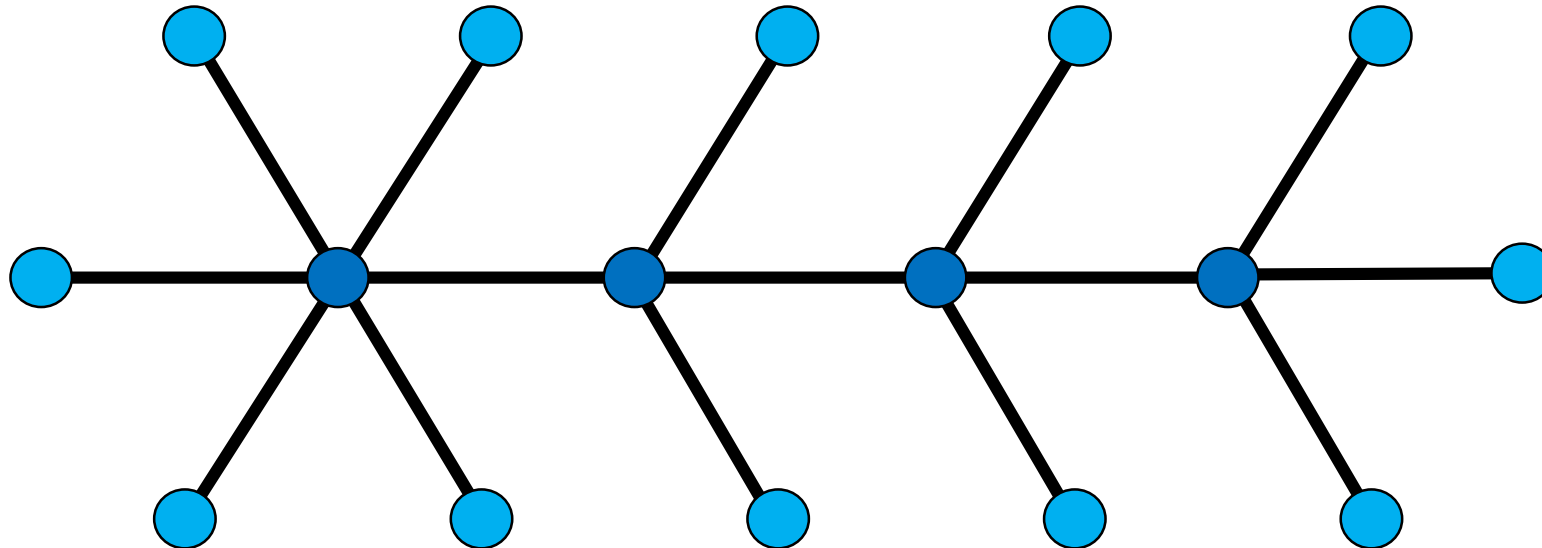
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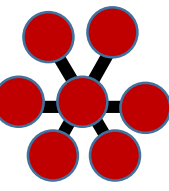




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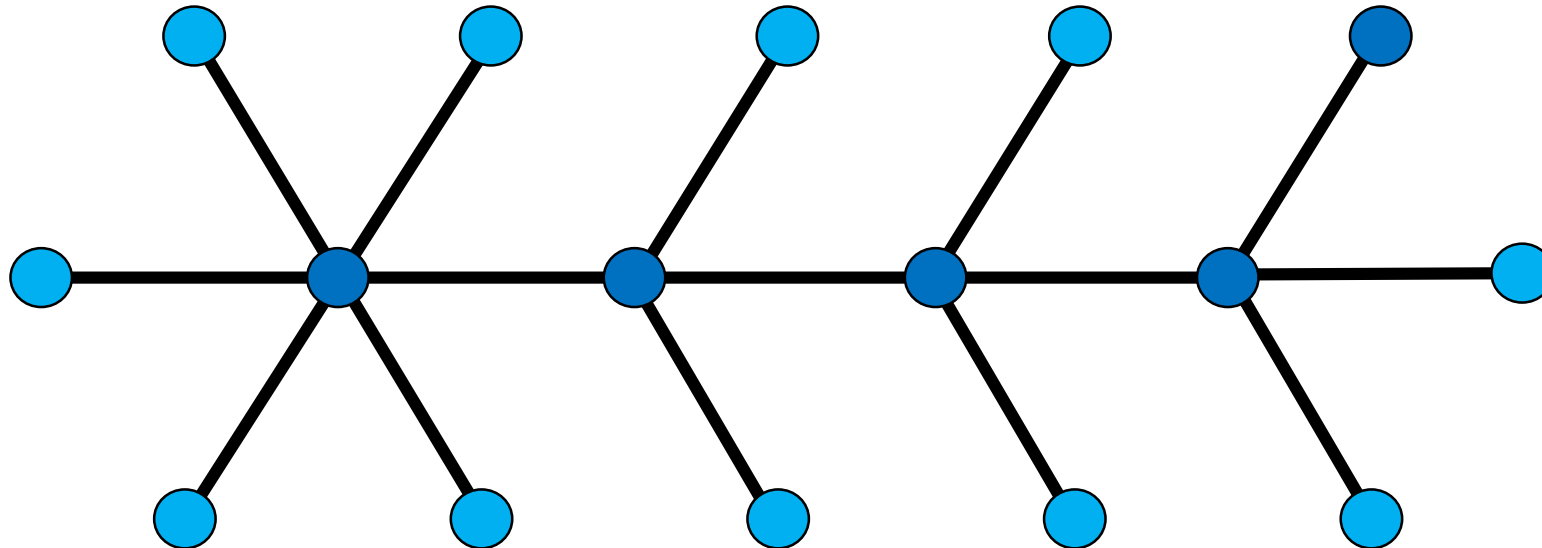
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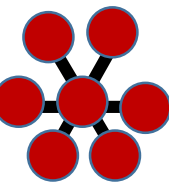




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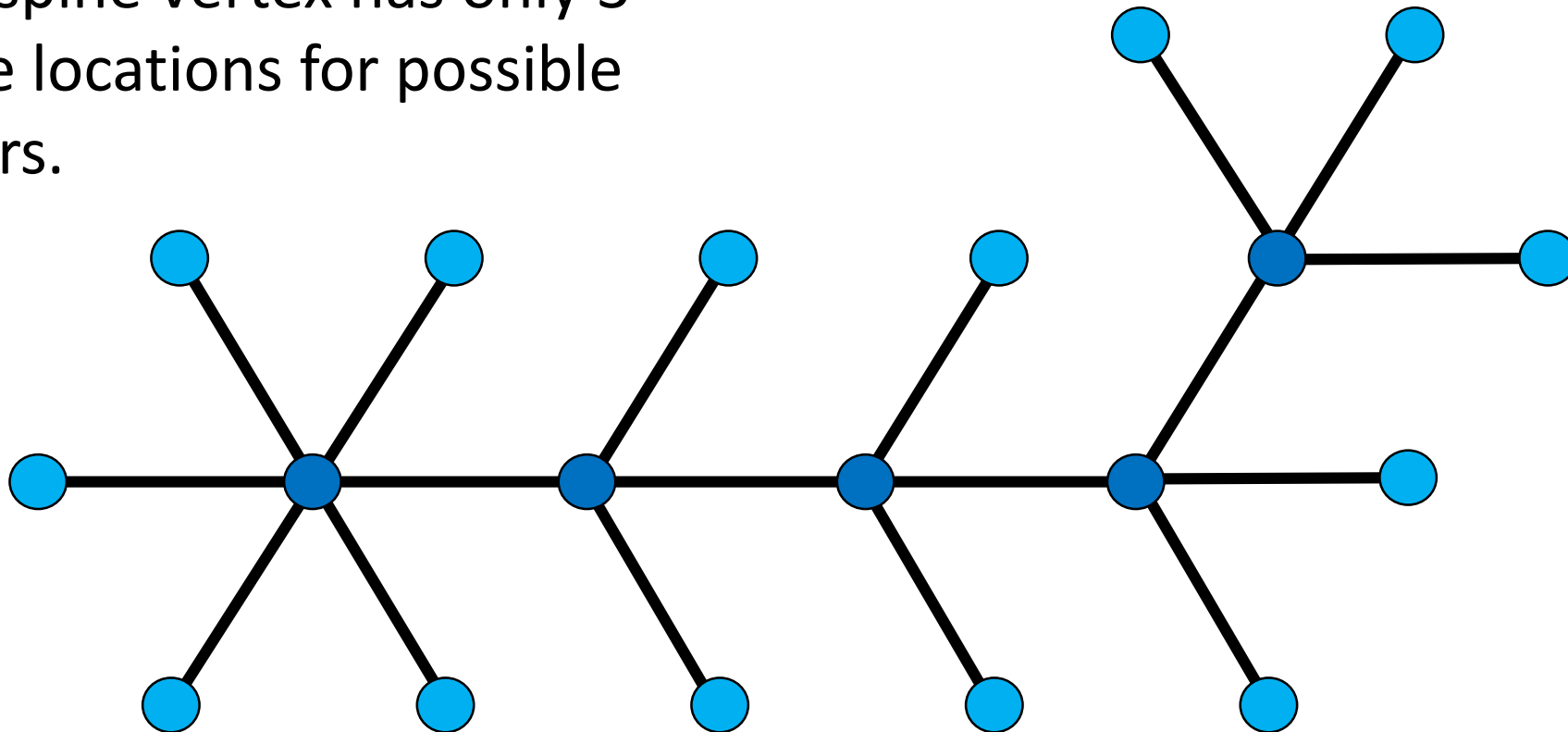
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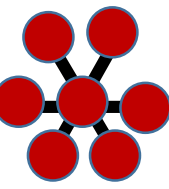




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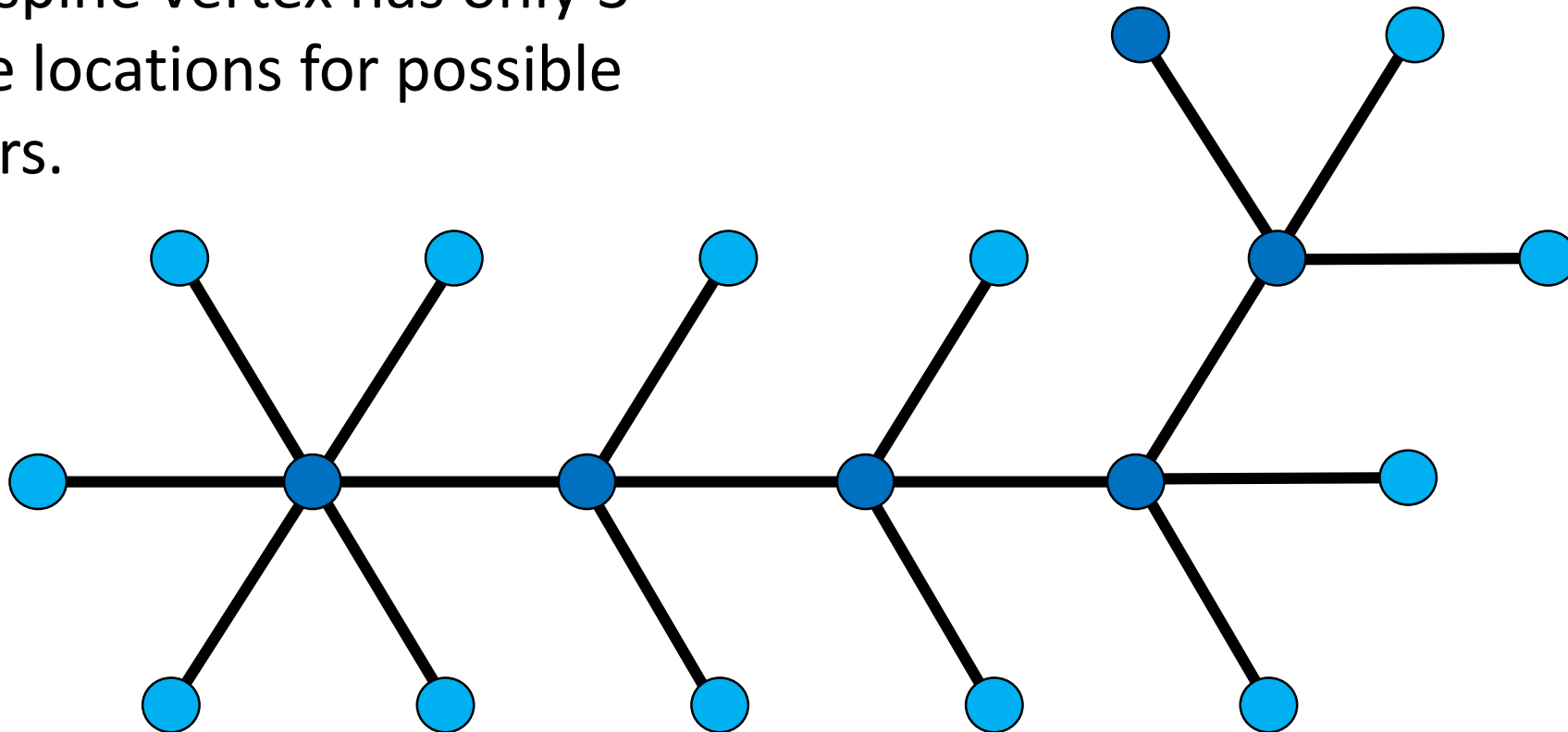
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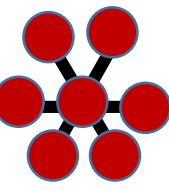




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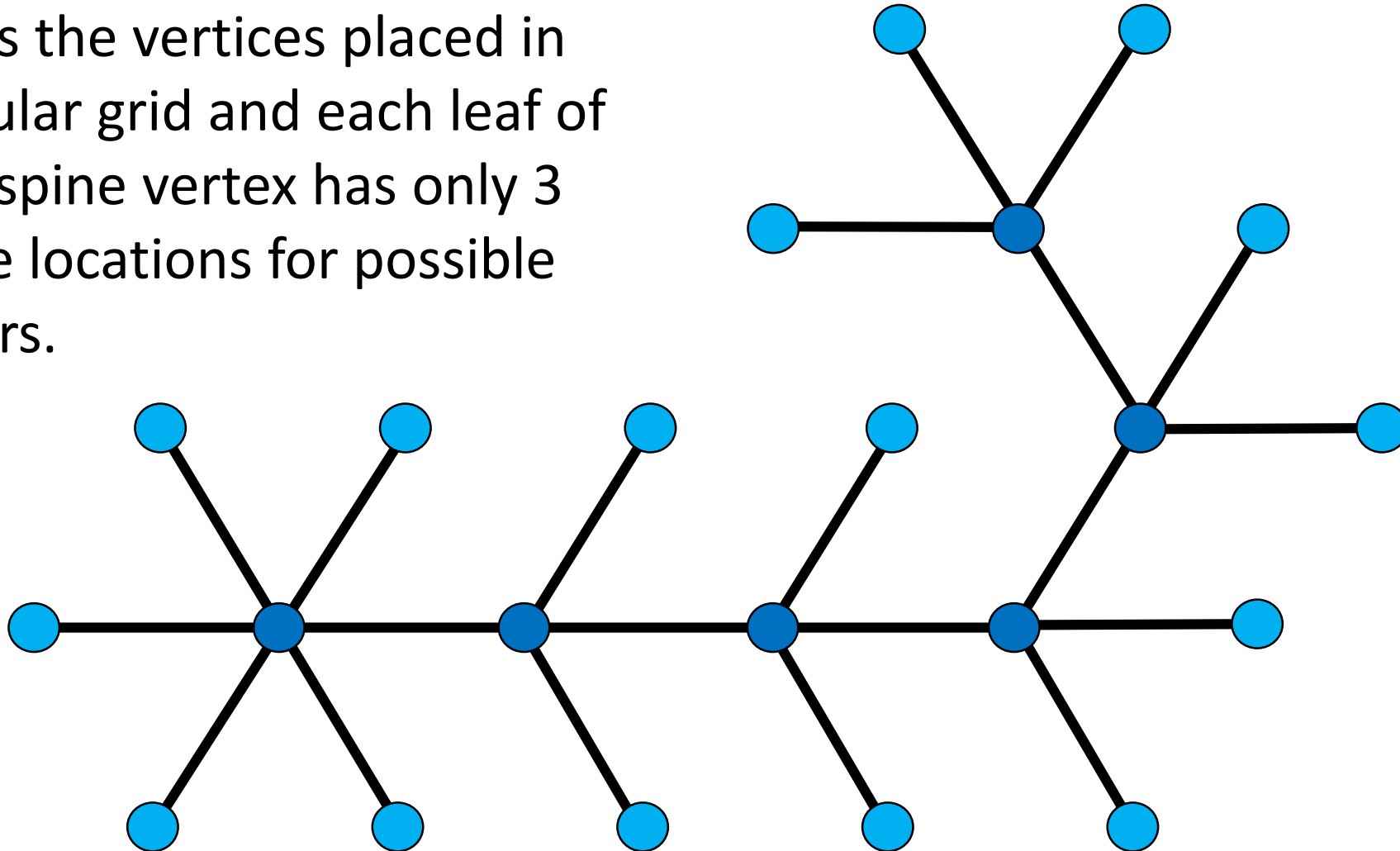
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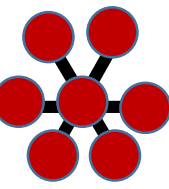




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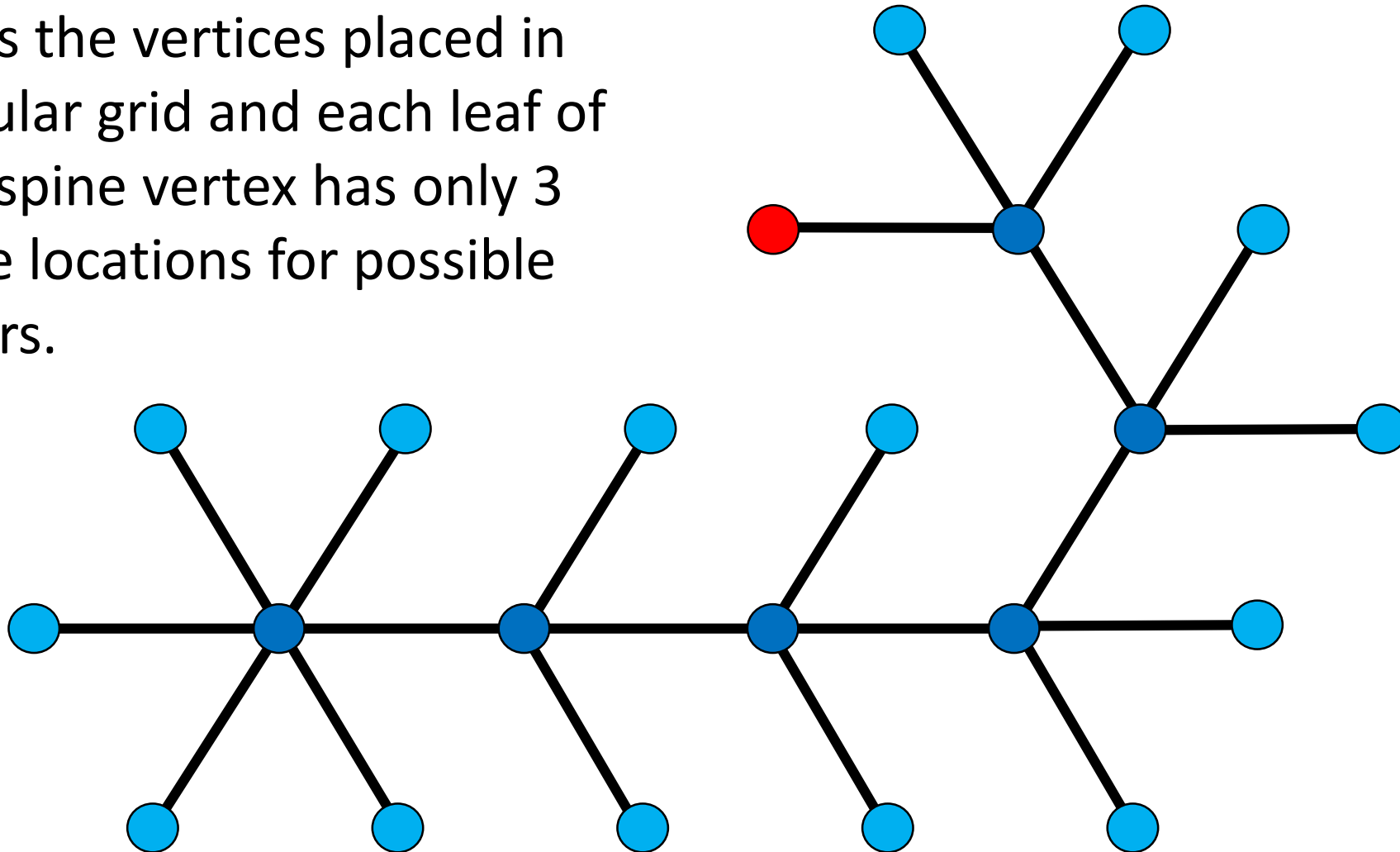
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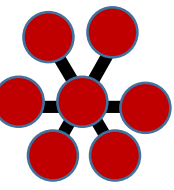




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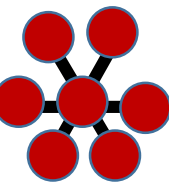




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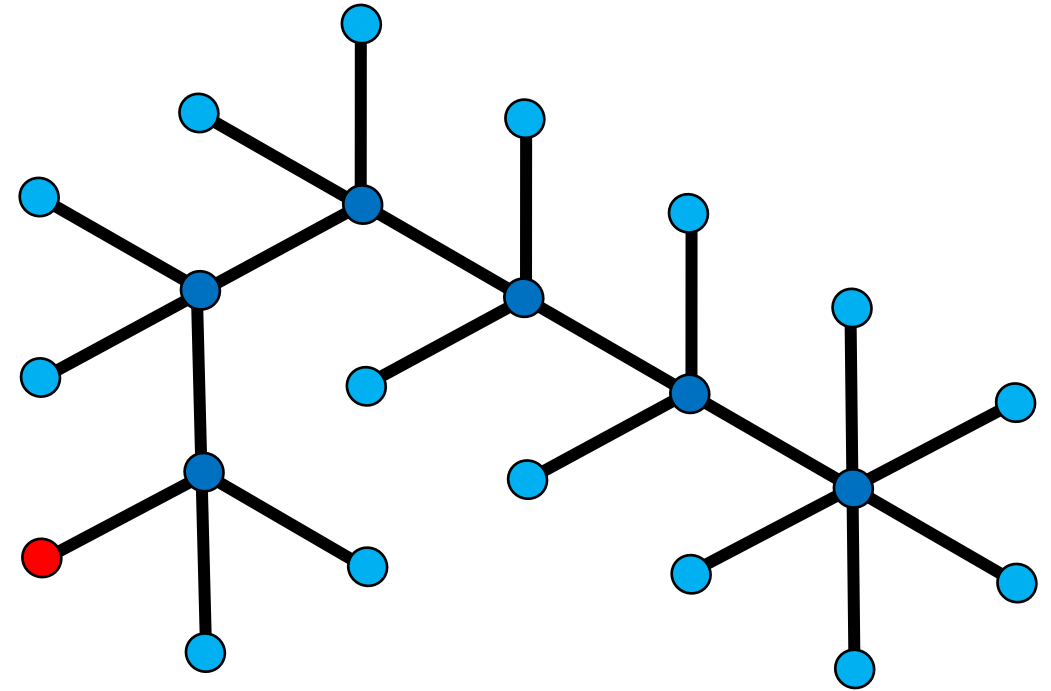
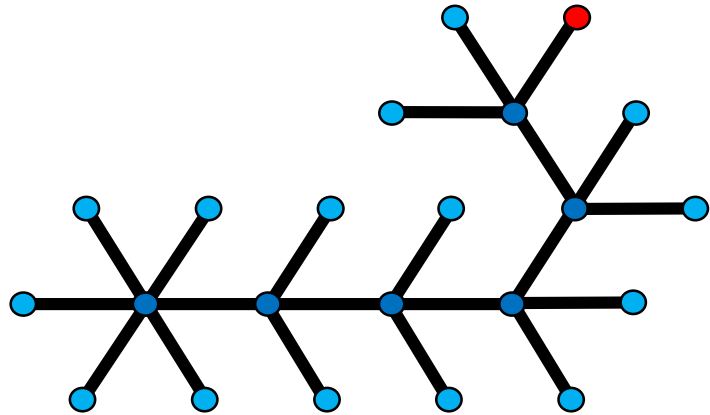
Lemma: Every EMST-drawing of 64..4 has the vertices placed in a triangular grid and each leaf of the last spine vertex has only 3 available locations for possible neighbors.

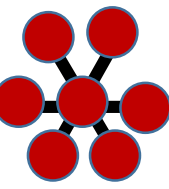
Corollary: 64..45 is not EMST-drawable. Hence neither 64..46.



1. Caterpillars as EMST

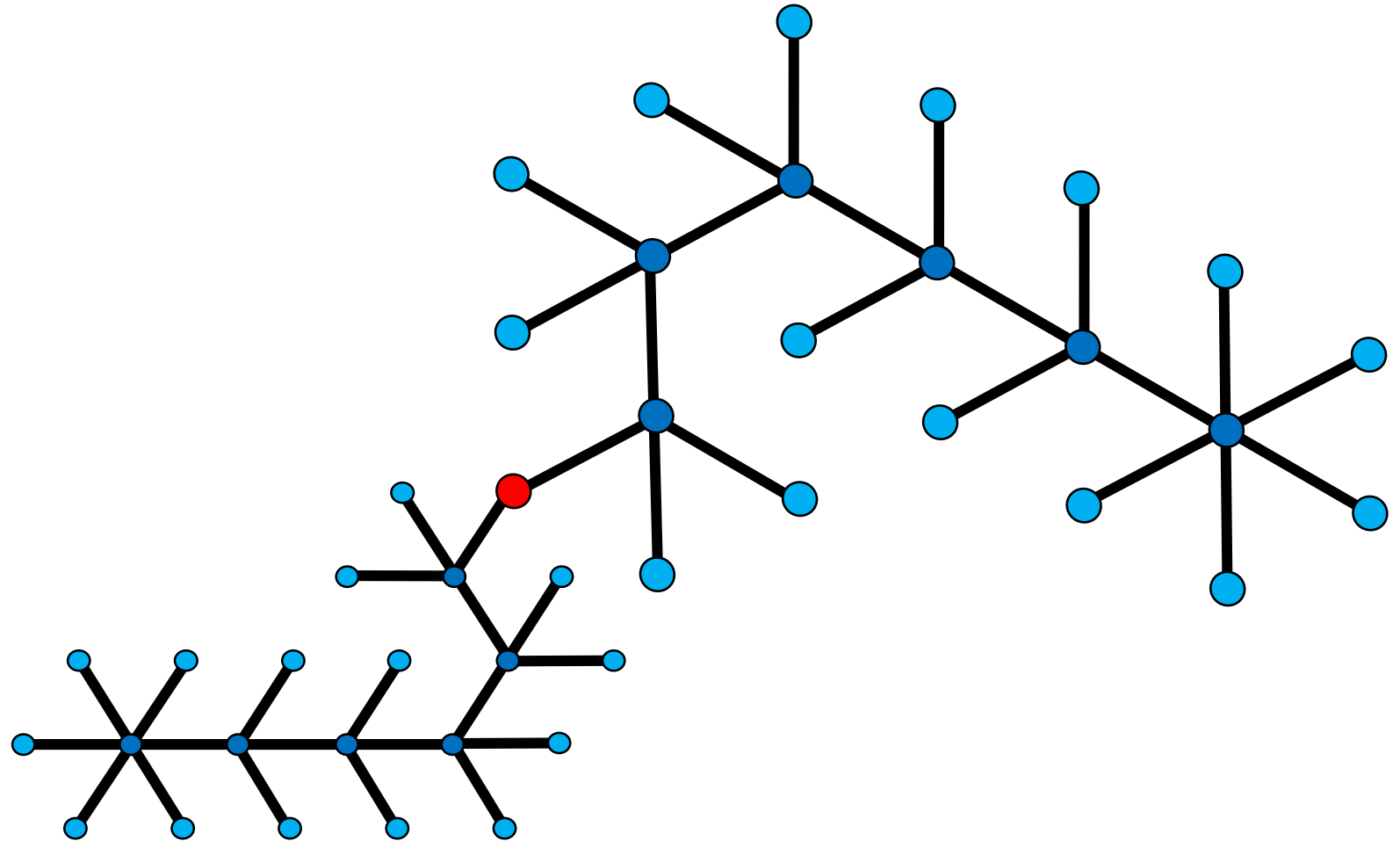
Lemma: 64..434..46 is not EMST-drawable.

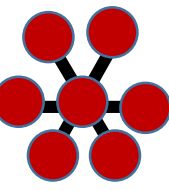




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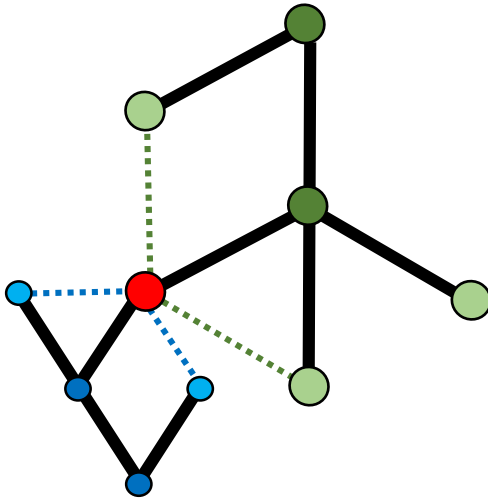
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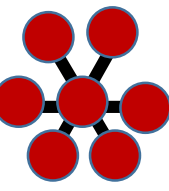




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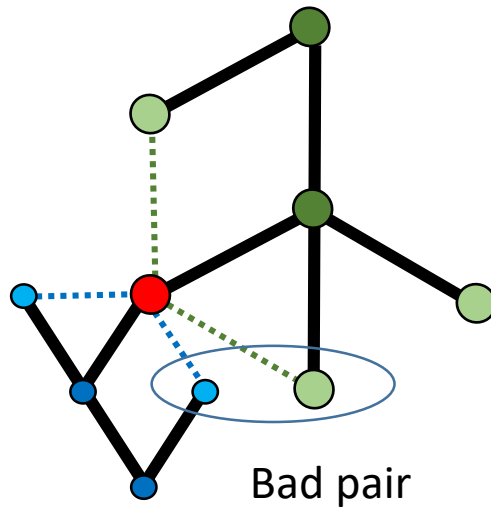
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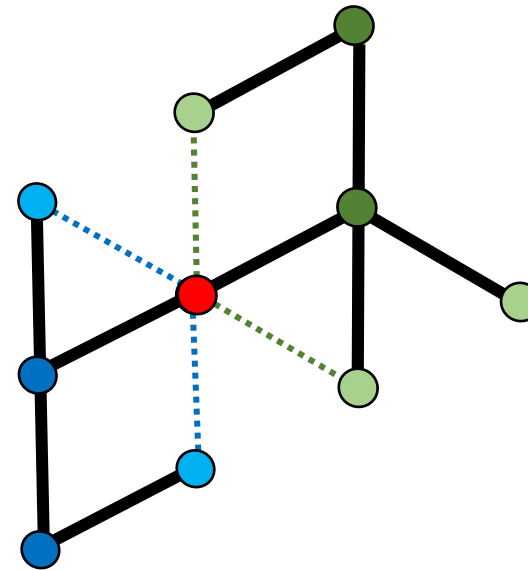


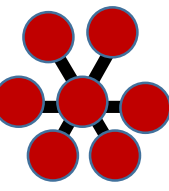
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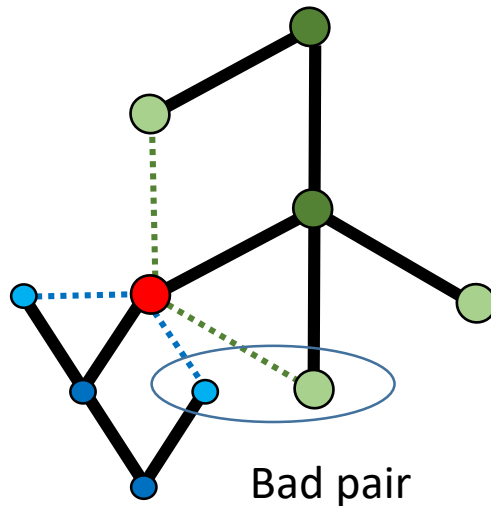
..... unless



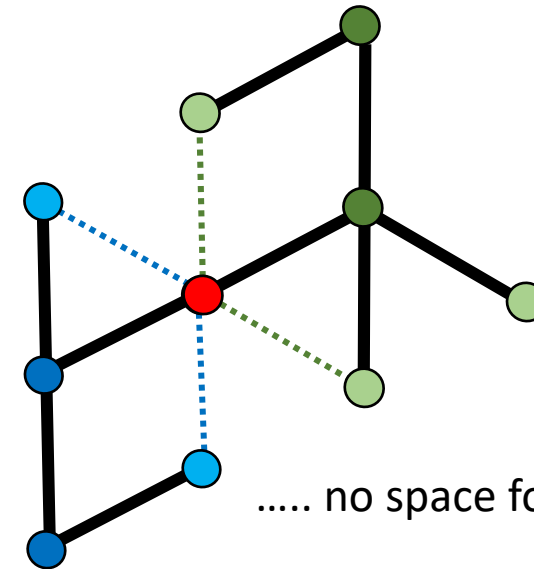


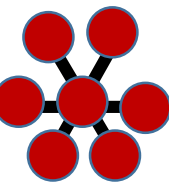
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..... unless



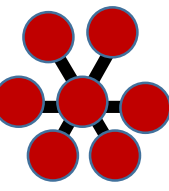


1. Caterpillars as EMST

Lemma: 64..424..46, 64..434..434..46 and 64..434..45 are EMST-drawable and can be combined.

$\tilde{T}$

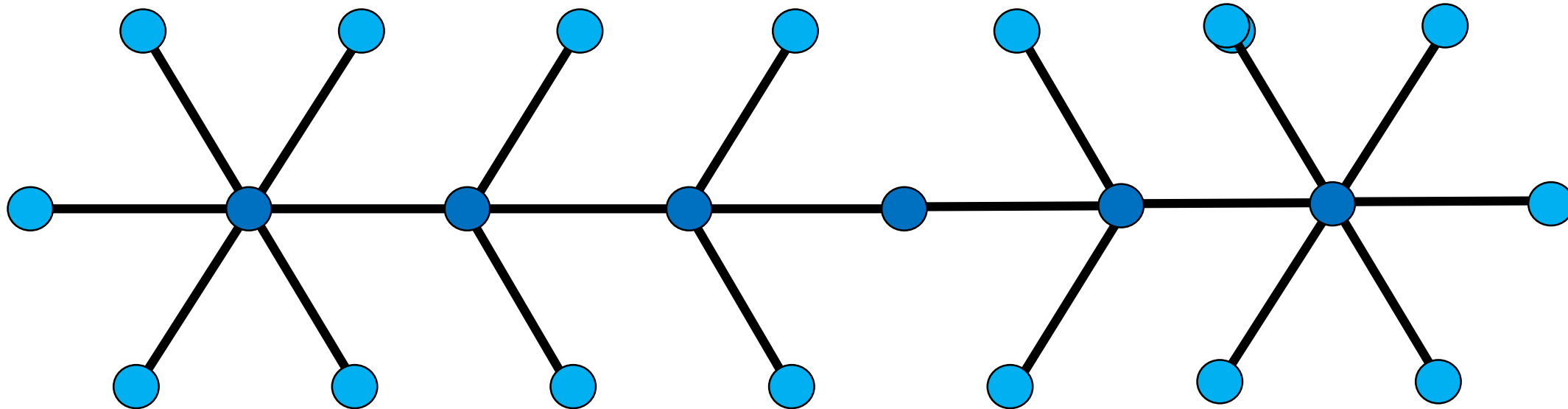


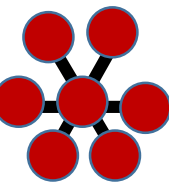


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64..424..46

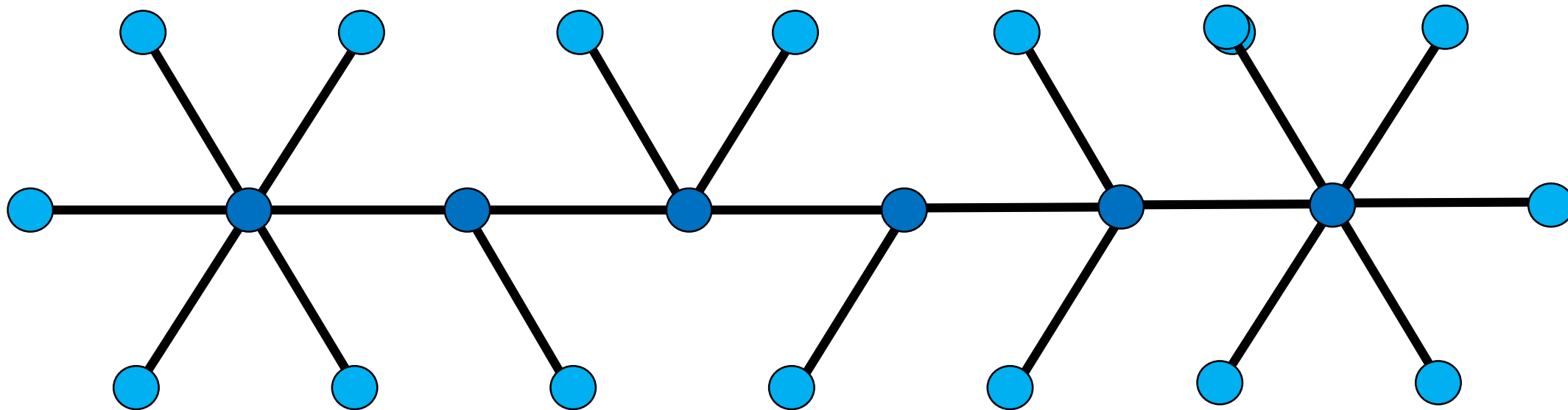


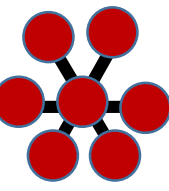


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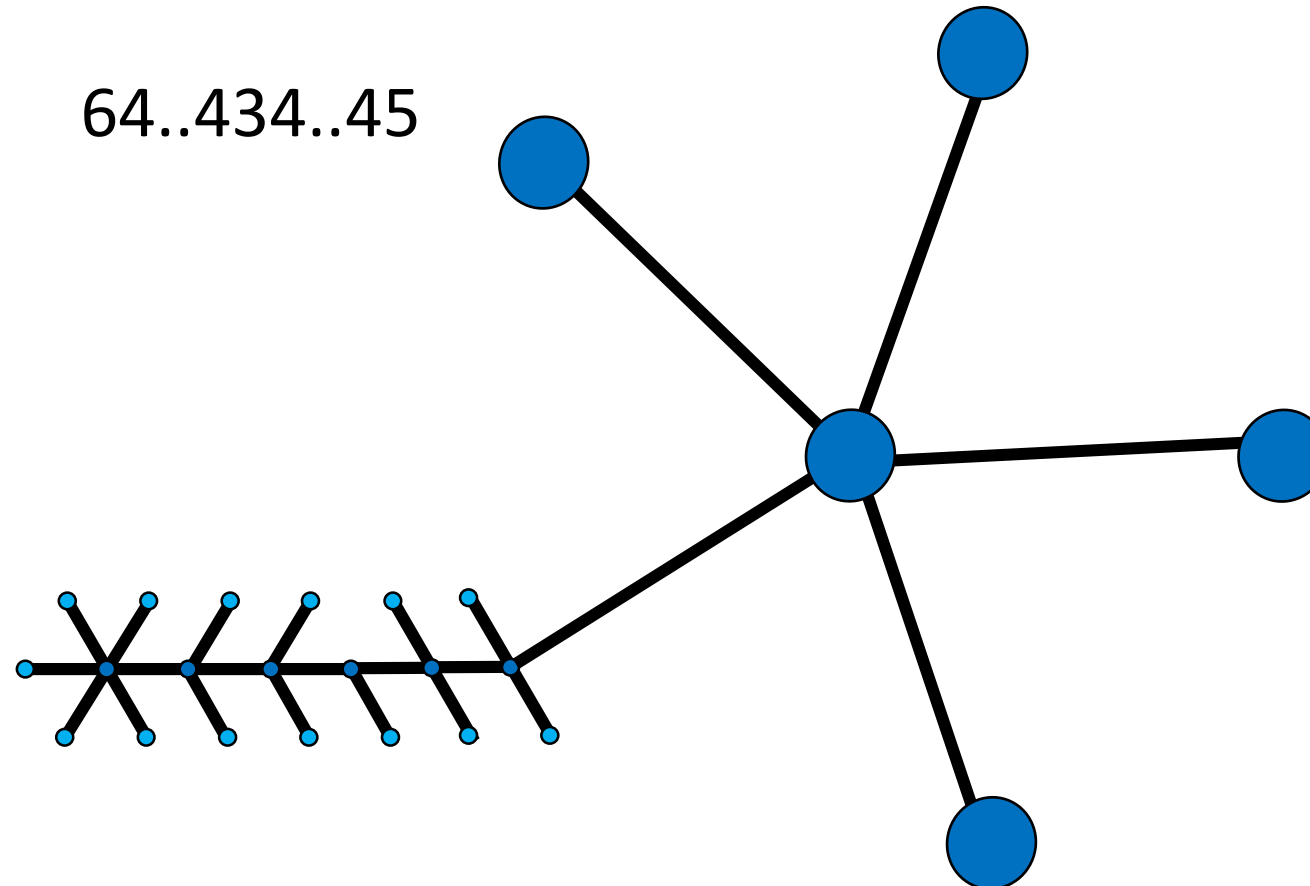
64..434..434..46

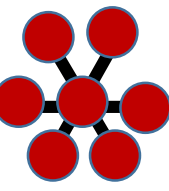




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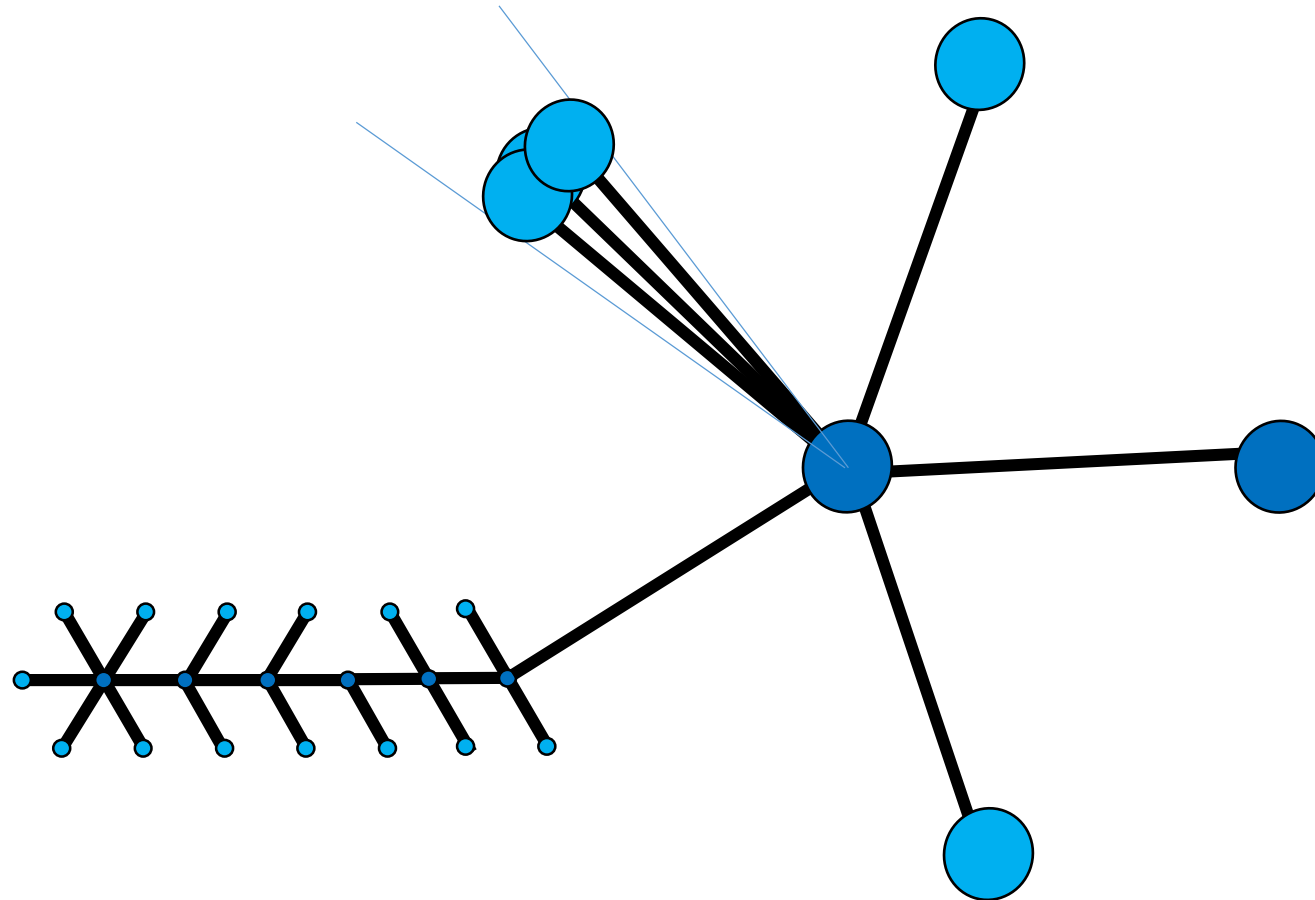
Lemma: 64..424..46, 64..434..434..46 and 64..434..45 are EMST-drawable and can be combined.





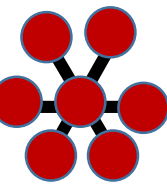
1. Caterpillars as EMST

Theorem 1': Every EMST-drawable caterpillar has a **strong drawing**, i.e. a drawing in which every spine vertex of degree 5 has a leaf with a **safe angle**.





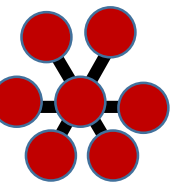
2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$



Warm up: For every caterpillar T of maximum degree 6, $\text{bp}(T)$ is upperbounded by its number of vertices of degree 6.



2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$

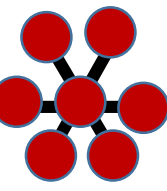


Warm up: For every caterpillar T of maximum degree 6, $\text{bp}(T)$ is upperbounded by its number of vertices of degree 6.

Proof: Delete one leaf at every vertex of degree 6, draw the resulting T' by Monma-Suri so that every vertex of degree 5 has a leaf with a safe angle, and return each deleted leaf in the safe angle at its parent. This creates exactly one bad pair in the neighborhood of each vertex of degree 6 (and no more).



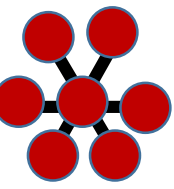
2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$



Theorem 2. The bad-pair number of a caterpillar of maximum degree 6 can be determined in polynomial time.



2. Determining $\text{bp}(T)$ for caterpillars of maxdeg 6



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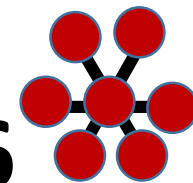
Algorithm:

1. Construct the reduced model $\tilde{T}$.
2. Color the edges of $\tilde{T}$ by green, red, (dark) blue and (dark) purple colors.
3. Recolor some blue edges by light blue, and some purple edges by light purple.
4. Color some vertices of $\tilde{T}$ red.
5. Output the number $\text{rv}(T)$ of red vertices.

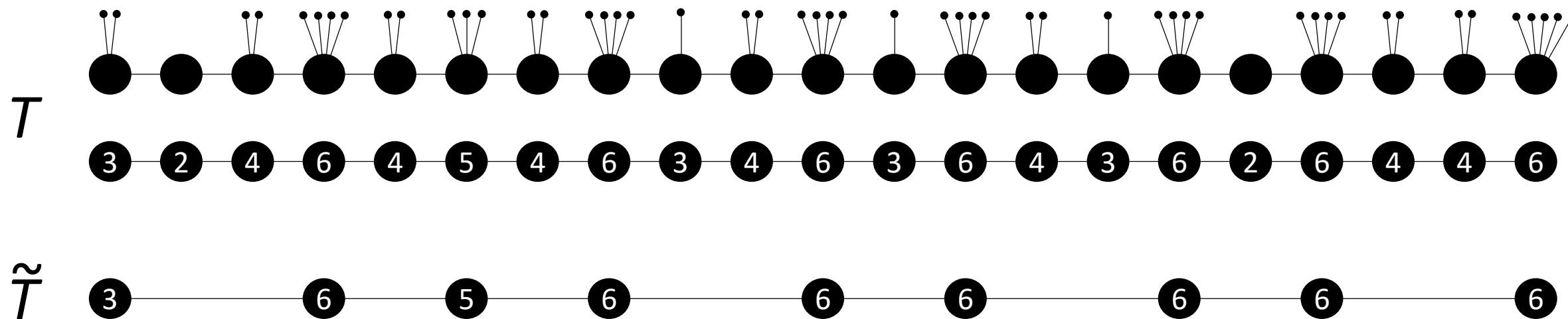
Claim: $\text{bp}(T) = \text{rv}(T)$.



2. Determining $\text{bp}(T)$ for caterpillars of maxdeg 6

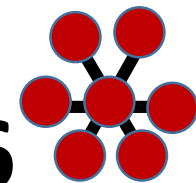


Phase 1. Constructing the reduced model.





2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$



Phase 2. Coloring the edges of the reduced model.

64..46



64..434..46



54..46

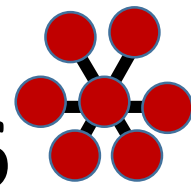


all other





2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$



Phase 2. Coloring the edges of the reduced model.

64..46



64..434..46



54..46



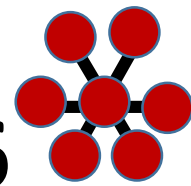
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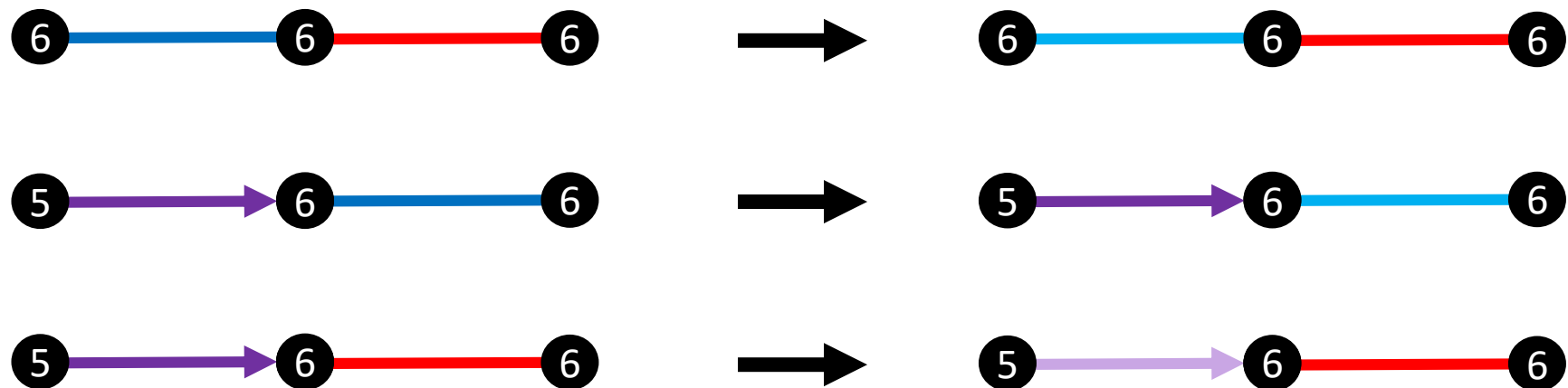
Fact: All edges are green if and only if $\text{bp}(T) = 0$.



2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$

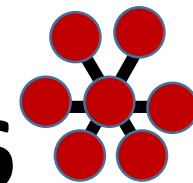


Phase 3. Recoloring some dark blue and dark purple edges.

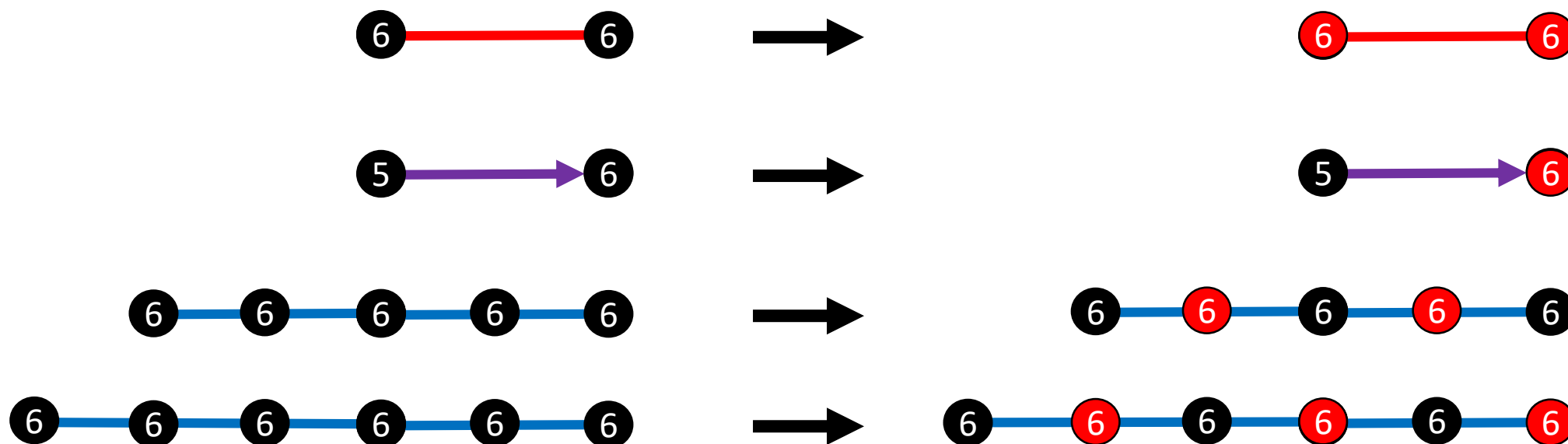




2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$

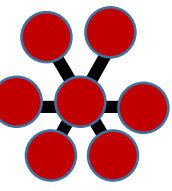


Phase 4. Coloring some vertices red.





2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$

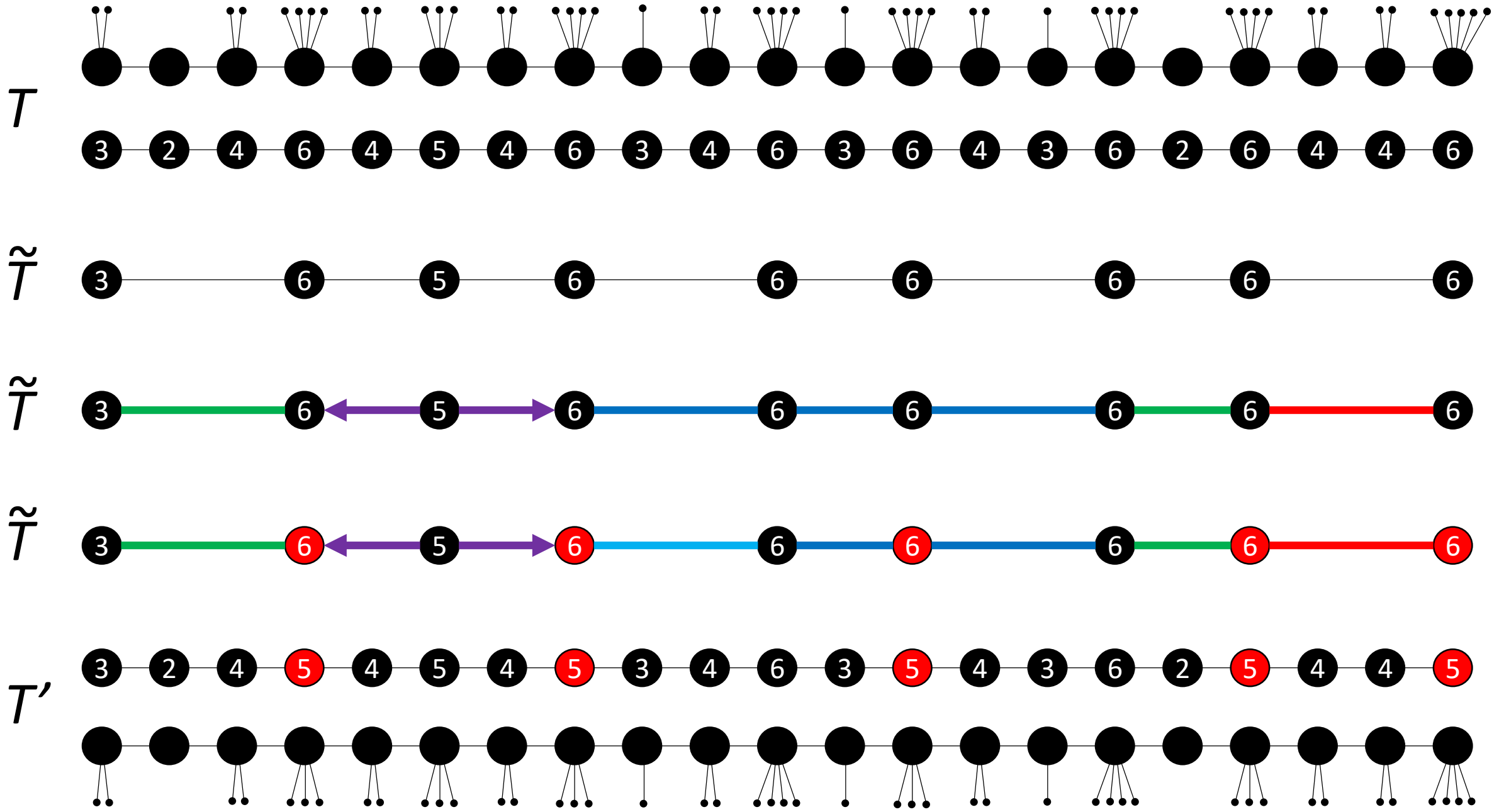


Phase 5. Output the number $\text{rv}(T)$ of red vertices of $\tilde{T}$.

Claim 1: $\text{bp}(T) \leq \text{rv}(T)$.

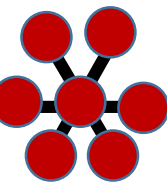
Proof: Delete one leaf at each red vertex. The reduced model of the resulting T' would have all edges green (if colored by the Algorithm again), and thus, T' is EMST-drawable.

Take a strong EMST-drawing of T' and add a leaf in the safe angle at each red vertex. This gives a drawing of T with $\text{rv}(T)$ bad pairs.





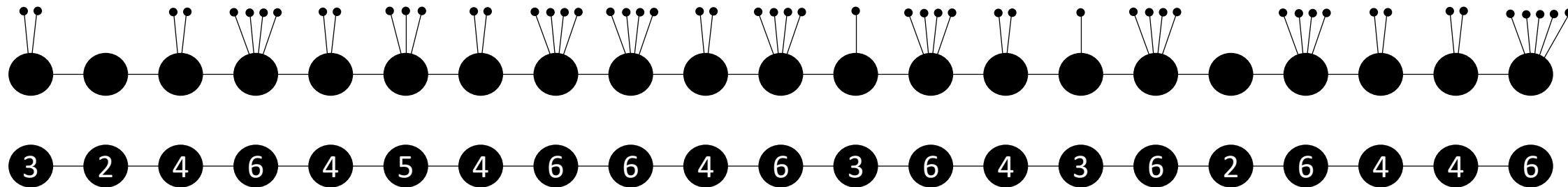
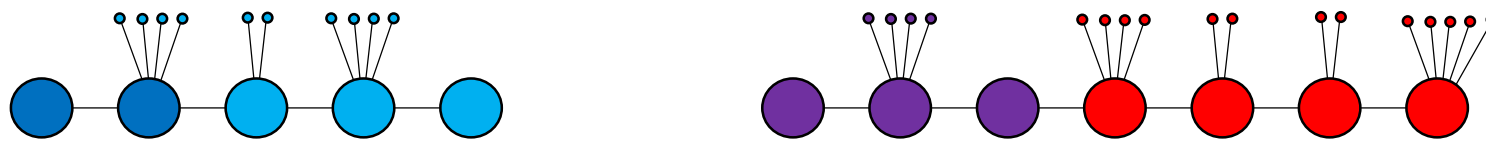
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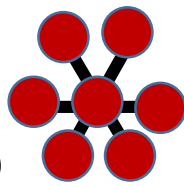
Claim 2: Consider an embedding Γ of a caterpillar T and let X, Y be two spine induced caterpillars of T . Let X^*, Y^* be the corresponding spine induced caterpillars in T .

- If X and Y are disjoint, then X^* and Y^* do not share any bad pair in Γ ;
- If X and Y share exactly one vertex, say u , then all bad pairs that are shared by X^* and Y^* in Γ are among the neighbors of u .

T  $\tilde{T}$  X  Y  X^*  Y^* 



2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$



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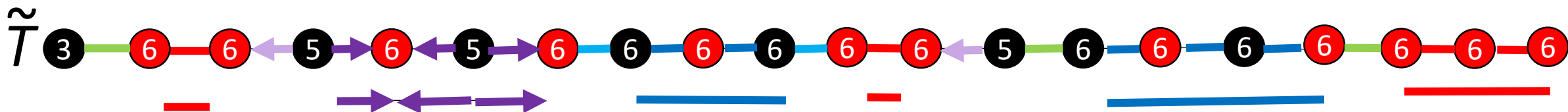
The maximal monochromatic subpaths of the reduced model that contain some red vertices are

red subpaths

alternating dark purple subpaths

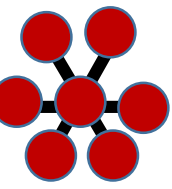
dark blue subpaths.

These paths are vertex-disjoint.





2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$



Phase 5. Output the number $\text{rv}(T)$ of red vertices of $\tilde{T}$.

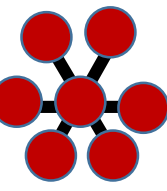
Claim 3: A red path with k vertices requires at least k bad pairs in any drawing.

A dark purple path with k sinks requires at least k bad pairs in any drawing.

A dark blue path with $2k$ vertices requires at least k bad pairs, a dark blue path with $2k+1$ vertices requires at least k bad pairs.



2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$

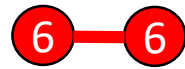


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Bad pairs

2

3

3

+

2

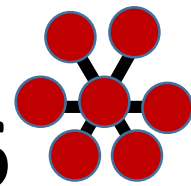
+

2

Needs



2. Determining $\text{bp}(T)$ for caterpillars of $\text{maxdeg } 6$

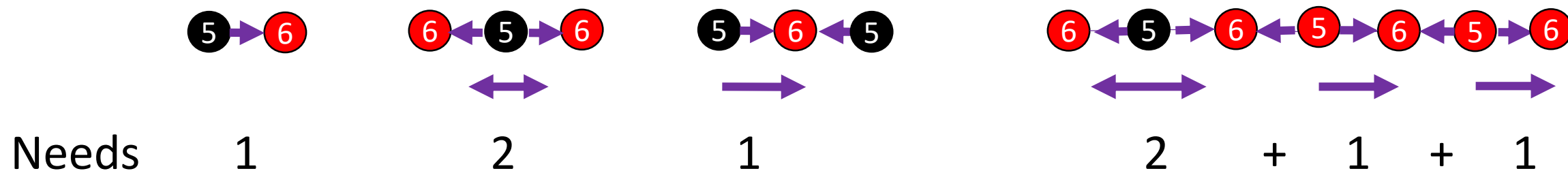


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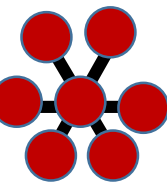
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Bad pairs

Needs

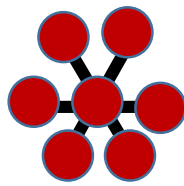
1 + 1 + 1



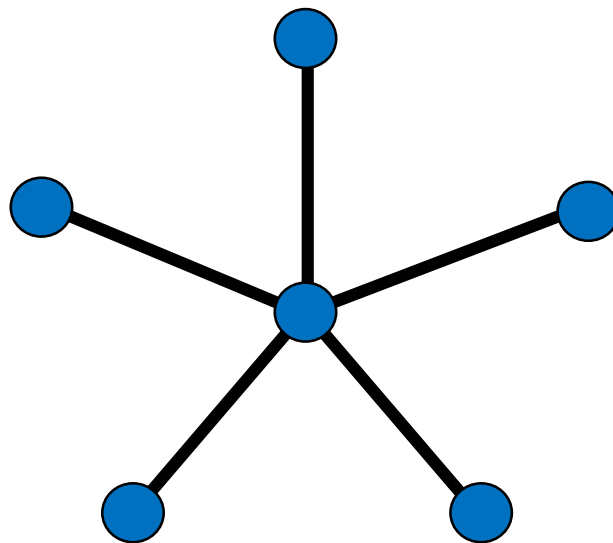
1 + 1 + 1



3. Determining the bad pair number of stars

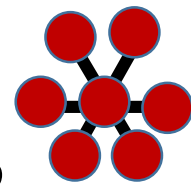


Theorem 3. For $s > 6$, $\text{bp}(K_{1,s}) = \binom{s}{2} - t(s,5)$, where $t(s,5)$ is the Turán number, i.e. the maximum number of edges of an s -vertex graph with no K_6 subgraph.



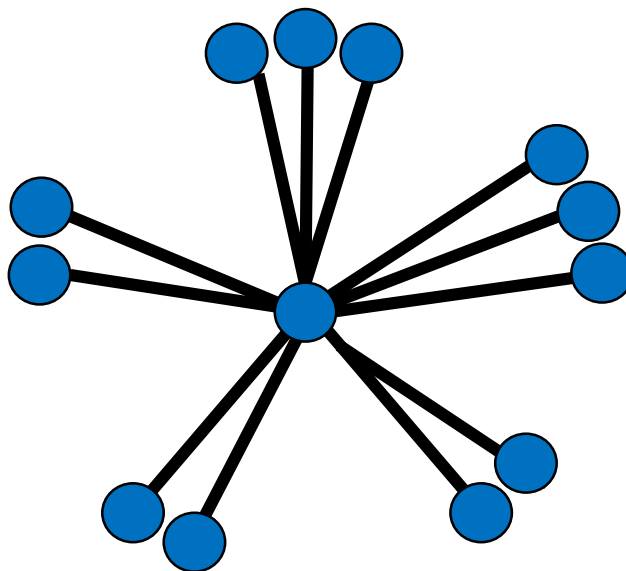


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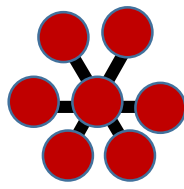
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$$\text{bp}(K_{1,s}) = s^2/10 + \Theta(s)$$



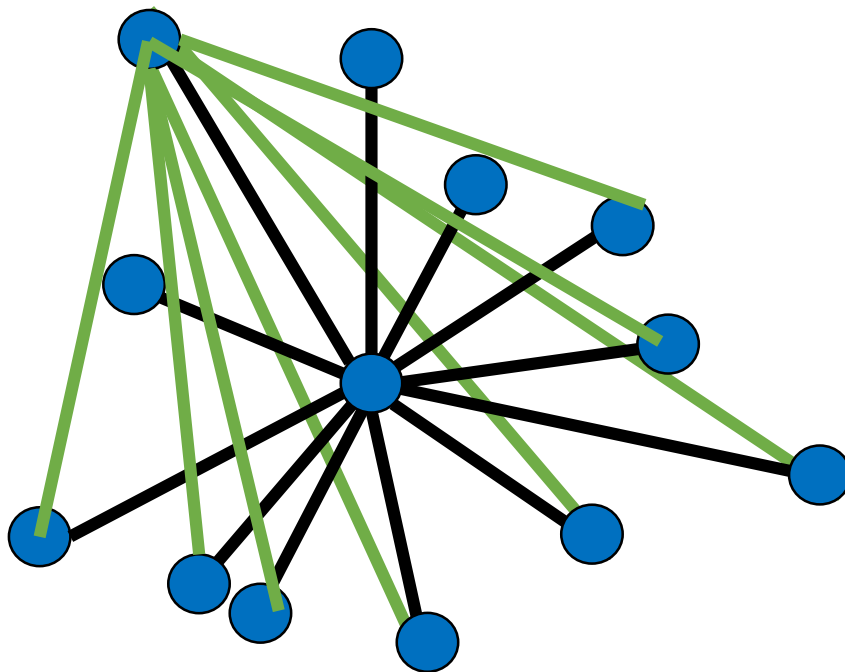


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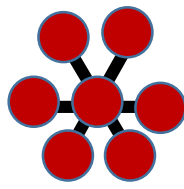
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Lower bound:
Consider the graph of
good pairs





3. Determining the bad pair number of stars



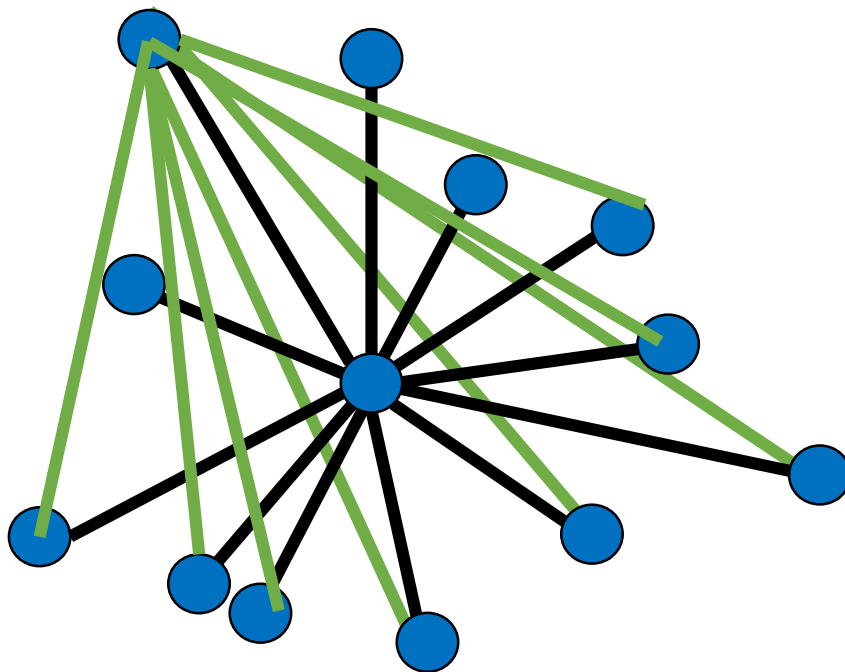
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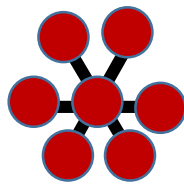
If it has no K_6 ,

Turán implies that the number of good pairs is $\leq t(s,5)$





3. Determining the bad pair number of stars



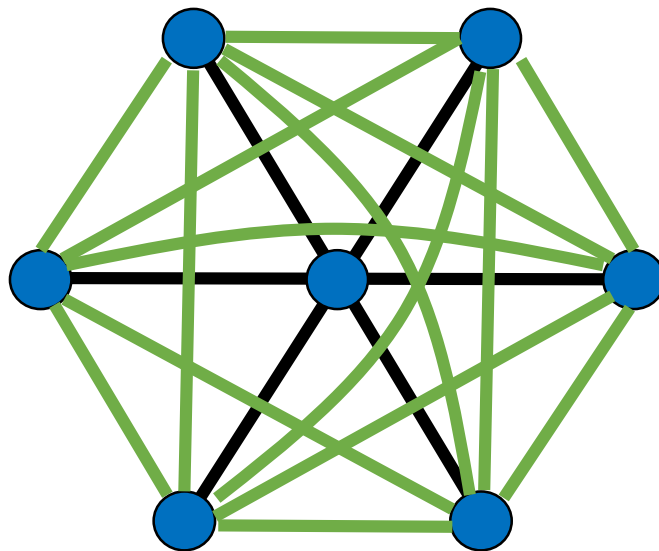
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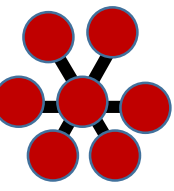
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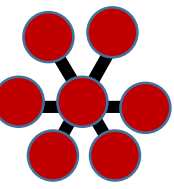
If it has a K_6 , then these leaves must be placed in the corners of a regular hexagon. Delete them and continue by induction.



4. Beyond caterpillars

Theorem 4. For a tree T with n vertices and maximum degree Δ ,
 $\text{bp}(T) \leq \Delta^2 n \log n$.

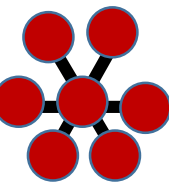
When $\Delta = o(\sqrt{n / \log n})$, this bound is subquadratic.



4. Beyond caterpillars

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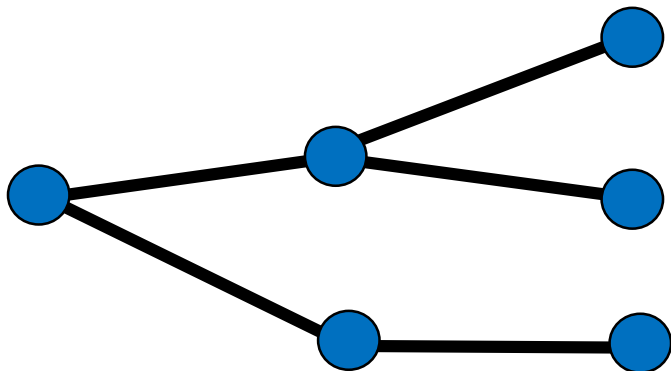
Sketch of proof: We show by induction on the height of T that
 $\text{bp}(T) \leq cn \log n$ for a constant $c = \frac{\Delta-1}{\log(\Delta/(\Delta-1))} < \Delta^2$ (for $\Delta > 2$).

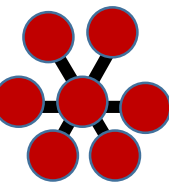


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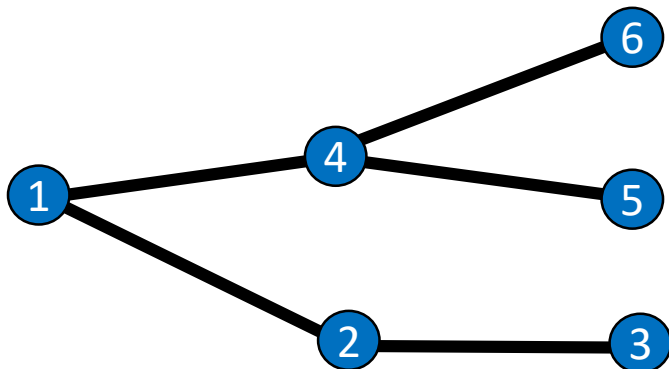


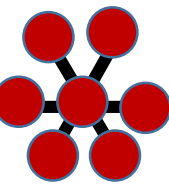


4. Beyond caterpillars

Theorem 4. For a tree T with n vertices and maximum degree Δ ,
 $\text{bp}(T) \leq \Delta^2 n \log n$.

Sketch of proof: We show by induction on the height of T that
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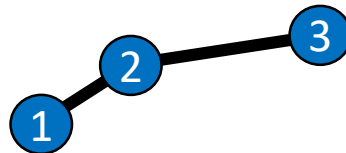
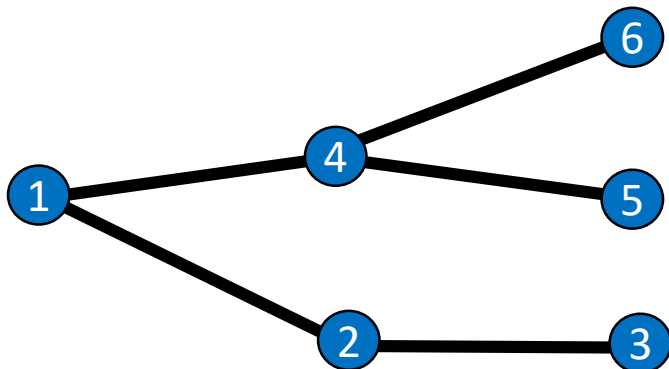


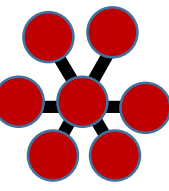


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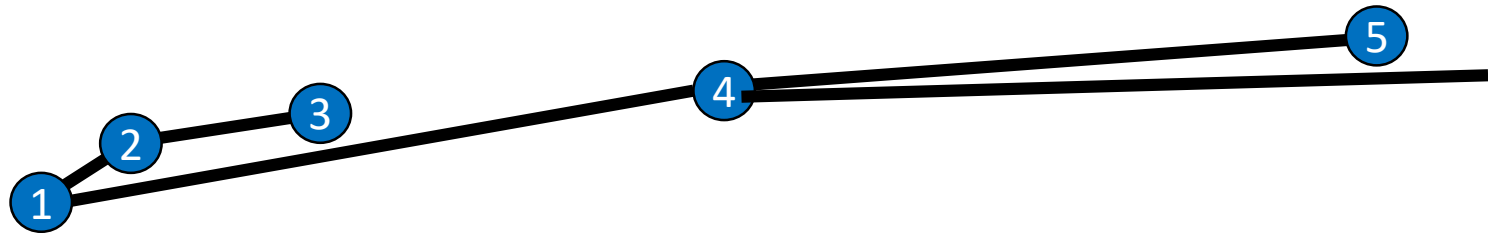
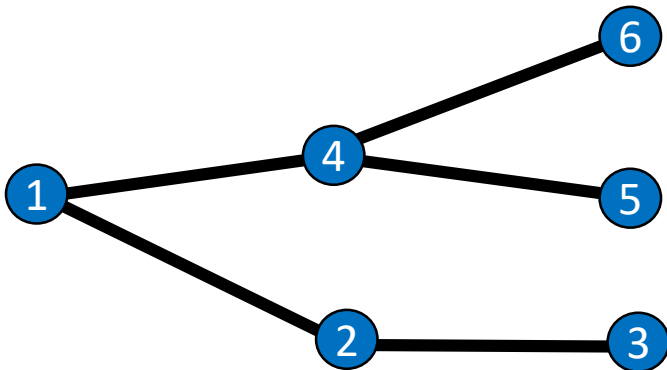


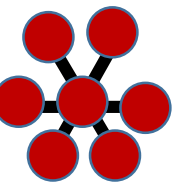


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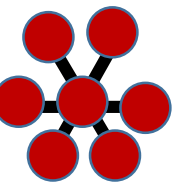


4. Directions of further research

Question 1. Trees of maximum degree 6 such that every vertex of degree 6 has at least two incident leaves. Can we compute $\text{bp}(T)$ in polynomial time?

Question 2. Can we compute $\text{bp}(T)$ in polynomial time for trees of bounded pathwidth?

Question 3. Study the bad pair number question for other definitions of proximity drawings (e.g., Gabriel drawings).



Thank You!