Recent Progress in the Computational Complexity of Graph Covers

J. Bok, J. Fiala, P. Hliněný, N. Jedličková, Jan Kratochvíl,
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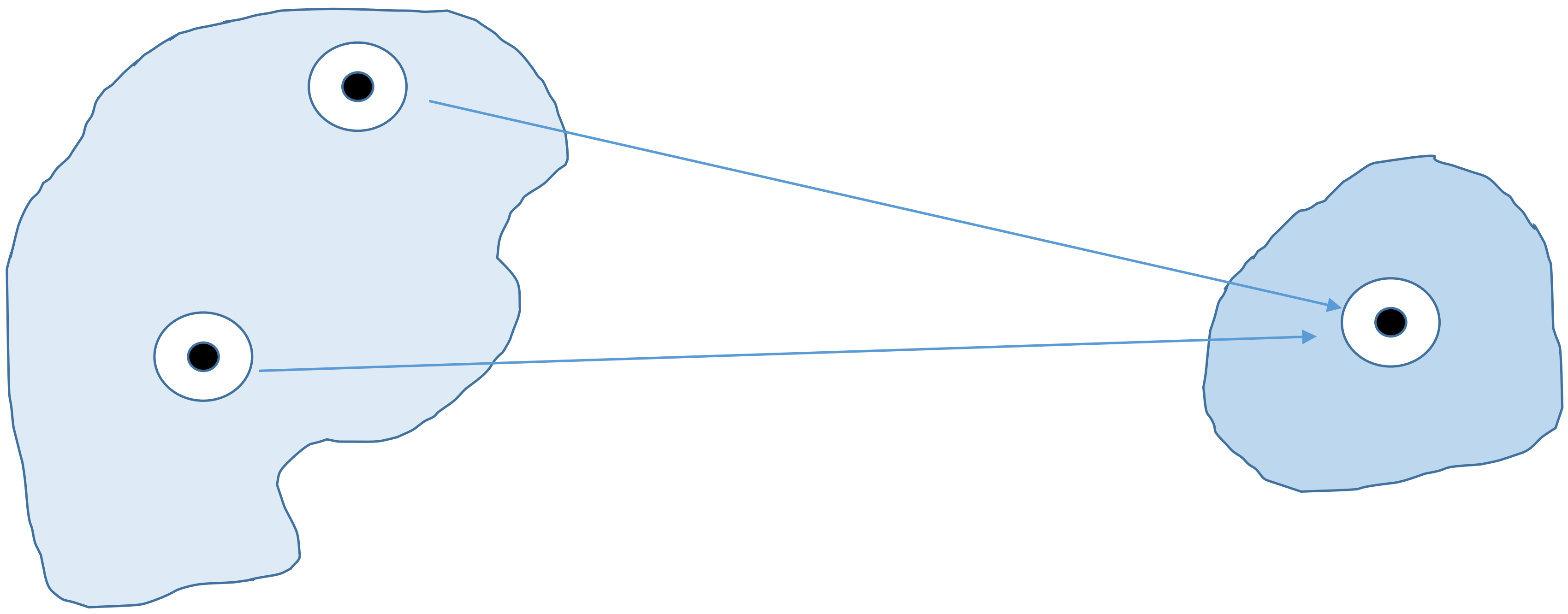
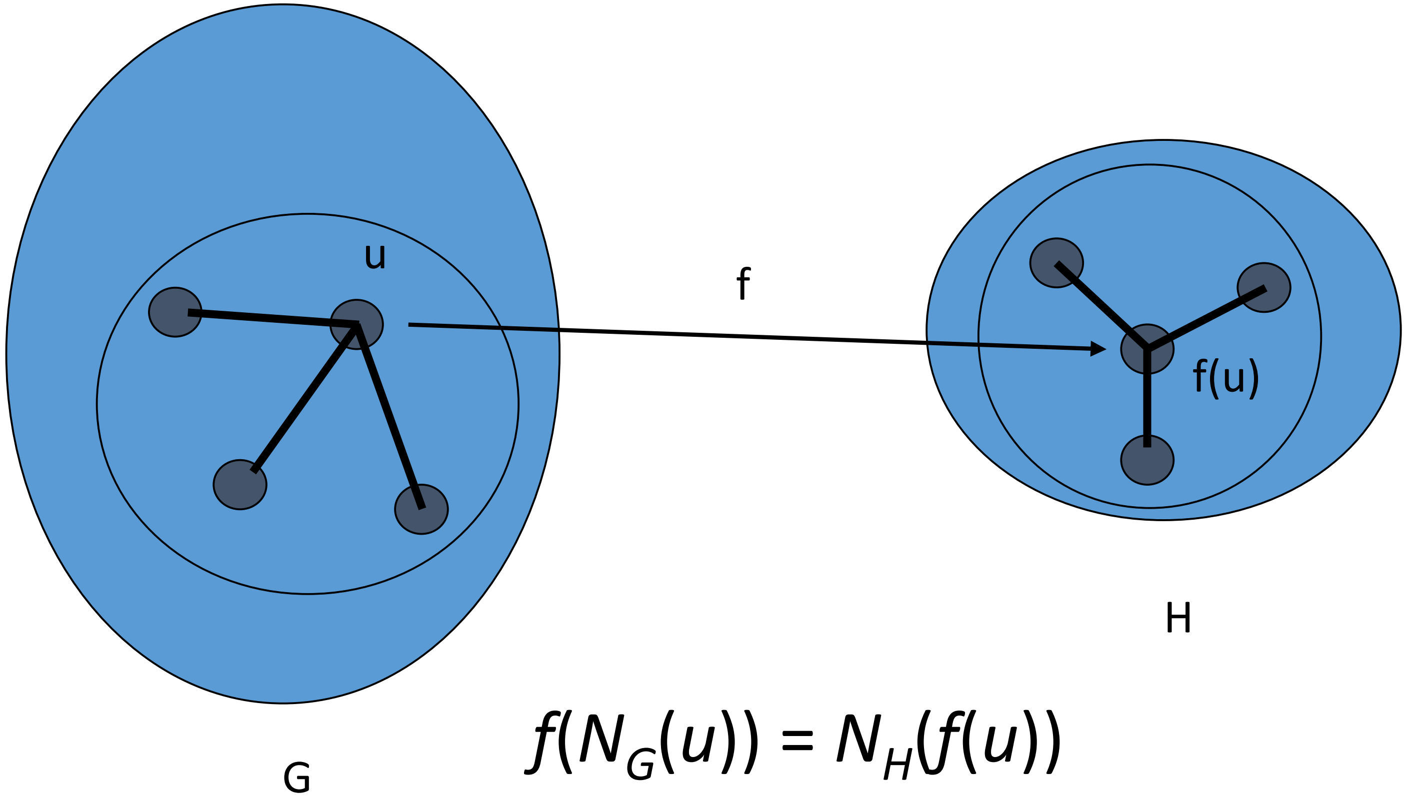
Motivation from topology

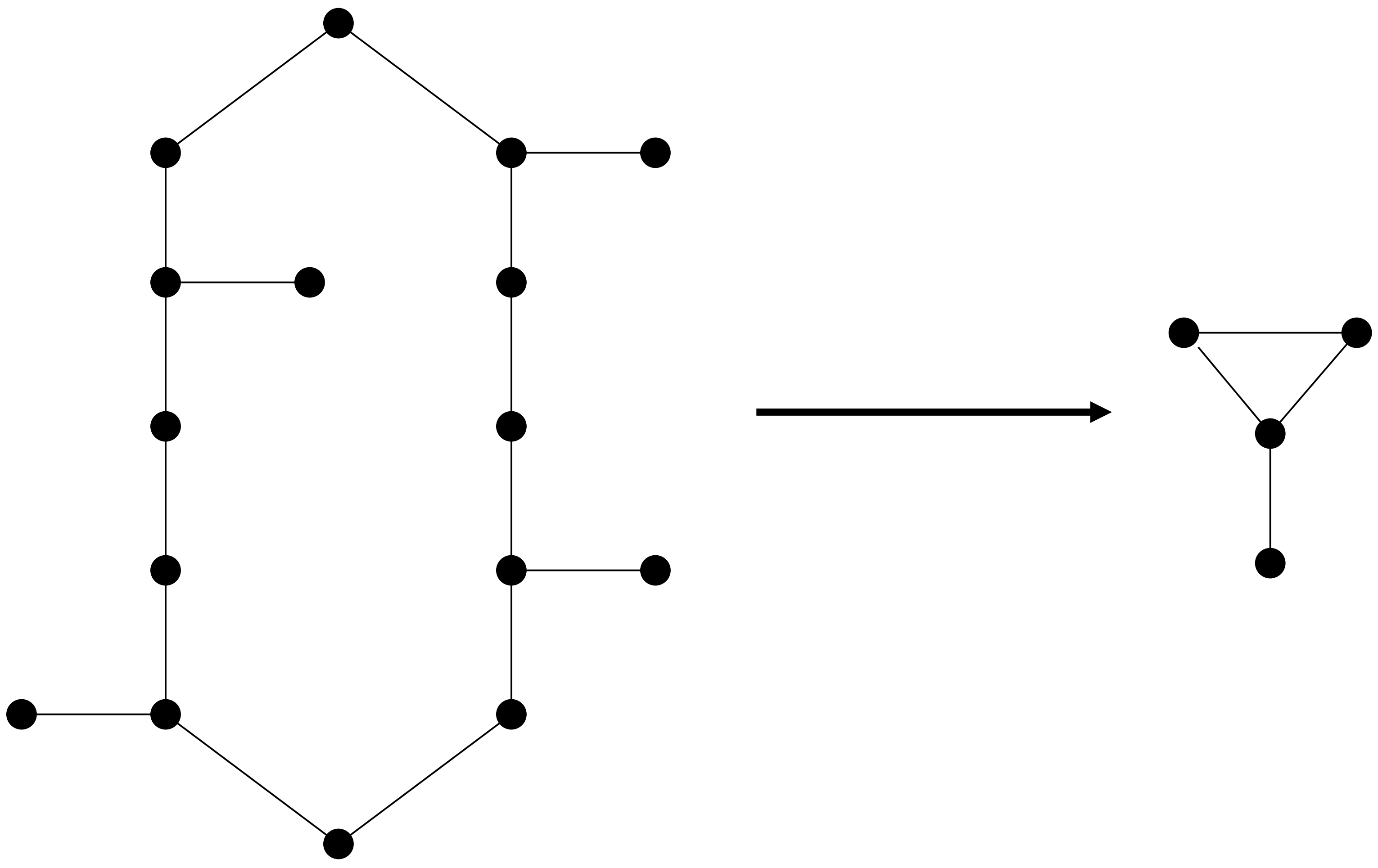


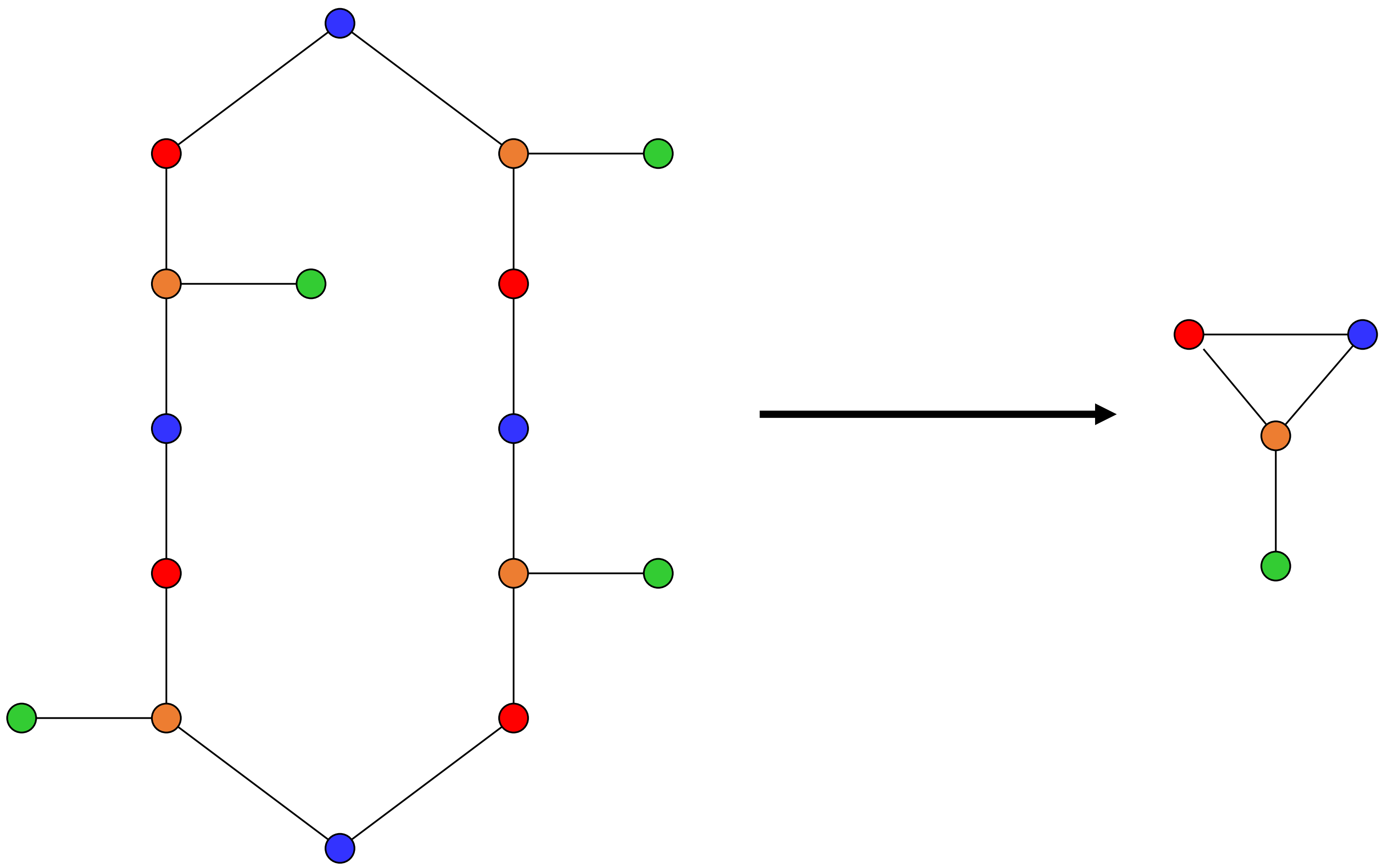
Definition of graph covering
(for connected simple graphs)

Definition: Mapping f: $V(G) \to V(H)$ is a *graph covering projection* if for every $u \in V(G)$, $f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$



$$f(N_G(u)) = N_H(f(u))$$





A bit of the history

- Topological graph theory, construction of highly symmetric graphs (Biggs 1974, Djokovic 1974, Gardiner 1974, Gross et al. 1977)
- Local computation (Angluin STOC 1980, Litovsky et al. 1992, Courcelle et al. 1994, Chalopin et al. 2006)
- Common covers (Angluin et al. 1981, Leighton 1982)
- Finite planar covers (Negami's conjecture 1988, Hliněný 1998, Archdeacon 2002, Hliněný et al. 2004)
- Regular covers and maps (Nedela et al. 1996, Malnic et al. 2000, …)

Computational complexity of graph covers

H-COVER
Input: A graph *G*
Question: Does *G* cover *H*?

Computational complexity of graph covers

- Thm (Bodlaender 1989): *H*-COVER is NP-complete if *H* is also part of the input.
- Abello, Fellows, Stilwell 1991: Initiated the study of computational complexity of the *H*-COVER problem for fixed *H*.
- Thm (Kratochvil, Proskurowski, Telle 1994): *H*-COVER is polynomial time solvable for every simple graph with at most 2 vertices per equivalence class in its degree partition.
- Thm (Fiala, Kratochvil, Proskurowski, Telle 1998): *H*-COVER is NP-complete for every simple regular graph of degree at least 3.
- Fiala, Kratochvil 2008: Relation to CSP
- Bílka, Jirásek, Klavík, Tancer, Volec 2011: NP-hardness of covering small graphs by planar inputs.
- Bok, Fiala, Hlineny, Jedlickova, Kratochvil 2021: Covering multigraphs with semi-edges

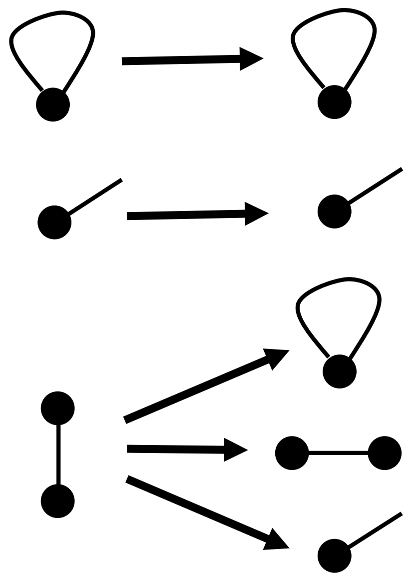
Outline of the presentation

- Multigraphs with semi-edges
- Strong Dichotomy Conjecture
- List Covering multigraphs with semi-simple vertices
- Complete characterization of List-H-Cover for cubic graphs H
- Open problems and directions of further research

1. Covers of multigraphs
(with multiple edges, loops and semi-edges)

Definition: A pair of mappings $f = (f_V, f_E): G \to H$ is a graph covering projection if

- $f_V: V(G) \to V(H)$ is a homomorphism,
- $f_E: E(G) \to E(H)$ is compatible with f_V, and it is a bijection of {edges incident with u} onto {edges incident with $f_V(u)$} for every $u \in V(G)$



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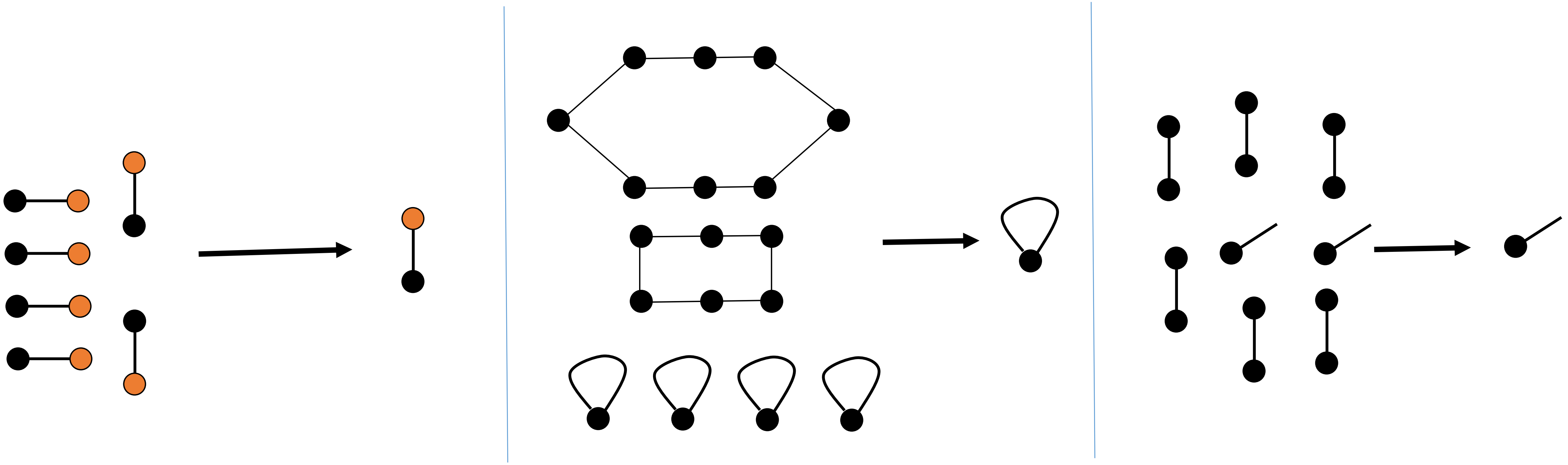
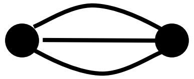
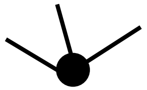


Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

 -COVER is NP-complete

Covers of multigraphs

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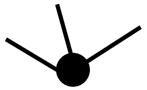
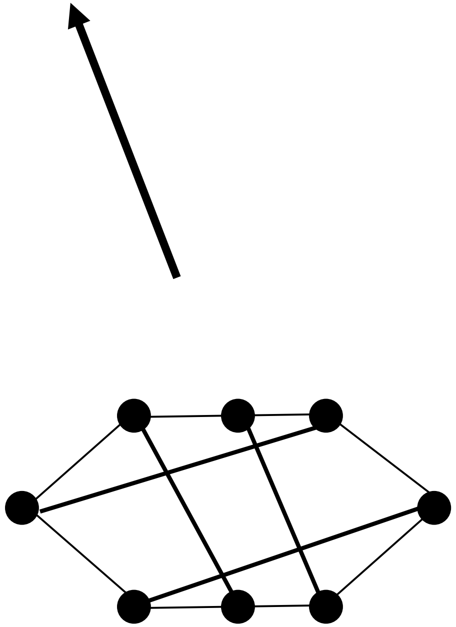
 -COVER is NP-complete



Covers of multigraphs

(few examples)

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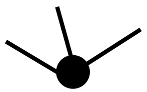
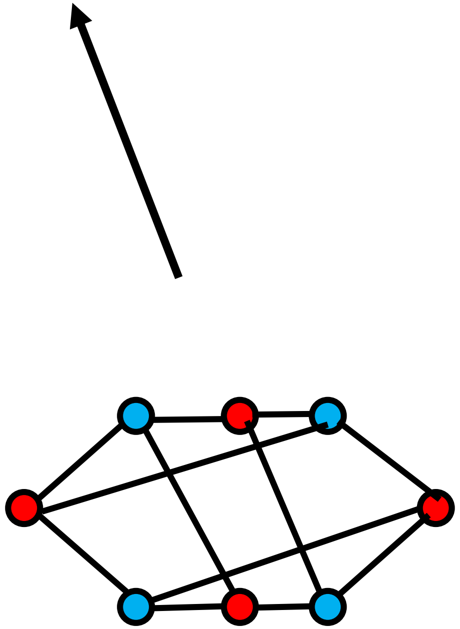
 -COVER is NP-complete



Covers of multigraphs

(few examples)

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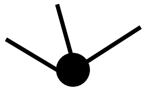
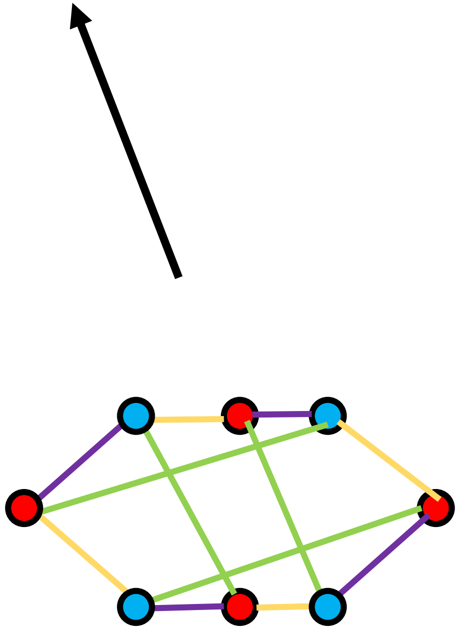
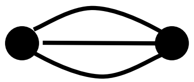
 -COVER is NP-complete

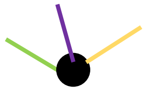


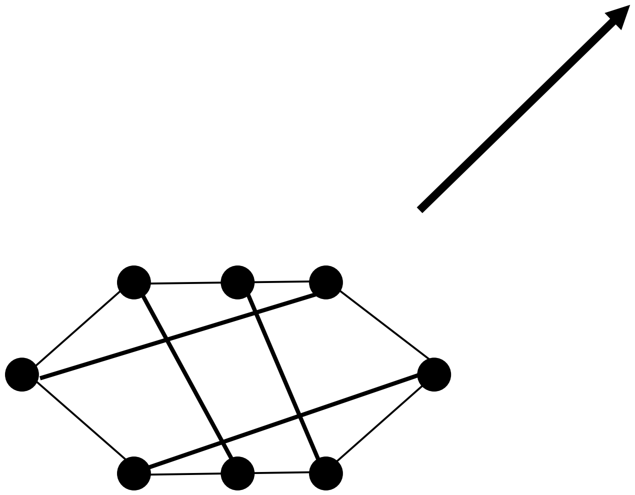
Konig-Hall

Covers of multigraphs

(few examples)

 -COVER is polynomial time solvable

 -COVER is NP-complete



Covers of multigraphs
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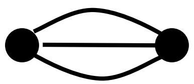
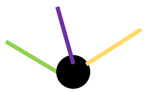
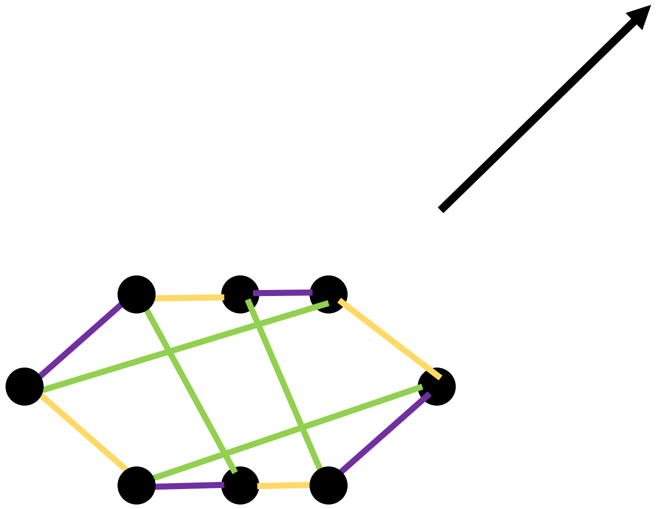
 -COVER is polynomial time solvable

 -COVER is NP-complete



Covers of multigraphs

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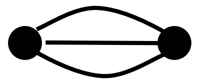
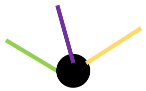
 -COVER is polynomial time solvable

= 3-edge-colorability of bipartite graphs

 -COVER is NP-complete

= 3-edge-colorability

Complexity of covering multigraphs

- Kratochvil, Proskurowski, Telle 1997: Complete characterization of the computational complexity of *H*-COVER for colored mixed 2-vertex multigraphs *H (no semi-edges at that time).*
- Kratochvil, Telle, Tesař 2016: Complete characterization of the computational complexity of *H*-COVER for 3-vertex multigraphs *H.*
- Bok, Fiala, Hliněný, Jedličková, Kratochvíl MFCS 2021: First results on the computational complexity of *H*-COVER for (multi)graphs *with semi-edges*. Full classification for 1-vertex and 2-vertex graphs *H* (report at CSGT 2020).
- Bok, Fiala, Jedličková, Kratochvíl, Seifrtová FCT 2021: Covers of disconnected multigraphs (also at CSGT 2021)
- Bok, Fiala, Jedličková, Kratochvíl, Rzazewski IWOCA 2022: List Covering version

2. Hoping for a stronger dichotomy

Strong dichotomy conjecture: For all connected graphs *H*, the *H*-COVER problem is either polynomial time solvable for general input graphs, or NP-complete for simple input graphs (i.e., no loops, no multiple edges, no semi-edges are allowed).

Or does there exist a connected graph *H* (loops, multiple edges and semi-edges allowed) such that the *H* -COVER problem is NP-complete for general inputs, but polynomial time solvable for simple graphs on the input?

3. List covering problems

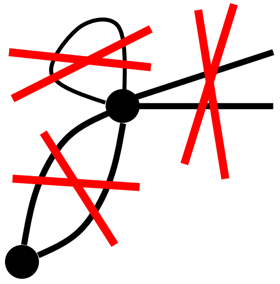
List-H-COVER
Input: A graph G, lists $L(u) \subseteq V(H)$ for $u \in V(G)$, $L(e) \subseteq V(H)$ for $e \in E(G)$.
Question: Does G allow a covering projection $f{:}G \to H$ such that $f(u) \in L(u)$ for every $u \in V(G)$ and $f(e) \in L(e)$ for every $e \in E(G)$?

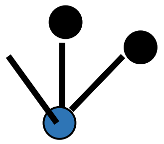
List covering problems

Partial cover (*locally injective homomorphism*) is a harder problem than graph cover, but a dichotomy has been proved for List-*H*-PartialCOVER [Fiala, Kratochvil WG 2006]

List covering problems

Theorem: If *H* is a *k*-regular graph, *k*≥3, with at least one *semi-simple vertex*, then List-*H*-COVER is NP-complete for simple input graphs.

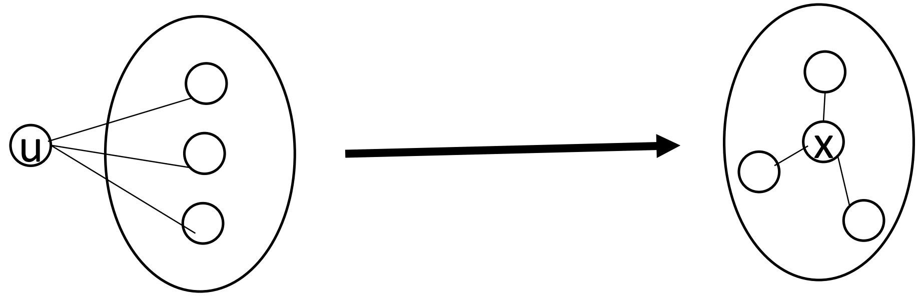




List covering problems

Sketch of proof: Revisit the reduction for *k*-edge-colorable *k*-regular graphs from Kratochvil, Proskurowski, Telle [JCTB 1997].

A graph *G* is a *multicover* of *H* if it covers *H* in many ways, in the sense that *G* has a vertex *u* such that for every vertex *x* of *H* and for every bijective mapping of the edges of *G* incident with *u* to the edges of *H* incident with *x*, there is a covering projection $G \rightarrow H$ that extends this mapping.



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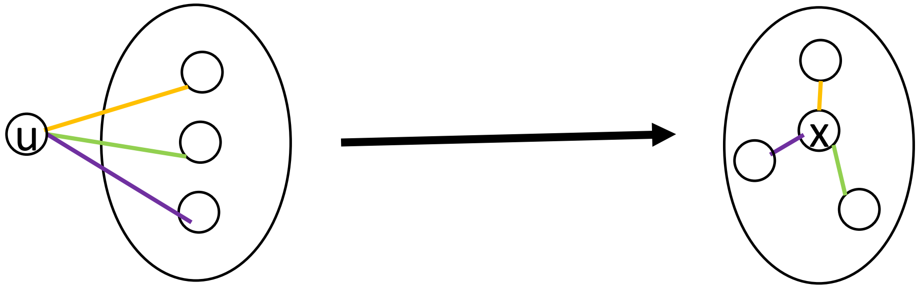

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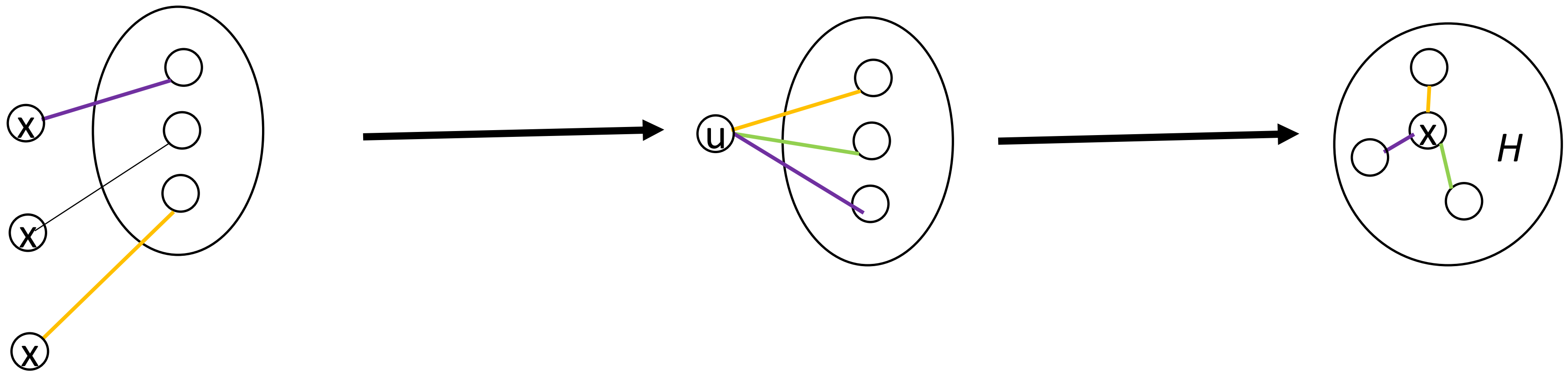
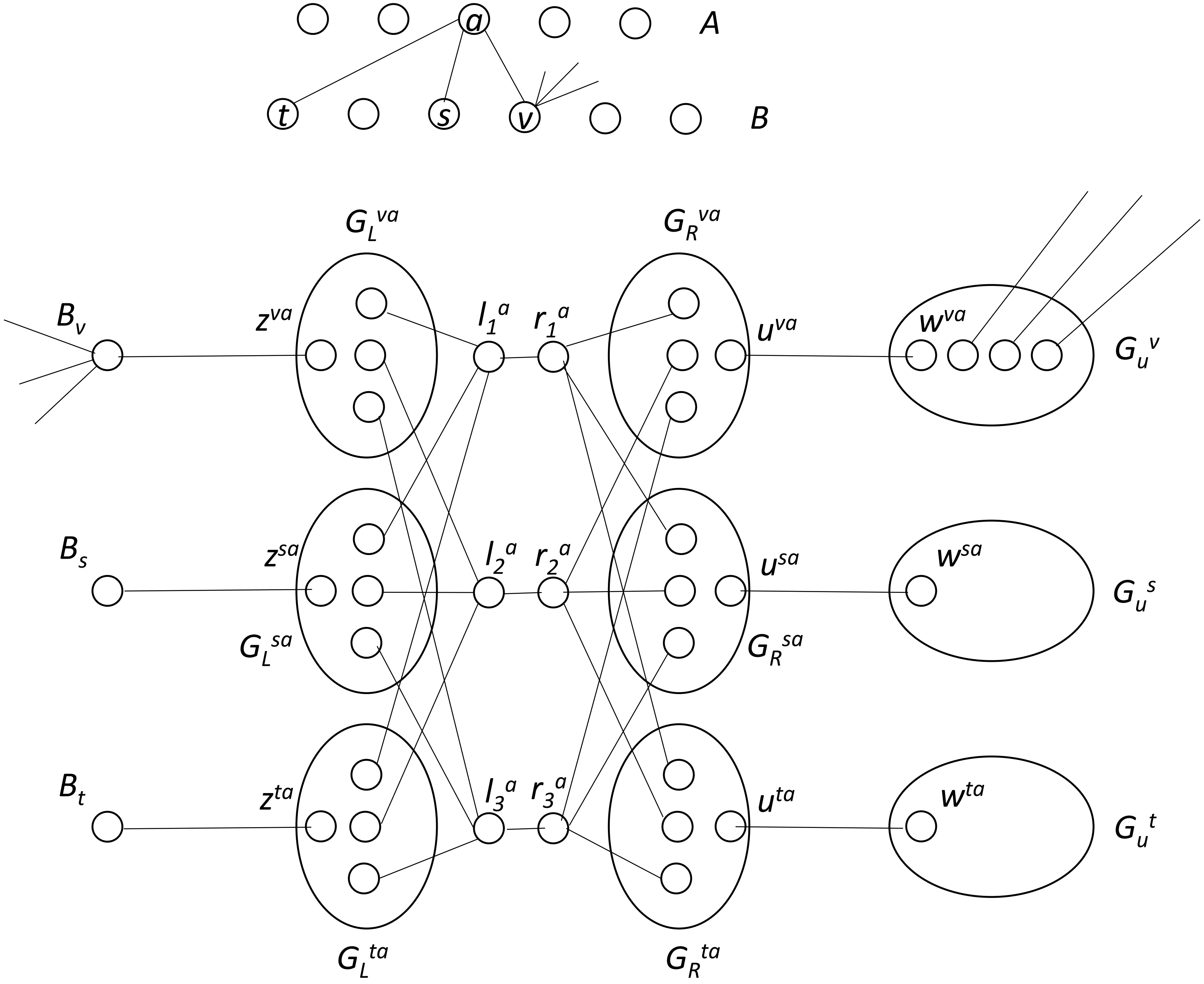
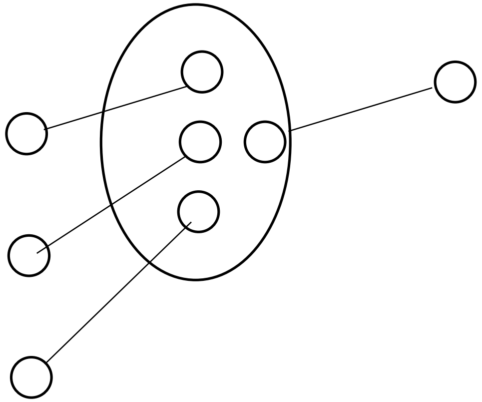




Reduction from k-edge colorability *of* k-regular (k-1)-uniform hypergraphs.

Multicover gadget:



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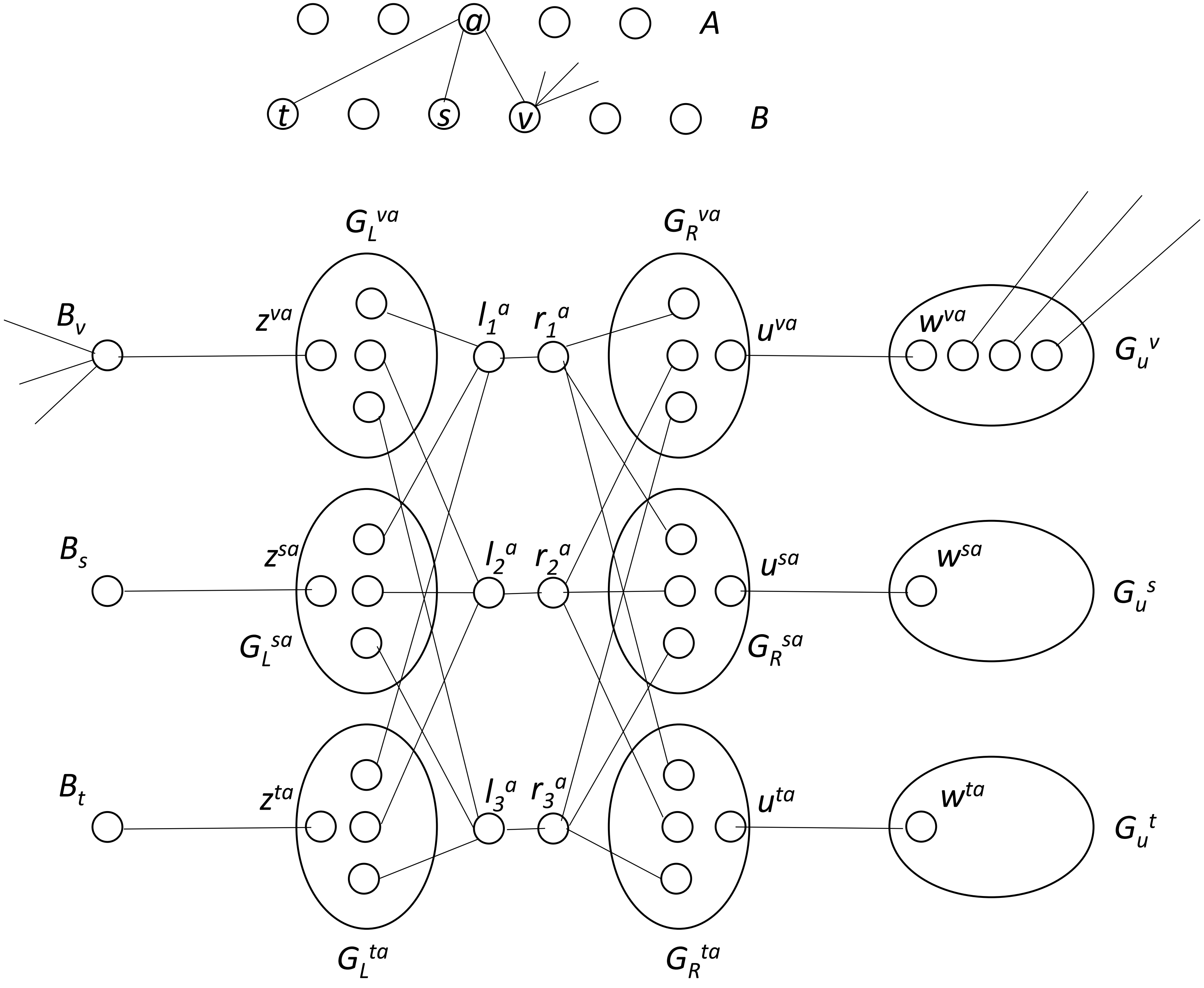

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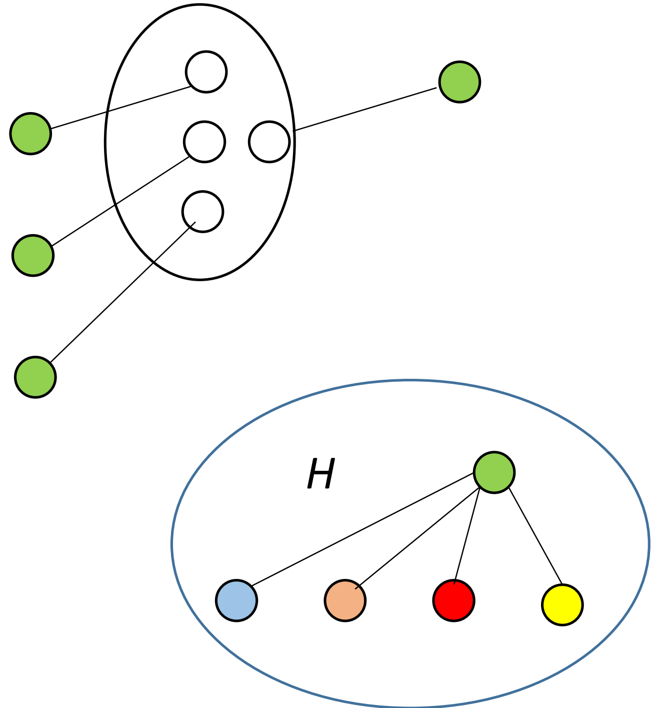

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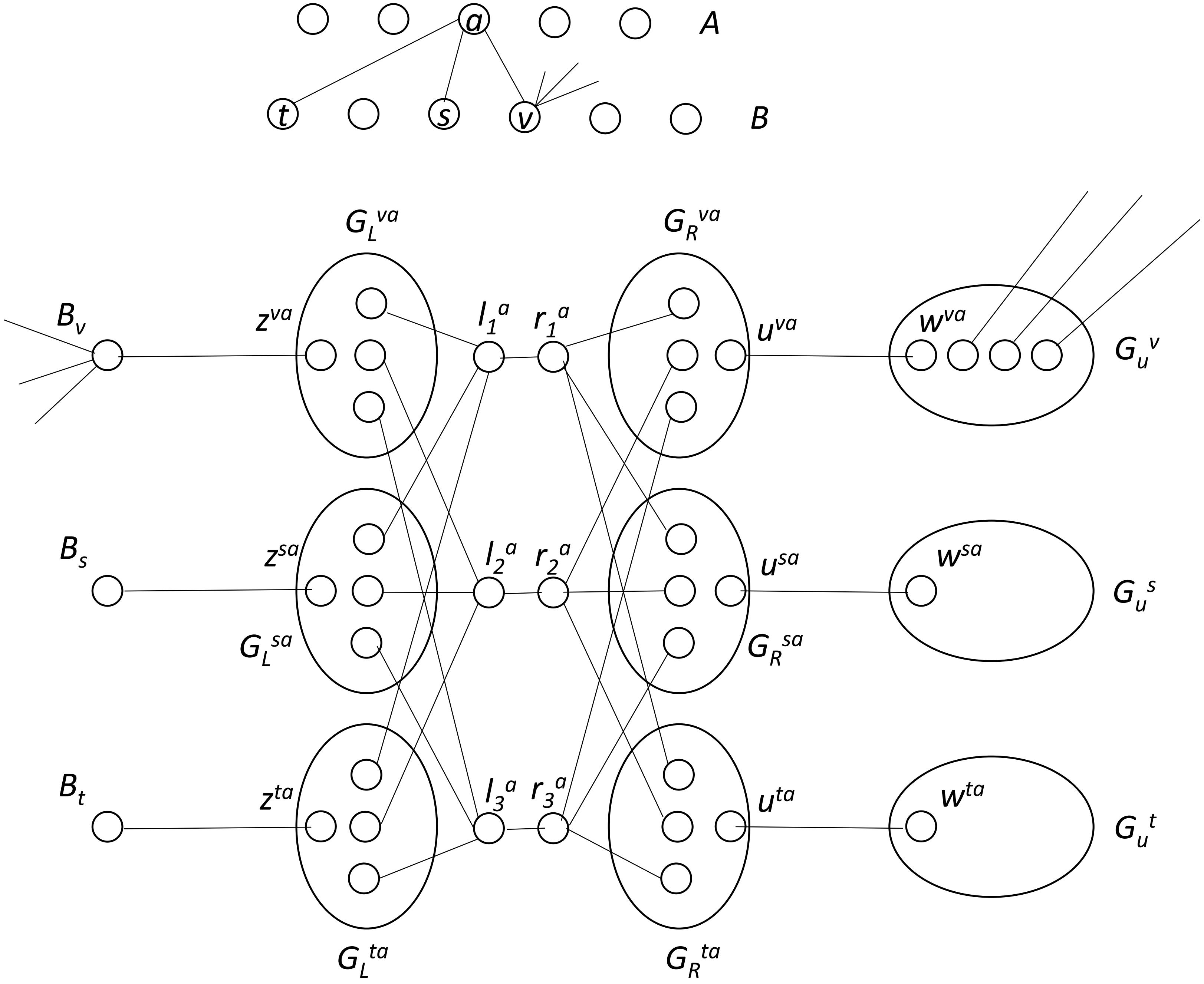

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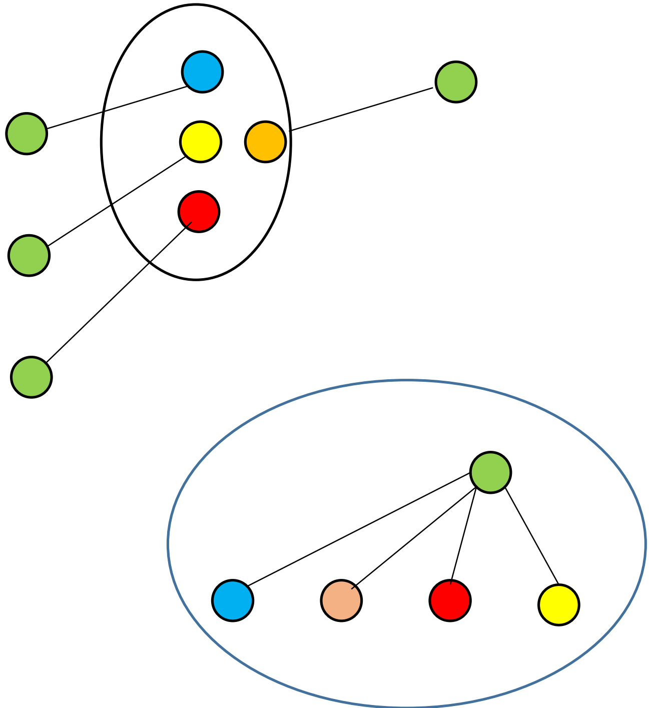

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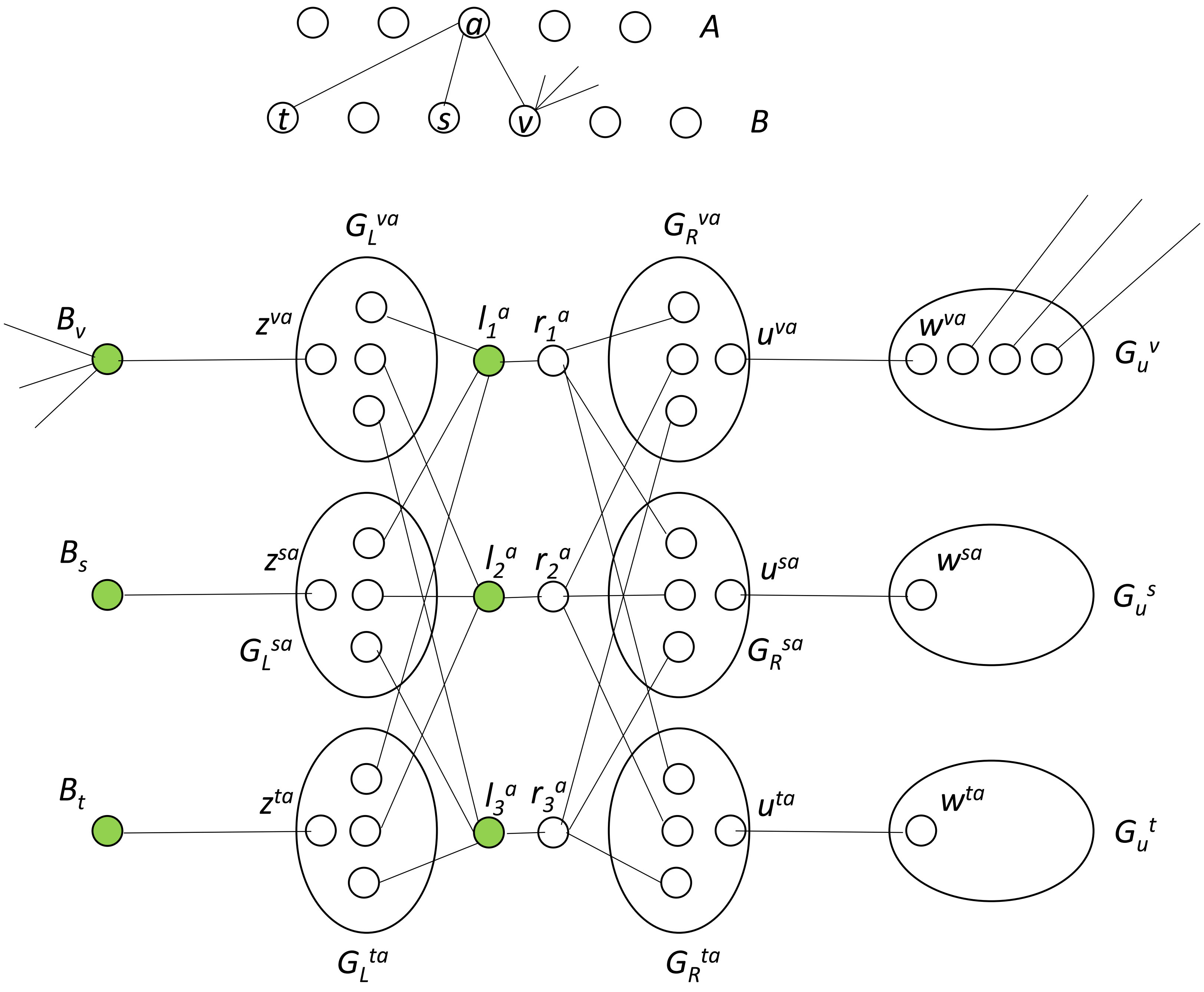
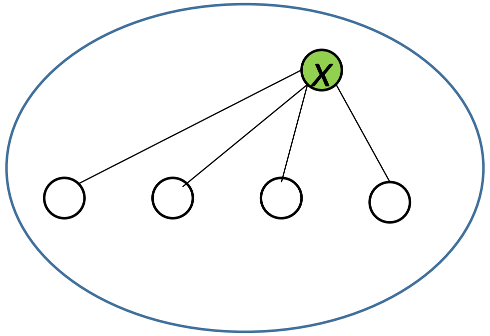
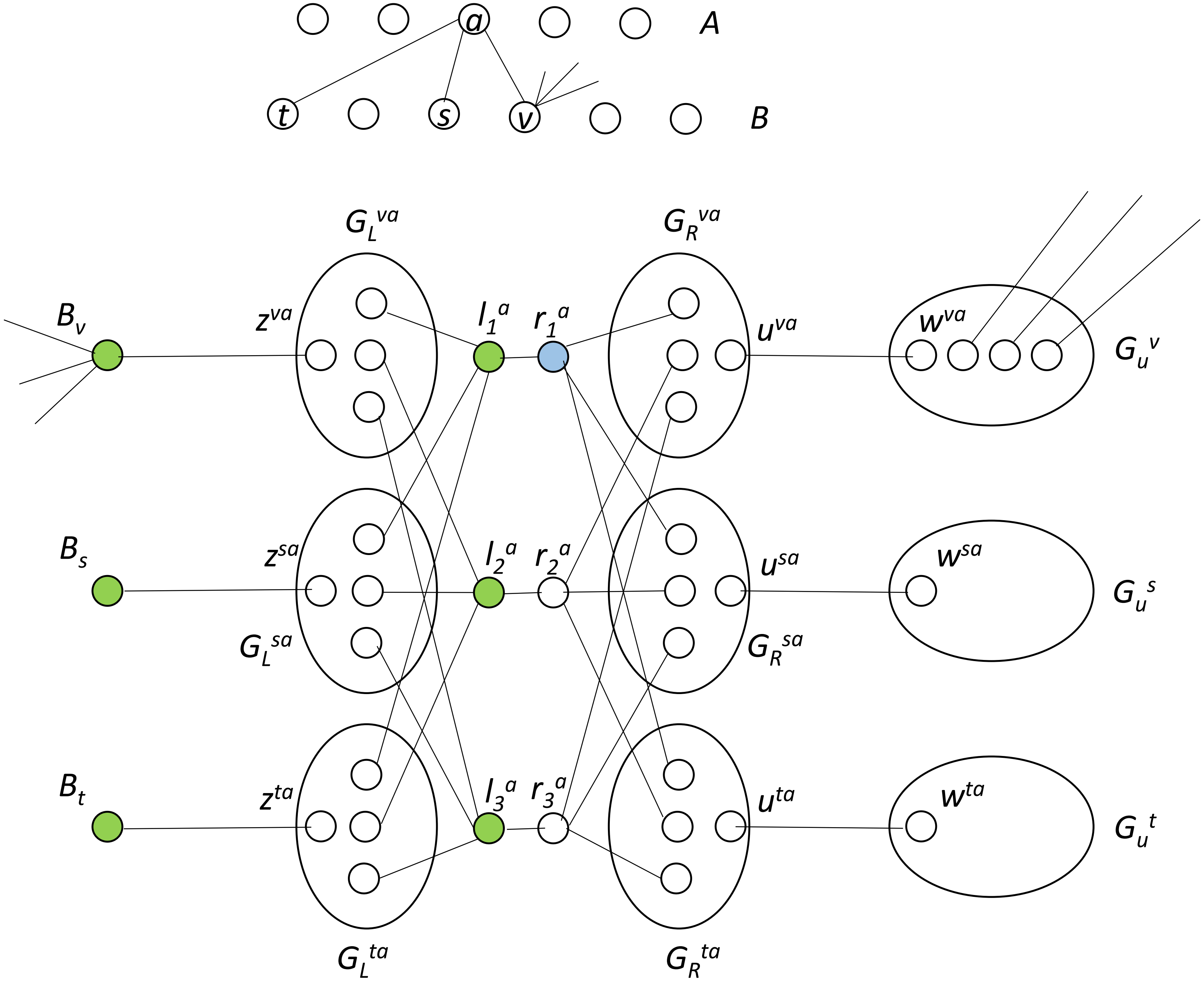
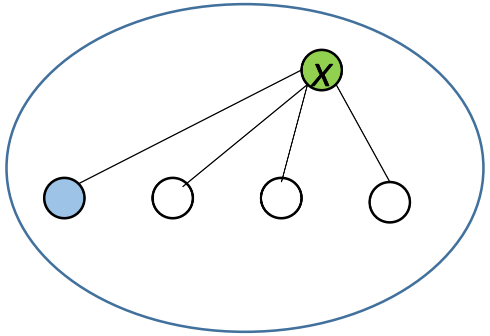






Reduction from *k*-edge colorability *of k*-regular (*k*-1)-uniform hypergraphs.



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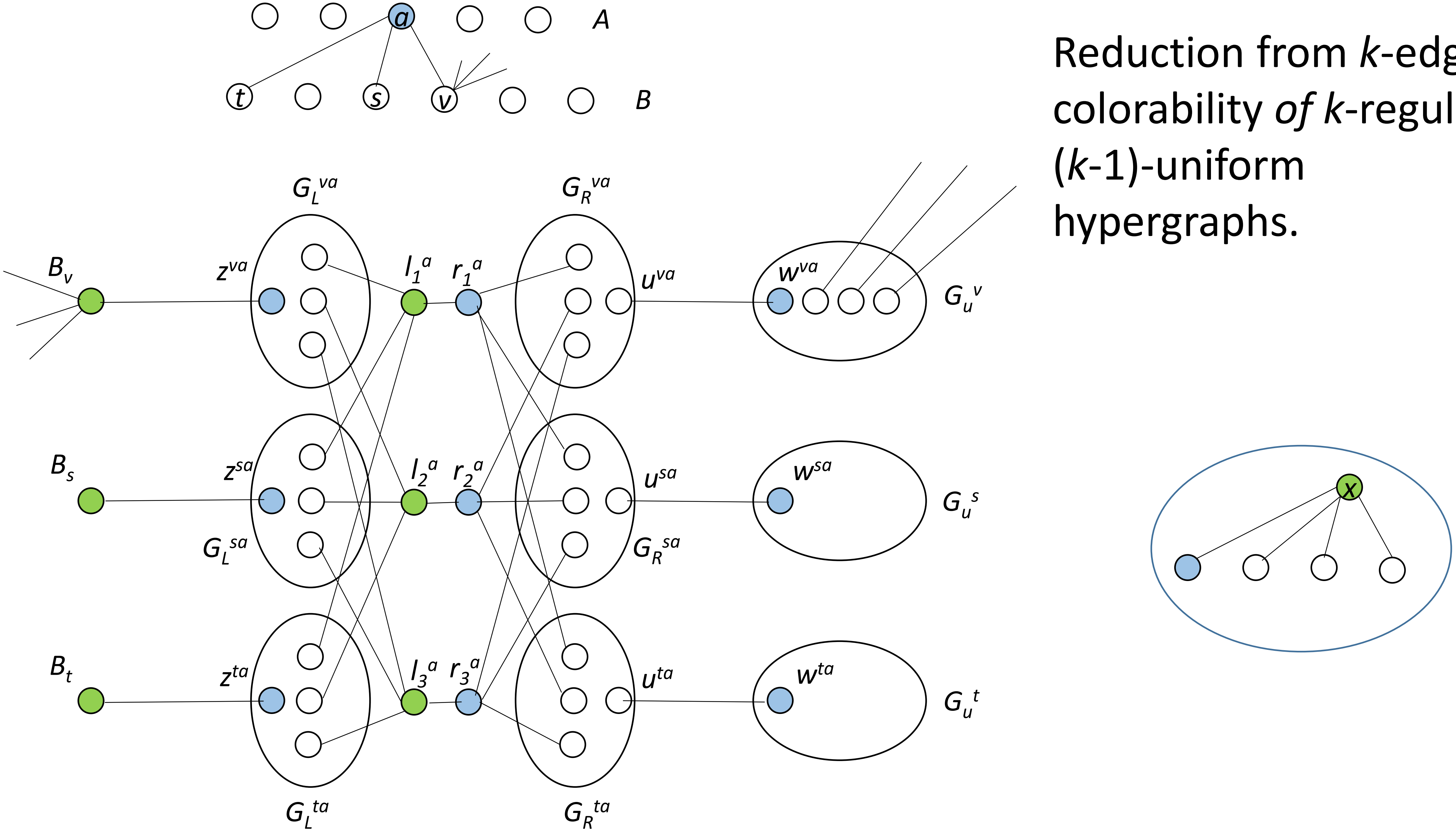

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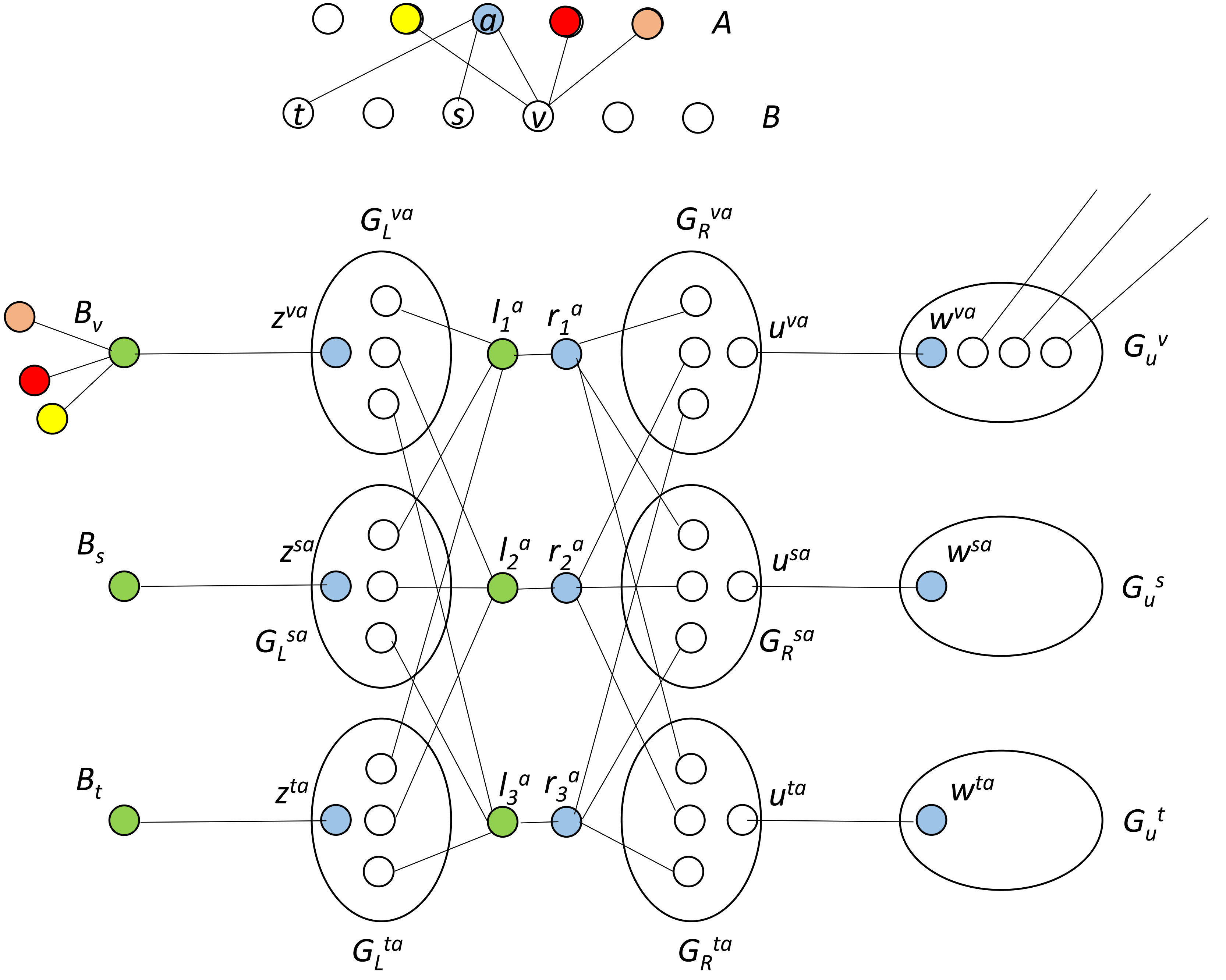
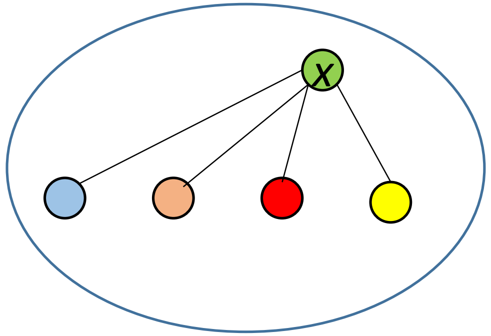
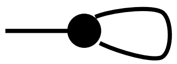




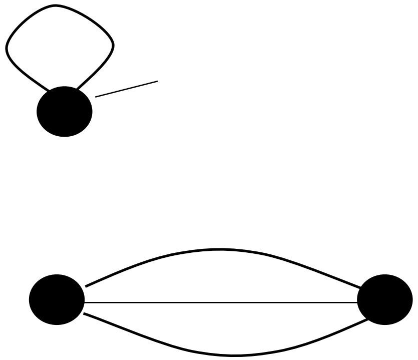
4. Strong Dichotomy for cubic graphs

Theorem (strong dichotomy for 3-regular target graphs): List-*H*-COVER is polynomial time solvable for  and NP-complete for all other target graphs *H*, even for simple inputs.

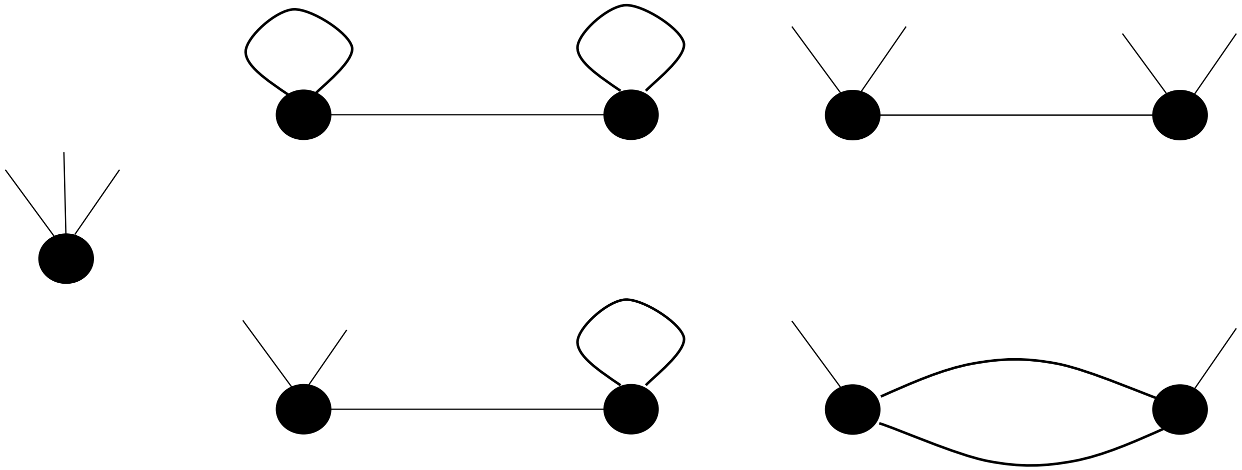
Strong Dichotomy for cubic graphs

If *H* has at most 2 vertices, the following is known from Bok, Fiala, Hlineny, Jedlickova, Kratochvil 2021 for the *H*-COVER problem:

Polynomial

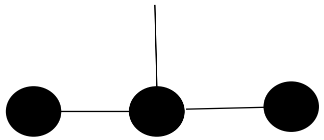


NP-complete without lists

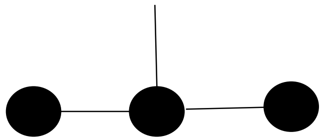


Fact: For every graph *H*, *H*-COVER α List-*H*-COVER

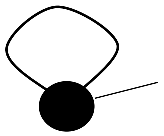
Strong Dichotomy for cubic graphs

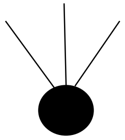
Case A: *H* has a vertex with 3 different neighbors  this is a semi-simple vertex and List-*H*-COVER is NP-complete for simple input graphs by the Theorem.

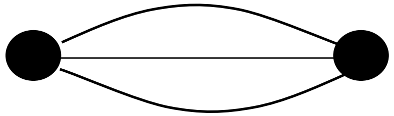
Strong Dichotomy for cubic graphs

Case A: *H* has a vertex with 3 different neighbors  this is a semi-simple vertex and List-*H*-COVER is NP-complete for simple input graphs by the Theorem.

Case B: H has a vertex whose all 3 neighbors are the same vertex

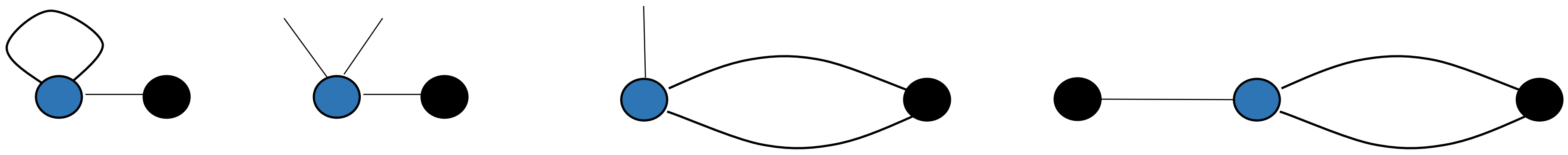
Case B1:  List-*H*-COVER is polynomial time solvable via perfect matching

Case B2:  *H*-COVER is NP-complete for simple inputs (3-edge-colorability)

Case B3:  List-*H*-COVER is NP-complete for simple inputs (via Precoloring extension for line graphs of cubic bipartite graphs, Fiala 1998)

Strong Dichotomy for cubic graphs

Case C: Every vertex of *H* has exactly 2 neighbors, one adjacent via a double edge or via a loop or via 2 semi-edges, and the other one via a single edge or via a semi-edge.



Strong Dichotomy for cubic graphs

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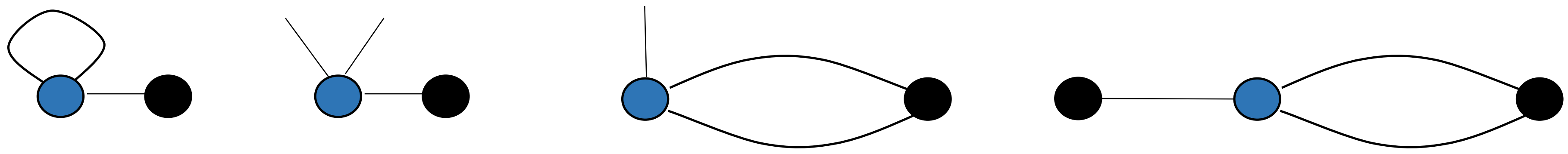
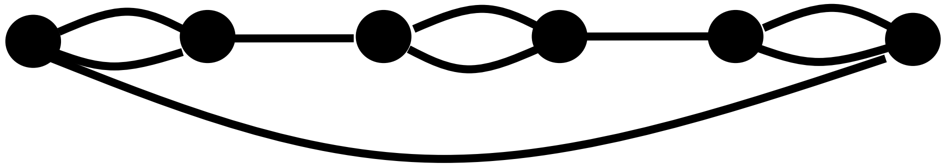
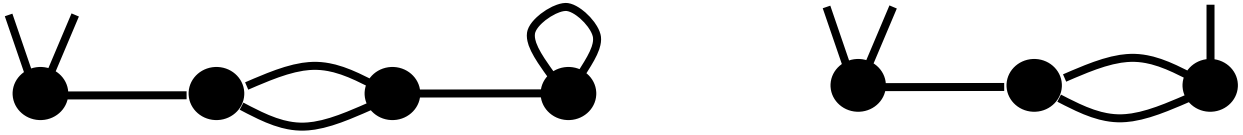


Case C1: *H* is a ring



Case C2: *H* is a sausage graph



Strong Dichotomy for cubic graphs

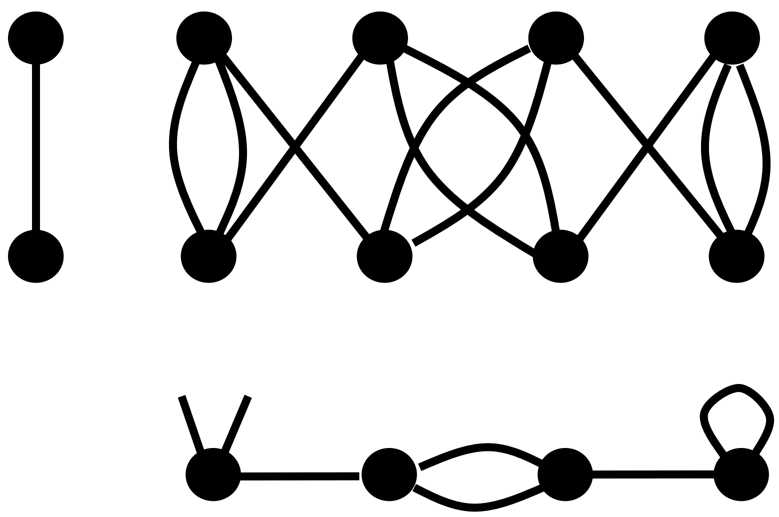
Cases C1 and C2:

Lemma 1: If *H* is a sausage graph with *k* vertices, then *k*-Ring-COVER α *H*-COVER

Lemma 2: For every $k \geq 2$, the *k*-Ring-COVER problem is NP-complete for simple input graphs

Strong Dichotomy for cubic graphs

Proof of Lemma 1: For every non-bipartite graph *H* and for every graph *G*, *G* covers $H \times K_2$ if and only if *G* is bipartite and covers *H* (Fiala 1998). And if *H* is a sausage with *k* vertices, then $H \times K_2$ is a *k*-Ring.



Research questions

Problem 1: Full characterization and strong dichotomy for List-*H*-COVER for *k*-regular target graphs *H* for $k \geq 4$?

Problem 2: Can we do without lists?

Problem 3: Can we do without semi-simple vertices?

Conjecture: Let *H* be a connected k-regular graph (loops, multiple edges and semi-edges allowed), with $k \geq 3$. Then both *H*-COVER and List-*H*-COVER are polynomial time solvable if *H* is a single-vertex graph with at most one semi-edge, *H*-COVER is solvable in polynomial time if H is a two-vertex graph with *k* parallel edges between its vertices, and both problems are NP-complete for simple input graphs otherwise.


Thank you!