# Graph Covers:
## Where Topology Meets Computer Science, and Simple Means Difficult

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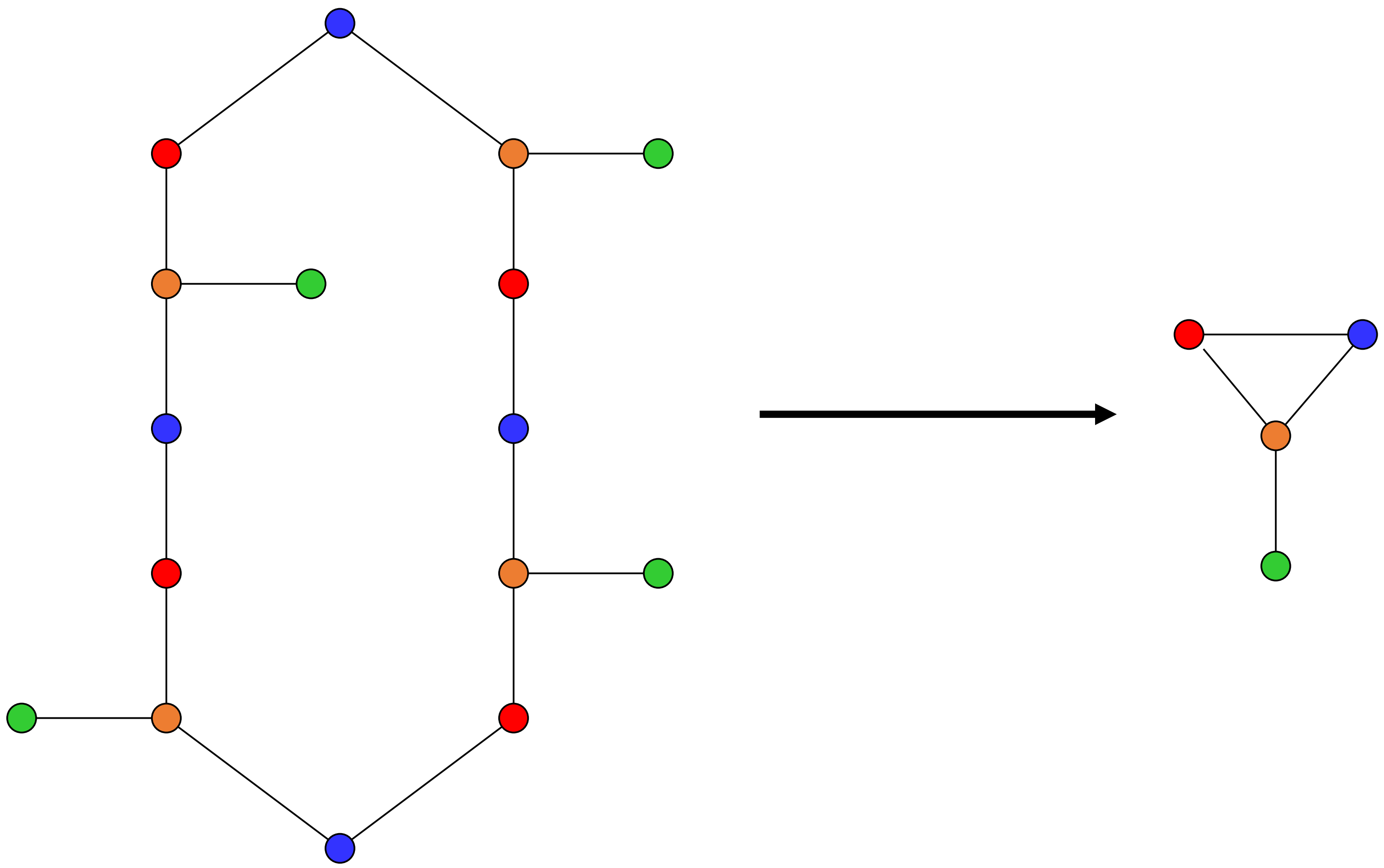
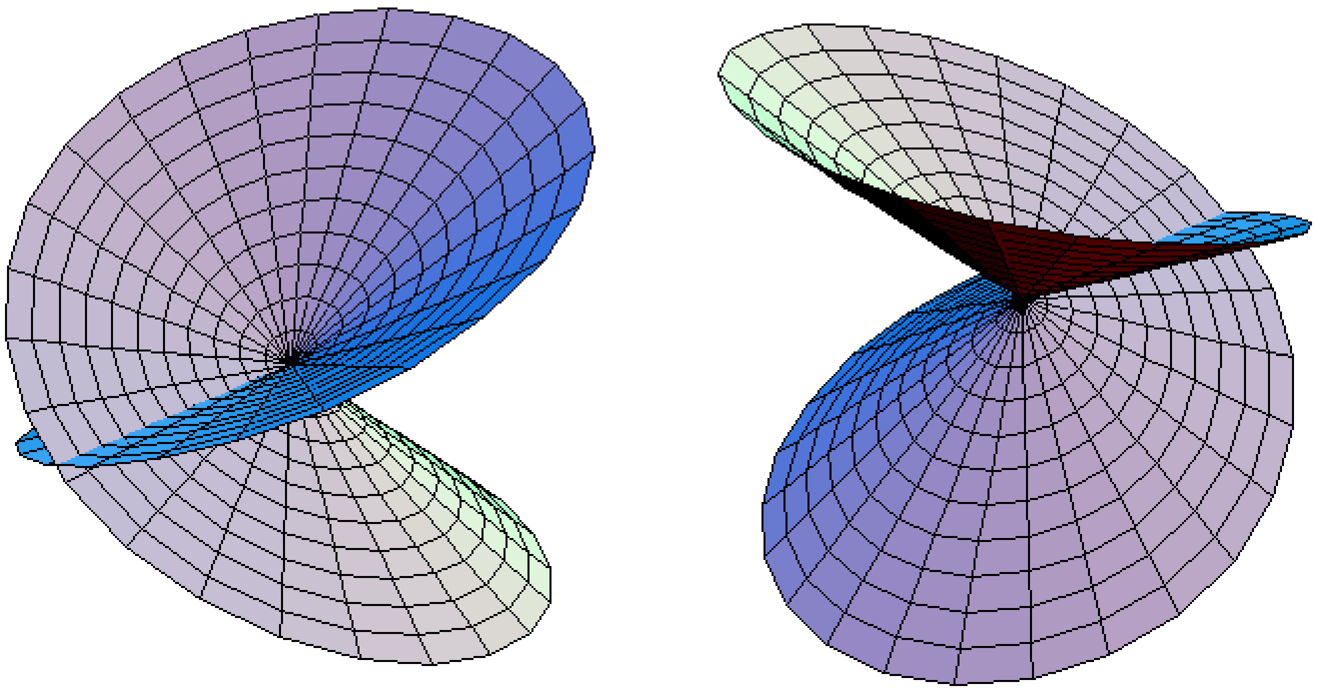


# Outline of the talk

- Motivation from topology
- Formal definitions
- Charming mathematical questions (Negami's conjecture)
- Computer science connections
- Computational complexity
- Going general (multiple edges, loops, semi-edges, orientations, colors)
- The Strong Dichotomy Conjecture
- The Empire strikes back (covers of disconnected graphs)
- Generalized snarks

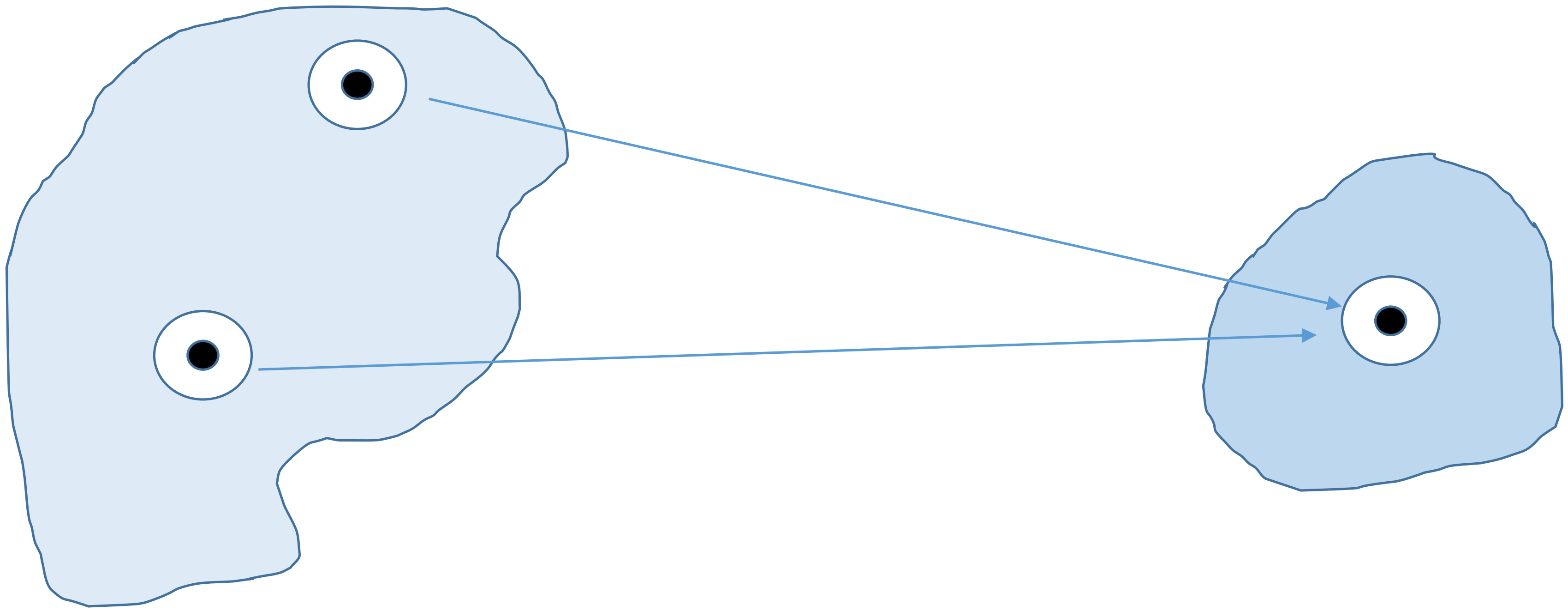
# Covering spaces in topology



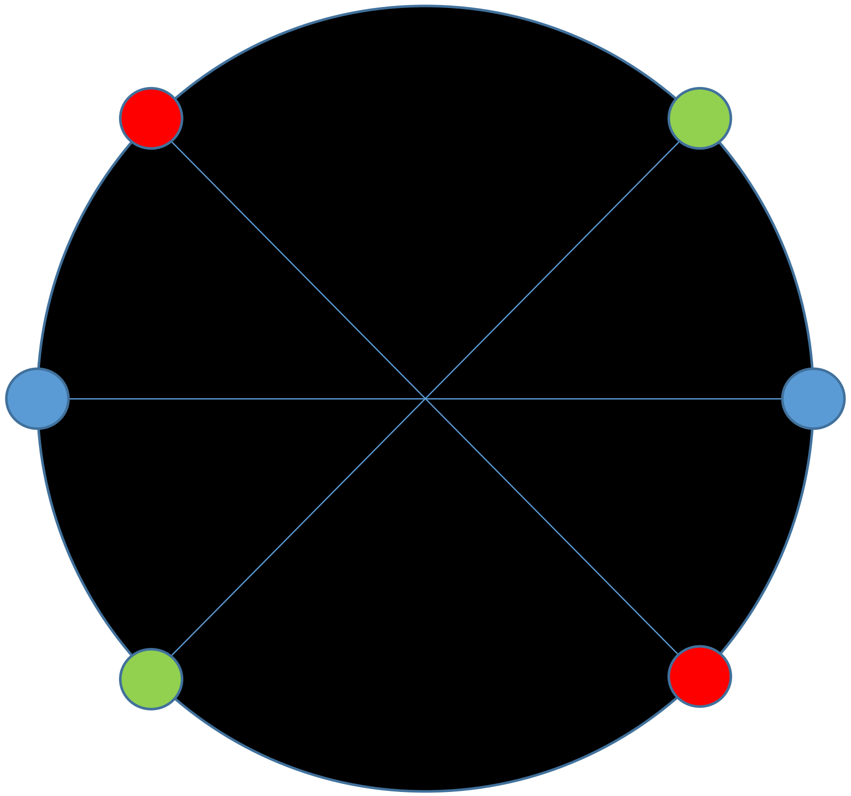
Wikipedia – Wikimedia commons

# Covering spaces in topology

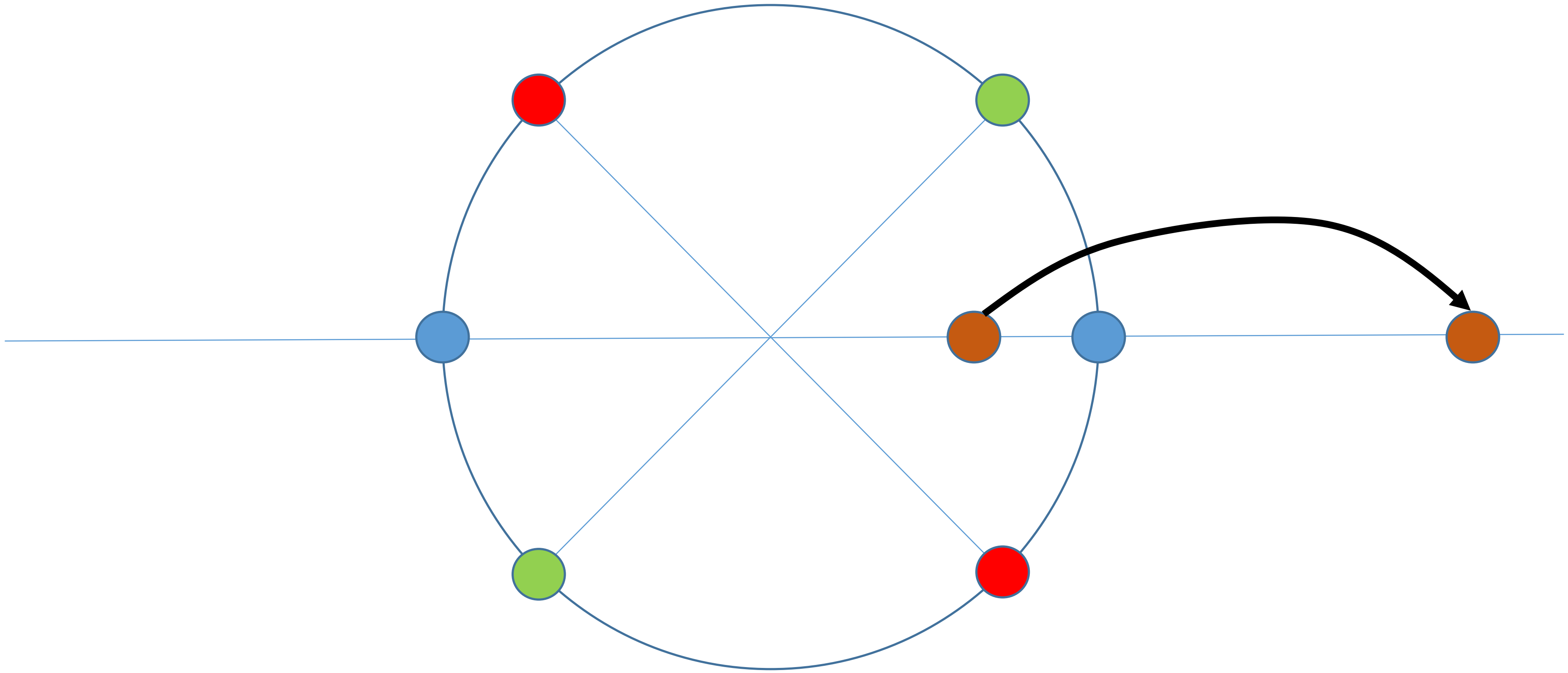
Euclidean and projective planes – the Euclidean plane is a double cover of the projective one



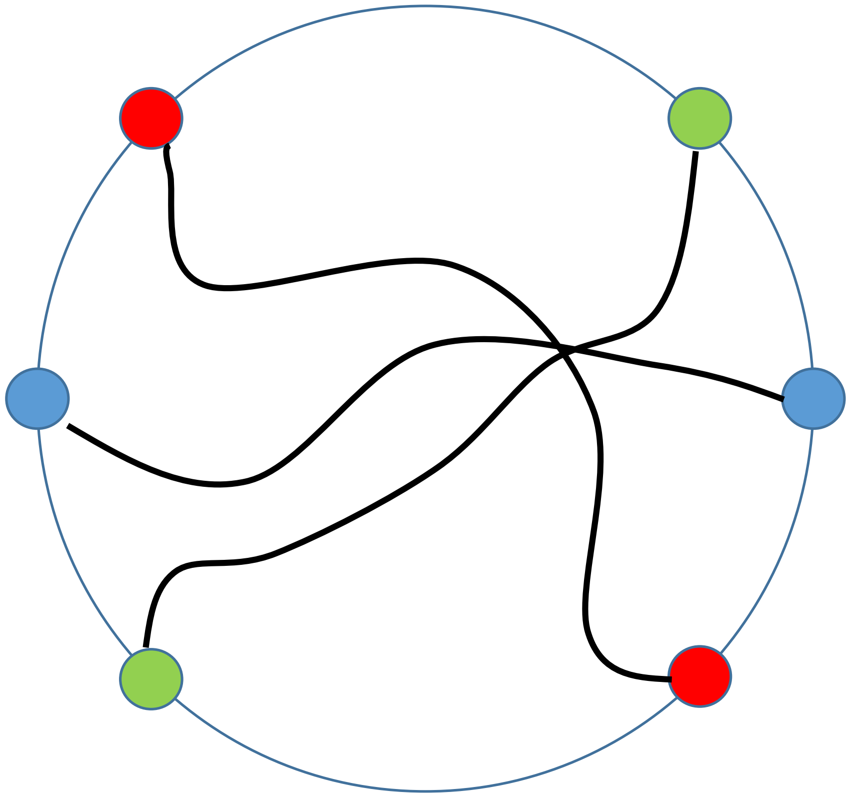
# The projective plane



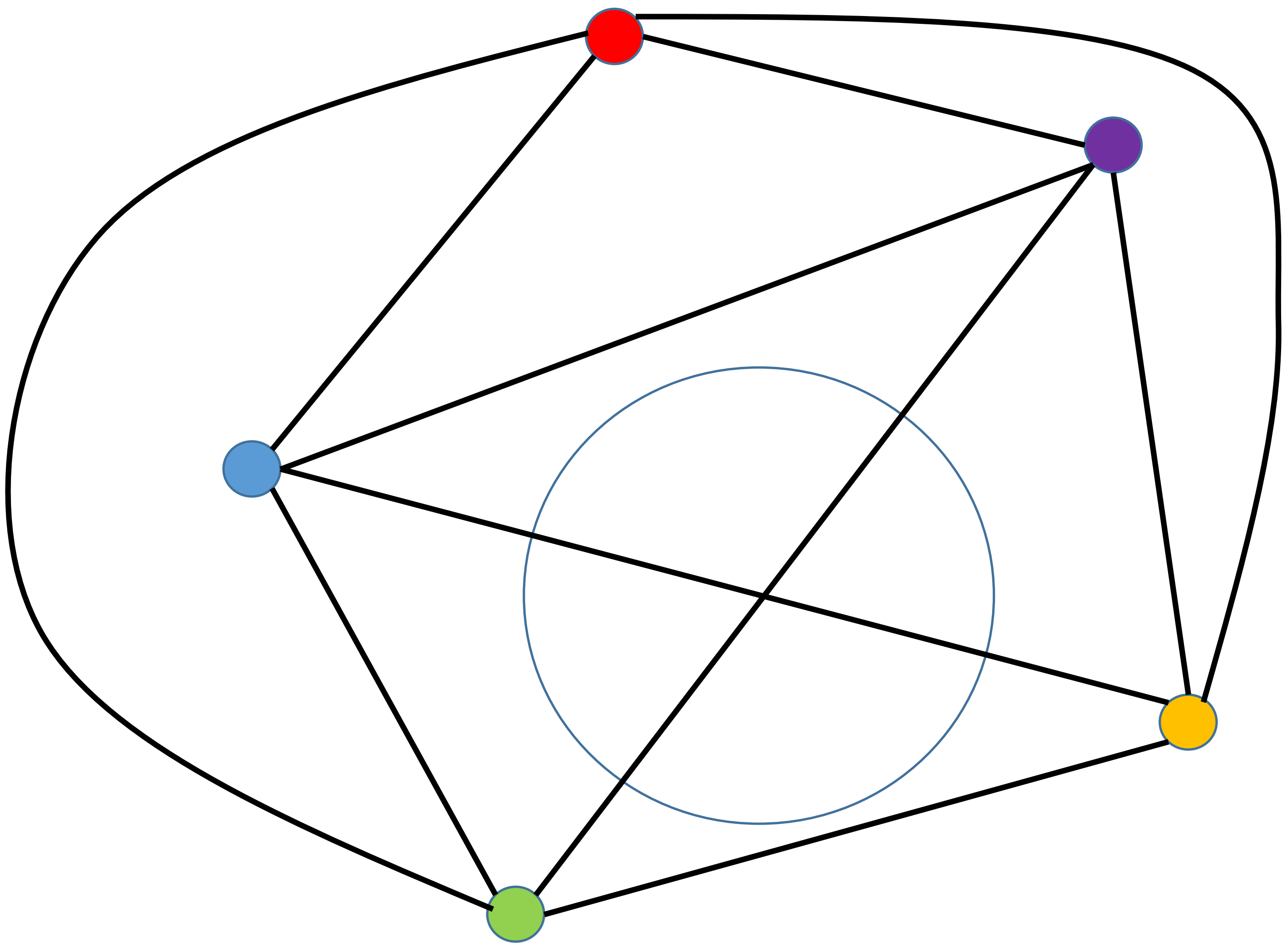
# The projective plane double covered by the Euclidean plane



# The projective plane as Euclidean plane with a cross-cap

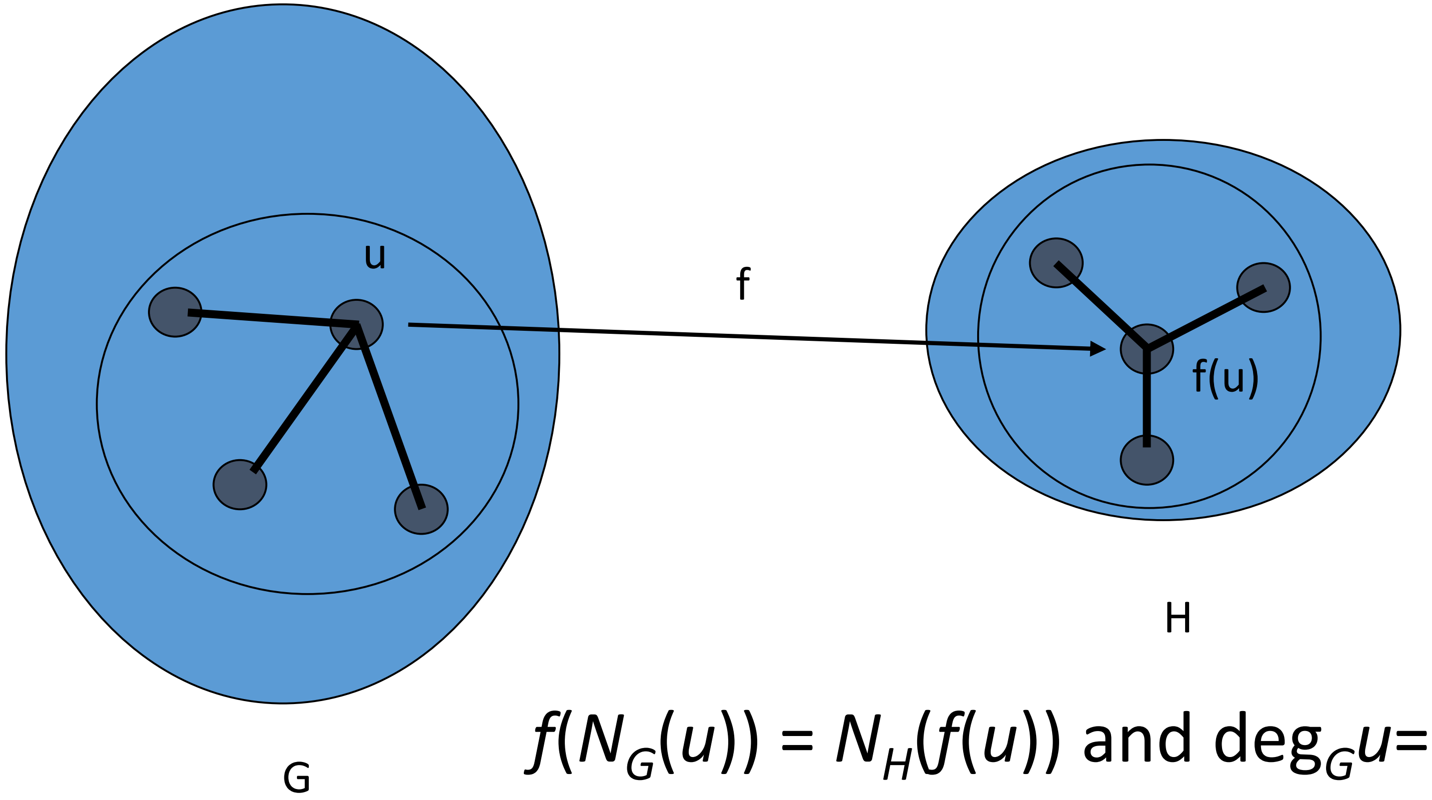


# $K_5$ is projective planar

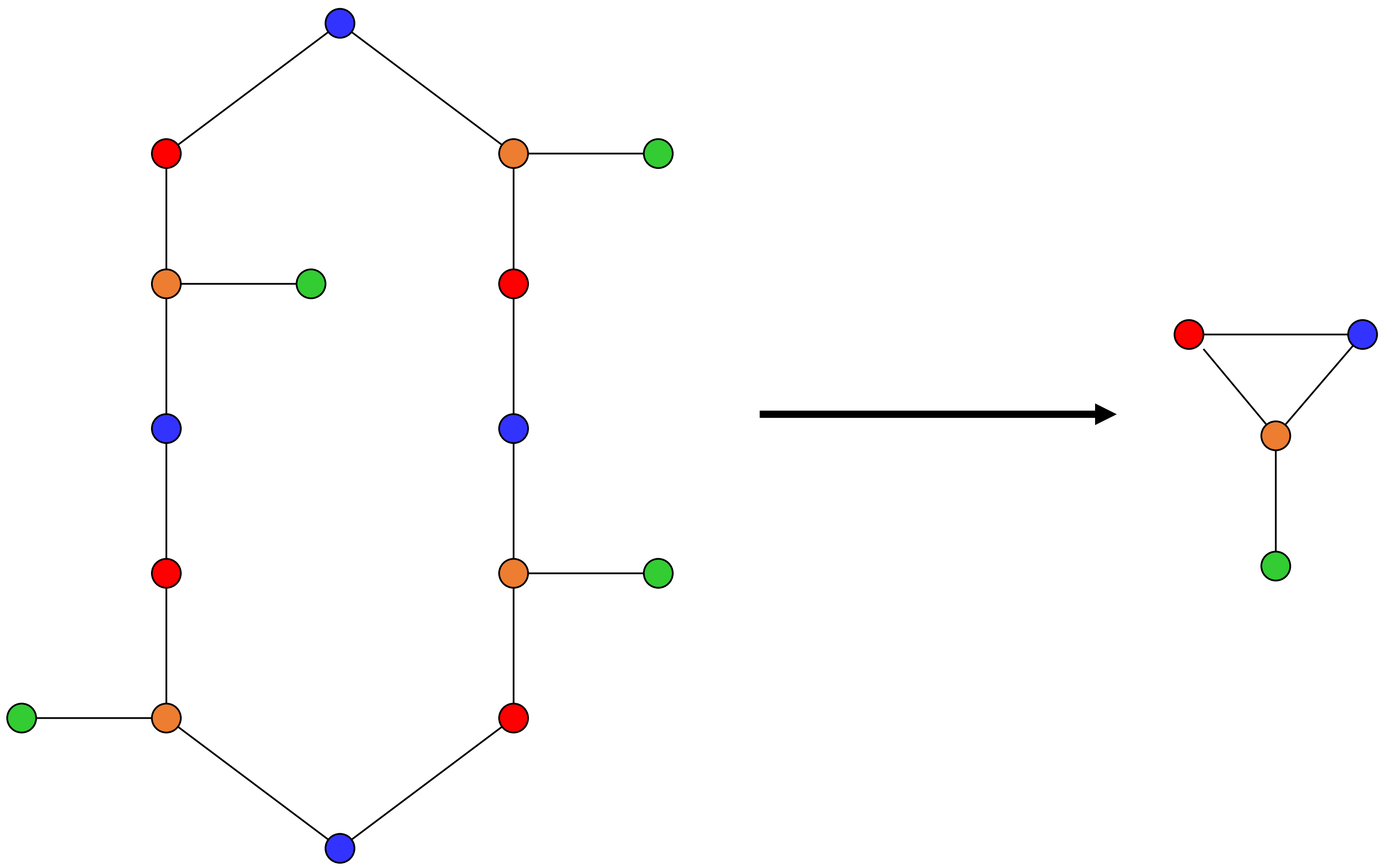


# Definition of graph covering
## (for connected simple graphs)

Definition: Mapping $f$: $V(G) \rightarrow V(H)$ is a *graph covering projection* if for every $u \in V(G)$, $f|N_G(u)$ is a bijection of $N_G(u)$ onto $N_H(f(u))$



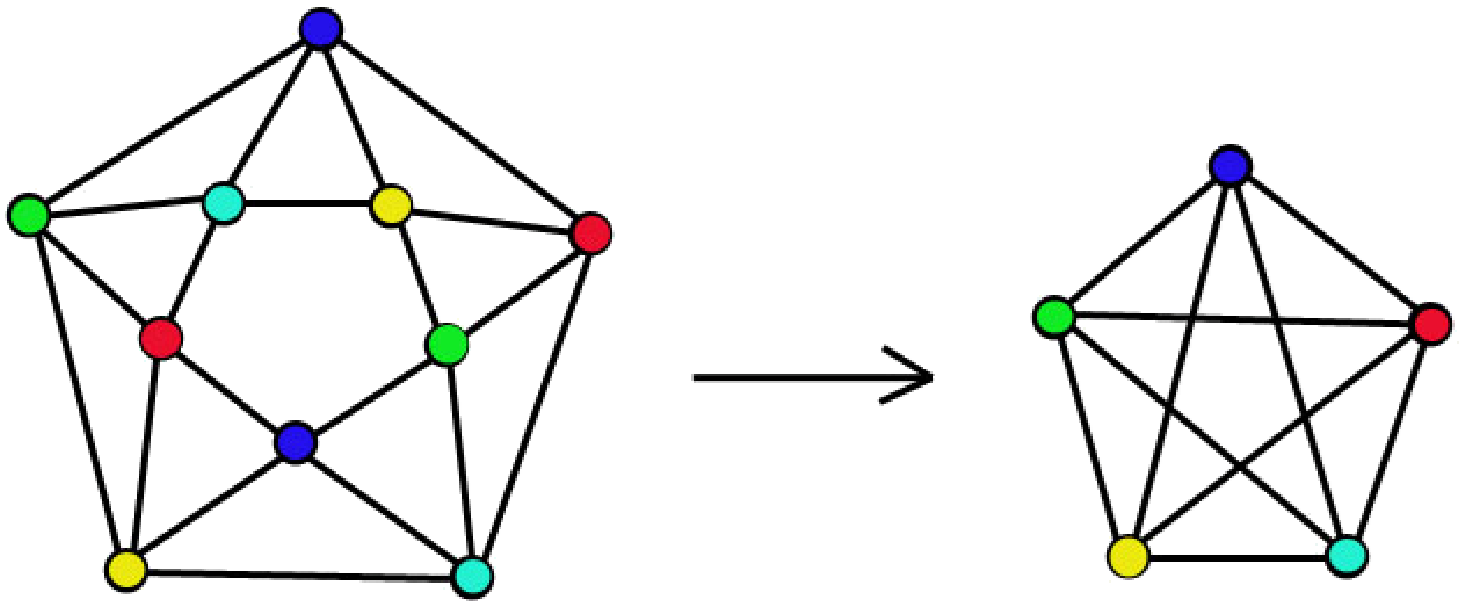

$f(N_G(u)) = N_H(f(u))$ and $\deg_G u = \deg_H f(u)$



# A bit of the history

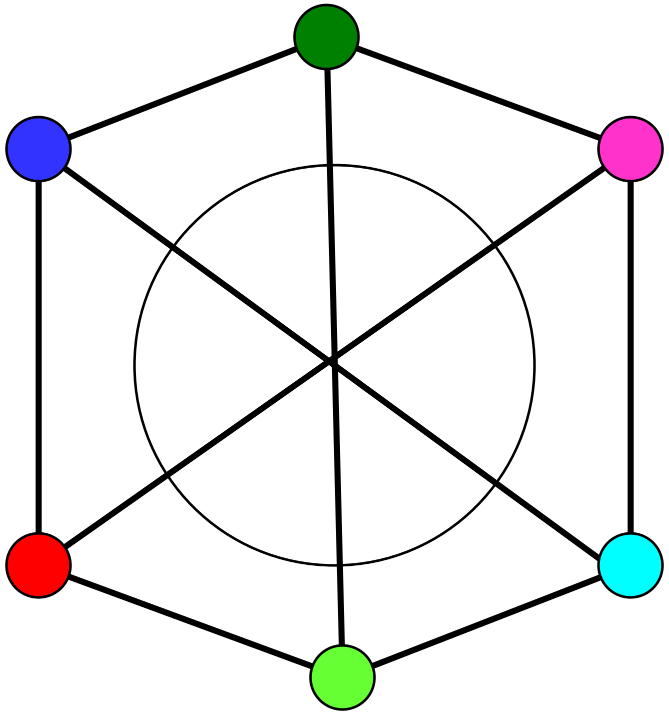
- Topological graph theory, construction of highly symmetric graphs (Biggs 1974, Djokovic 1974, Gardiner 1974, Gross et al. 1977)
- Local computation (Angluin STOC 1980, Litovsky et al. 1992, Courcelle et al. 1994, Chalopin et al. 2006)
- Common covers (Angluin et al. 1981, Leighton 1982)
- Finite planar covers (Negami’s conjecture 1988, Hliněný 1998, Archdeacon 2002, Hliněný et al. 2004)

# Negami's conjecture

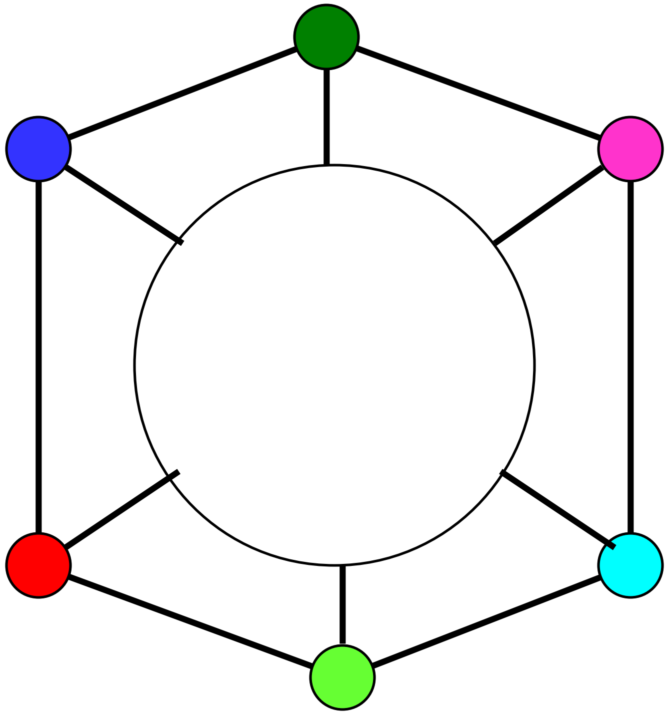


# Negami's conjecture

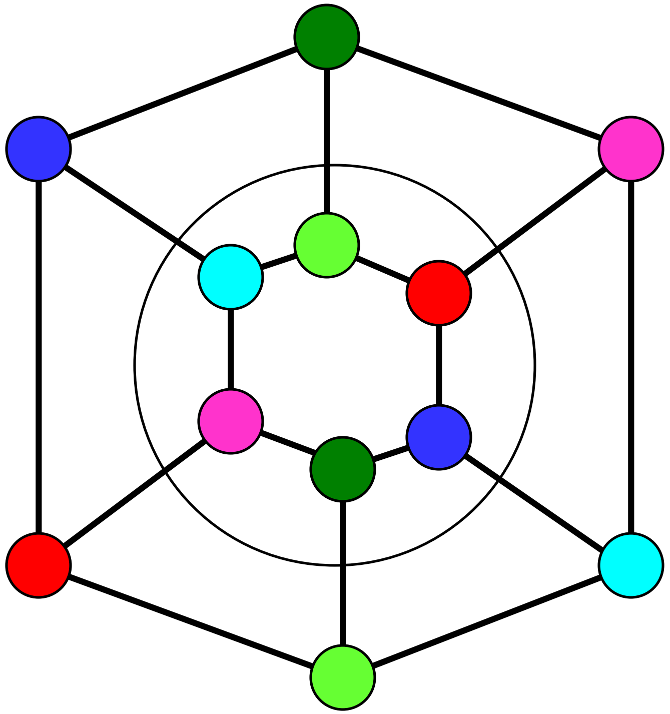
Conjecture (Negami 1988): A graph has a finite planar cover if and only if it is projective planar.



$K_{3,3}$



$K_{3,3}$



A planar cover of $K_{3,3}$

# Negami's conjecture

Attempts to prove via forbidden minors for projective planar graphs: Both *PlanarCoverable* and *ProjectivePlanar* are classes closed in the minor order. Moreover,

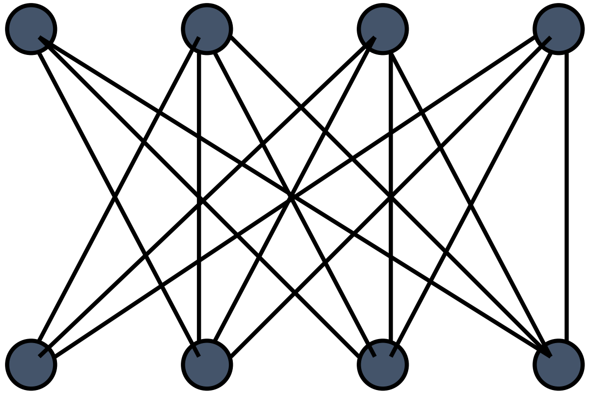
$$\textit{ProjectivePlanar} \subseteq \textit{PlanarCoverable}.$$

Need to show that no forbidden minor for the projective plane has a finite planar cover.

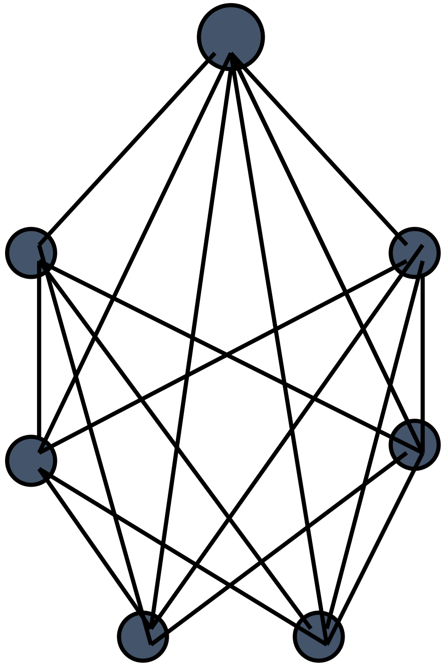
# Negami's conjecture

Thm (Negami, Fellows, Archdeacon 1990): Conjecture is true for graphs not containing $K^{--}_{4,4}$ and $K_{1,2,2,2}$ as minors.

# The terrible two



$K^{--}_{4,4}$



$K_{1,2,2,2}$

# Negami's conjecture

Thm (Negami, Fellows, Archdeacon 1990): Conjecture is true for graphs not containing $K^{--}_{4,4}$ and $K_{1,2,2,2}$ as minors.

P. Hliněný (1998): $K^{--}_{4,4}$ does not have a finite planar cover.

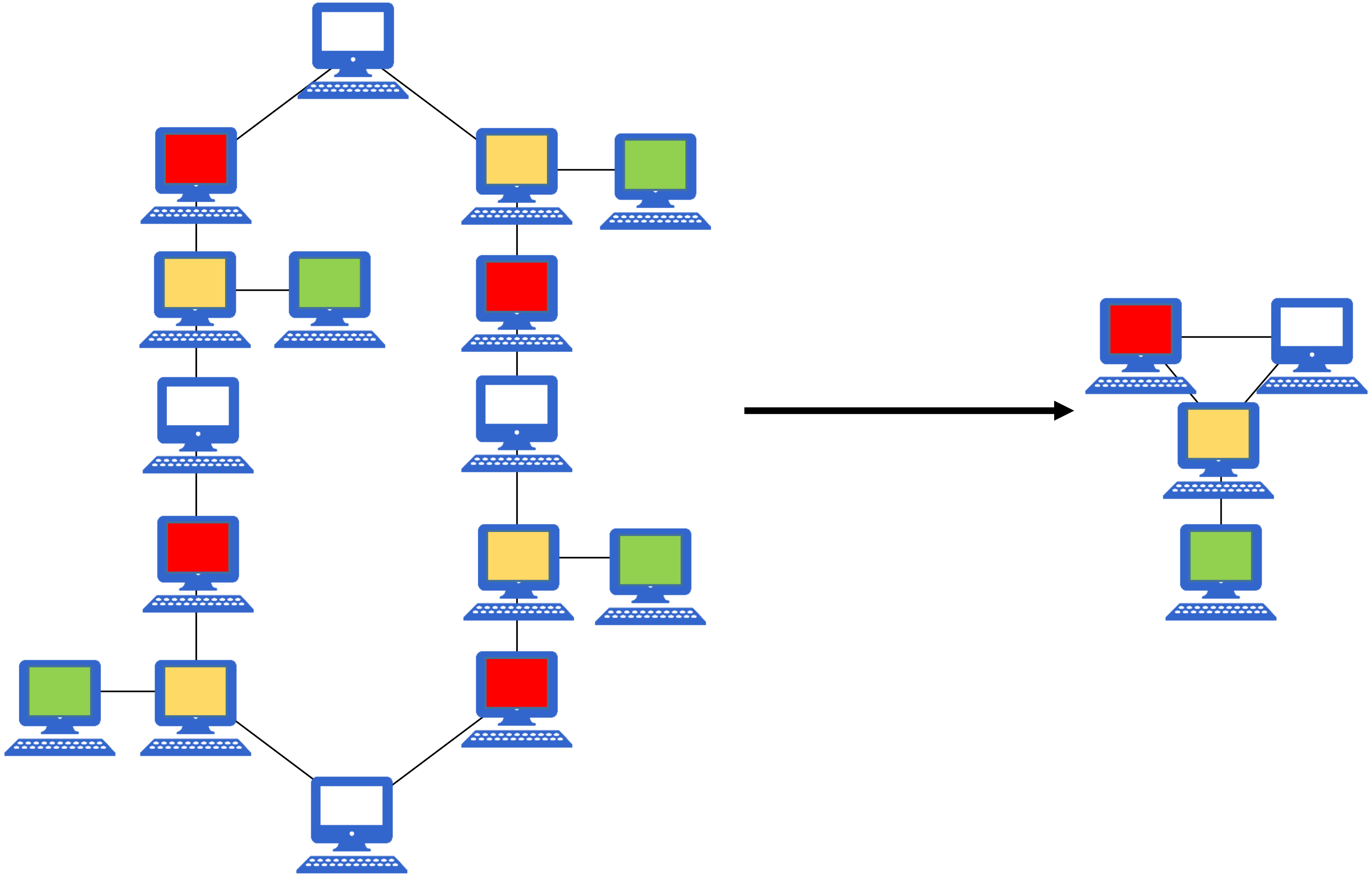
P. Hliněný, R. Thomas (2002): Only finite number of counterexamples exist (if any).


HBO


Computer Science

HBO

# Model of local computation

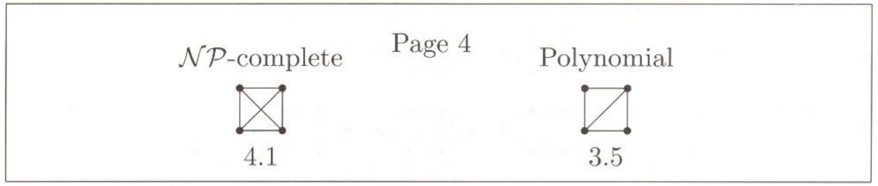


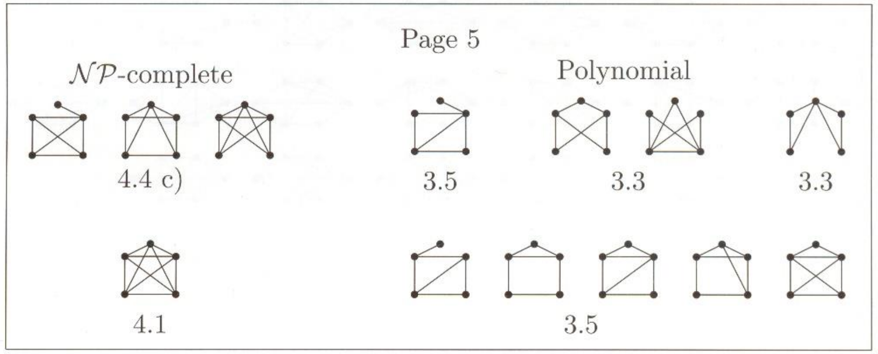
# Computational complexity of graph covers

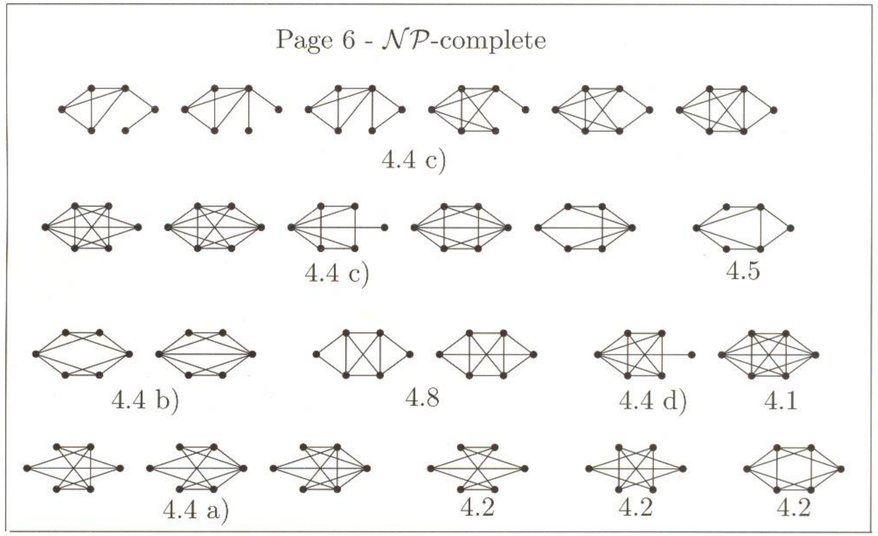
*H*-COVER
Input: A graph *G*
Question: Does *G* cover *H*?

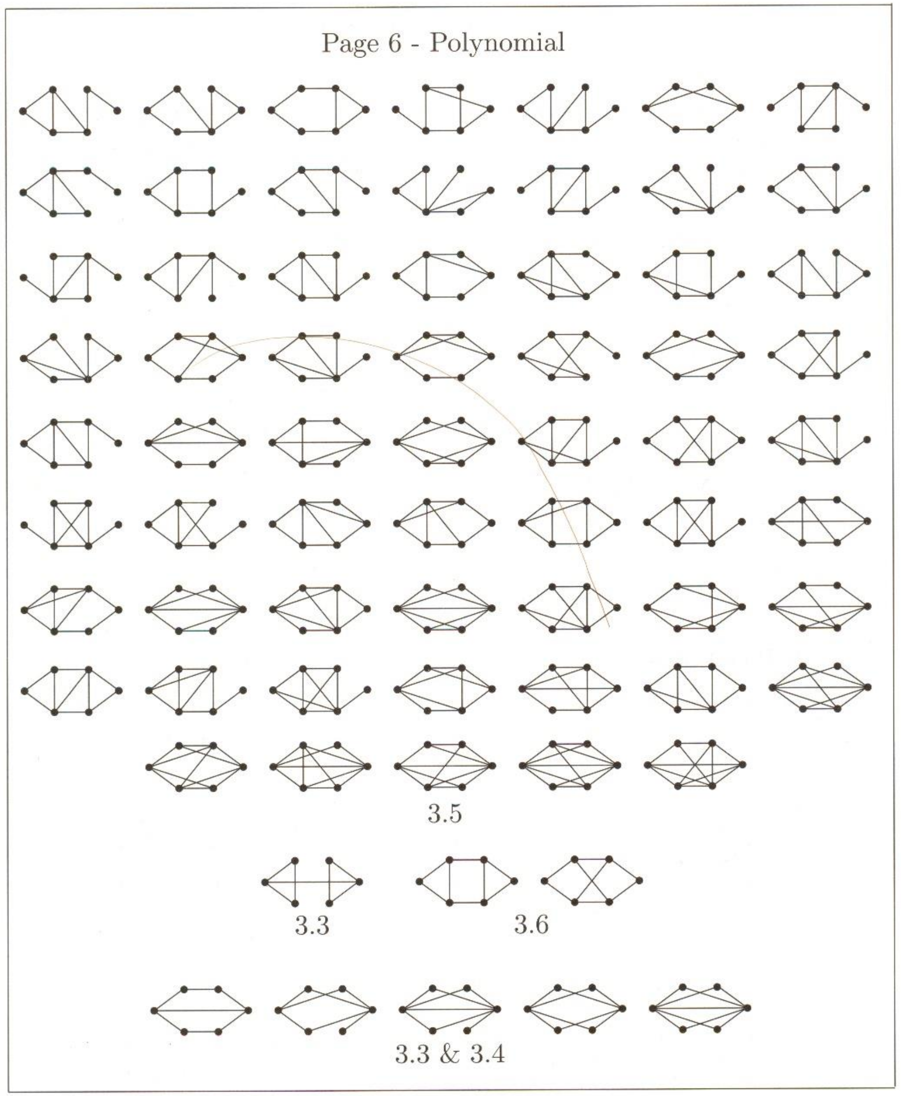
# Computational complexity of graph covers

- Thm (Bodlaender 1989): *H*-COVER is NP-complete if *H* is also part of the input.
- Abello, Fellows, Stilwell 1991: Initiated the study of computational complexity of the *H*-COVER problem for fixed *H*.
- Thm (Kratochvil, Proskurowski, Telle 1994): *H*-COVER is polynomial time solvable for every simple graph with at most 2 vertices per equivalence class in its degree partition.
- Thm (Fiala, Kratochvil, Proskurowski, Telle 1998): *H*-COVER is NP-complete for every simple regular graph of valency at least 3.
- Fiala, Kratochvil 2008: Relation to CSP
- Bílka, Jirásek, Klavík, Tancer, Volec 2011: NP-hardness of covering small graphs by planar inputs.


Page 4
𝒩𝒫-complete
Polynomial
4.1
3.5

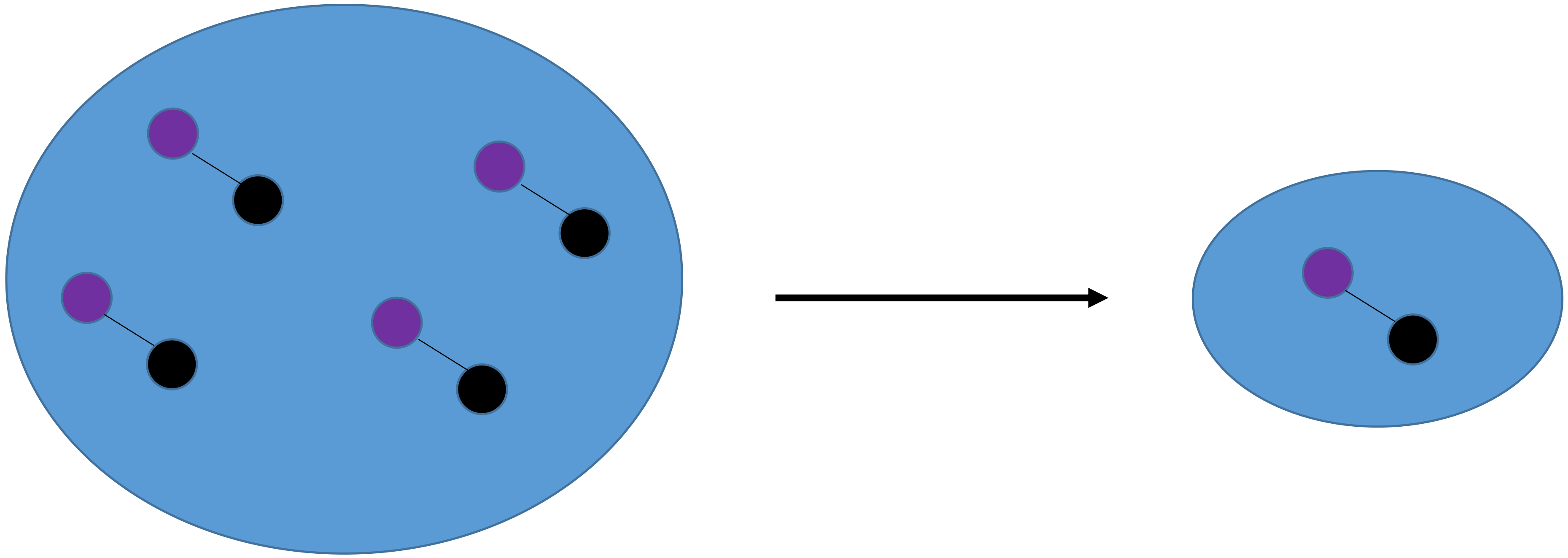

Page 5
𝒩𝒫-complete
Polynomial
4.4 c)
3.5
3.3
3.3
4.1
3.5


Page 6 - 𝒩𝒫-complete
4.4 c)
4.4 c)
4.5
4.4 b)
4.8
4.4 d)
4.1
4.4 a)
4.2
4.2
4.2

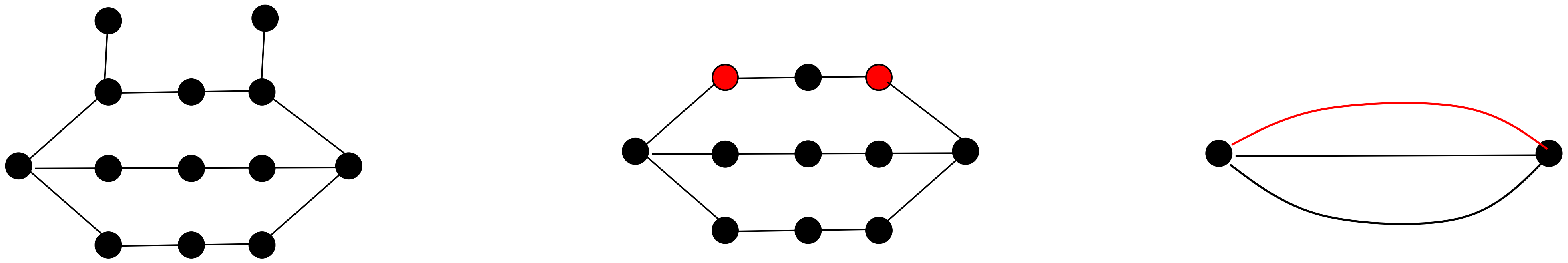

Page 6 - Polynomial
3.5
3.3
3.6
3.3 & 3.4

# A few facts on graph covers

- Every covering projection to a connected graph is equitable
- A (rooted) tree is covered only by an isomorphic tree
- A path is covered only by a path of the same length



# Reduction to colored graphs



Kratochvil, Proskurowski, Telle 1997: Apply the same reductions to *G* and *H*. Every covering projection must respect the colors. To fully understand the complexity of *H*-COVER for all simple graphs, it is necessary and suffices to understand its complexity for colored mixed multigraphs of minimum degree $\geq 3$.

# Complexity of covering multigraphs

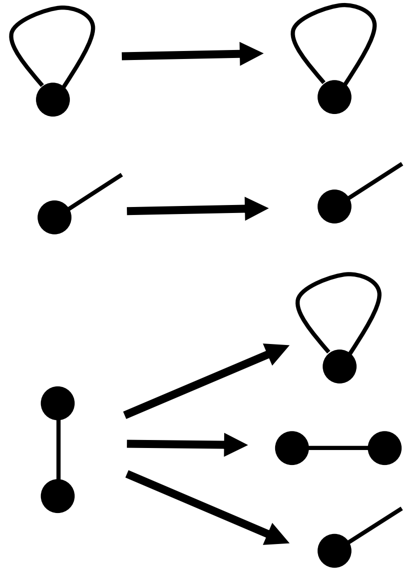
- Kratochvil, Proskurowski, Telle 1997: Complete characterization of the computational complexity of *H*-COVER for colored mixed 2-vertex multigraphs *H.*
- Kratochvil, Telle, Tesař 2016: Complete characterization of the computational complexity of *H*-COVER for 3-vertex multigraphs *H.*
- Bok, Fiala, Hliněný, Kratochvíl MFCS 2021: First results on the computational complexity of *H*-COVER for (multi)graphs with **semi-edges**. Full classification for 1-vertex and 2-vertex graphs *H*.

# Covers of general graphs

(with multiple edges, loops and semi-edges)

Definition: A pair of mappings $f = (f_V, f_E)$: $G \to H$ is a graph covering projection if

- $f_V$:$V(G) \to V(H)$ is a homomorphism,
- $f_E$:$E(G) \to E(H)$ is compatible with $f_V$, and it is a bijection of {edges incident with $u$} onto {edges incident with $f_V(u)$} for every $u \in V(G)$



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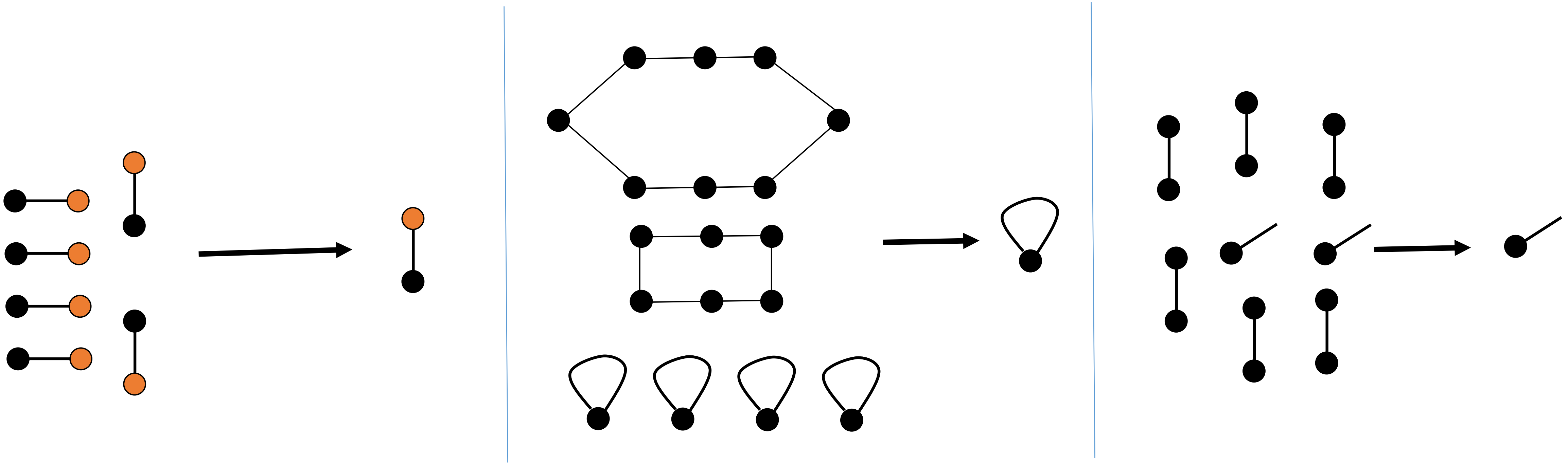

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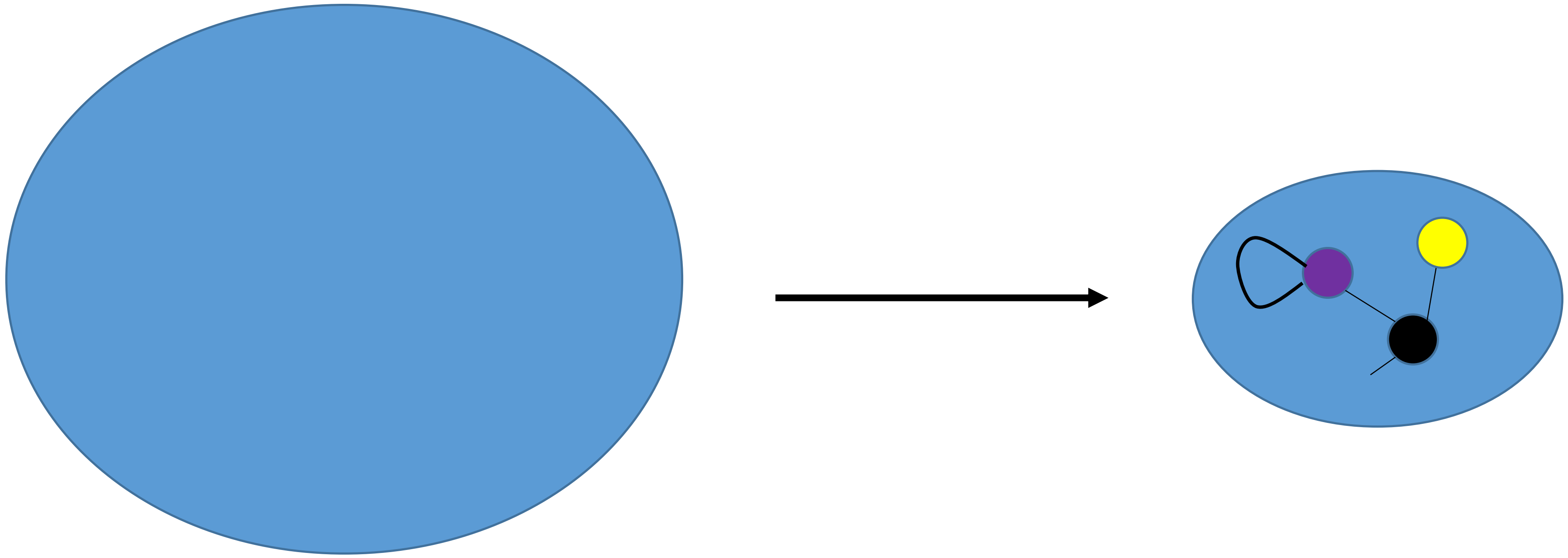

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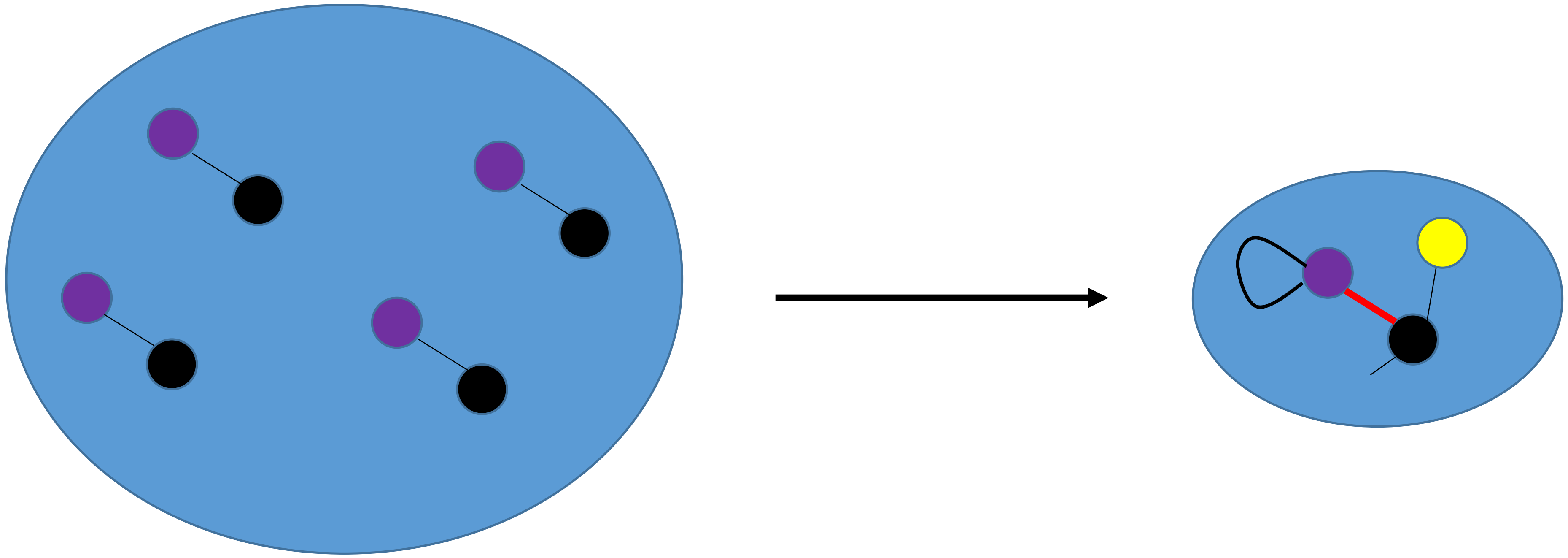

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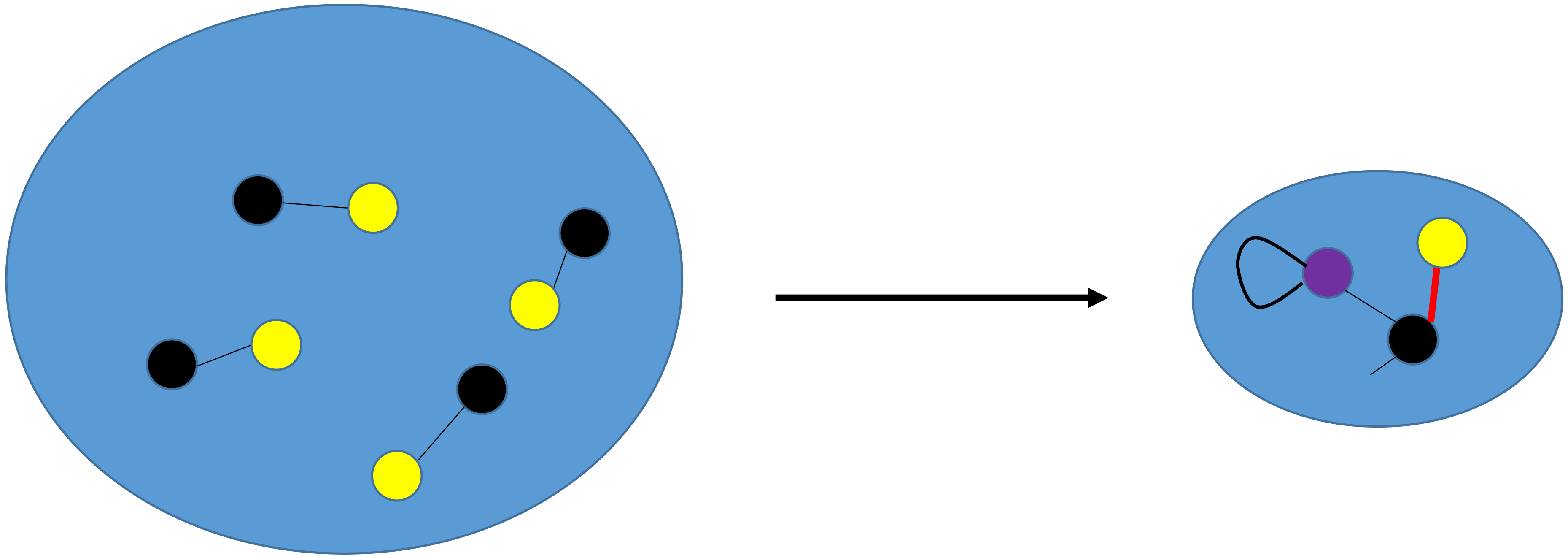

# Covers of general graphs

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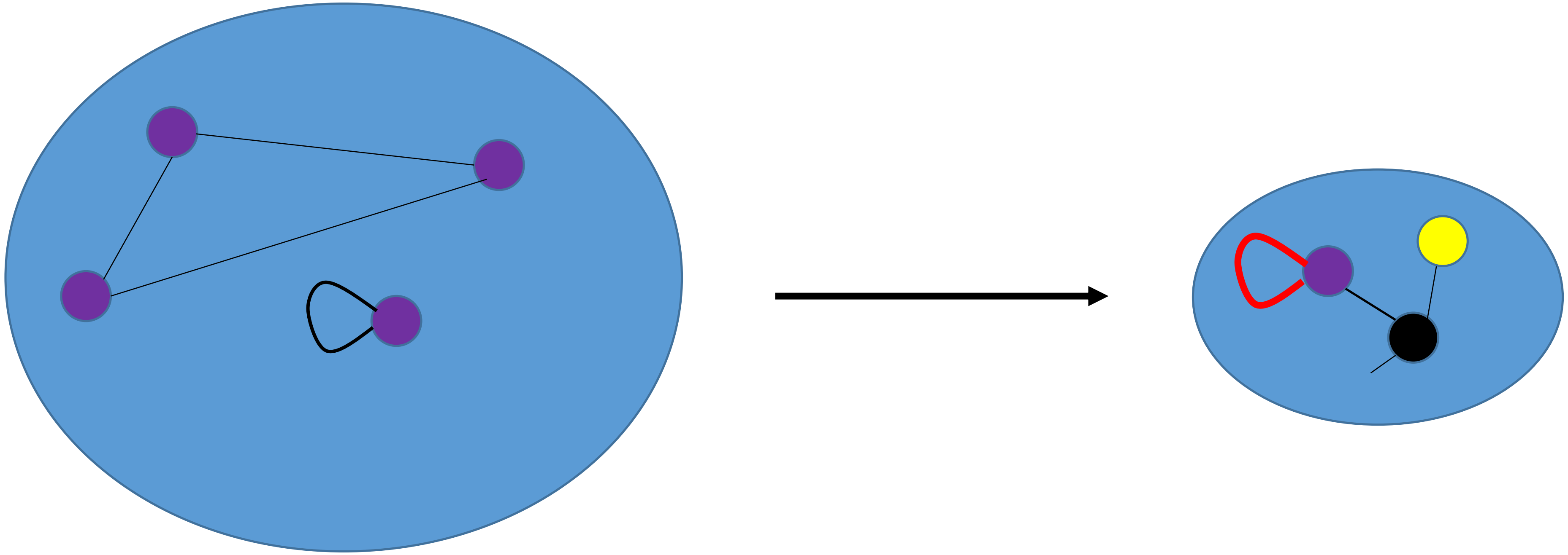

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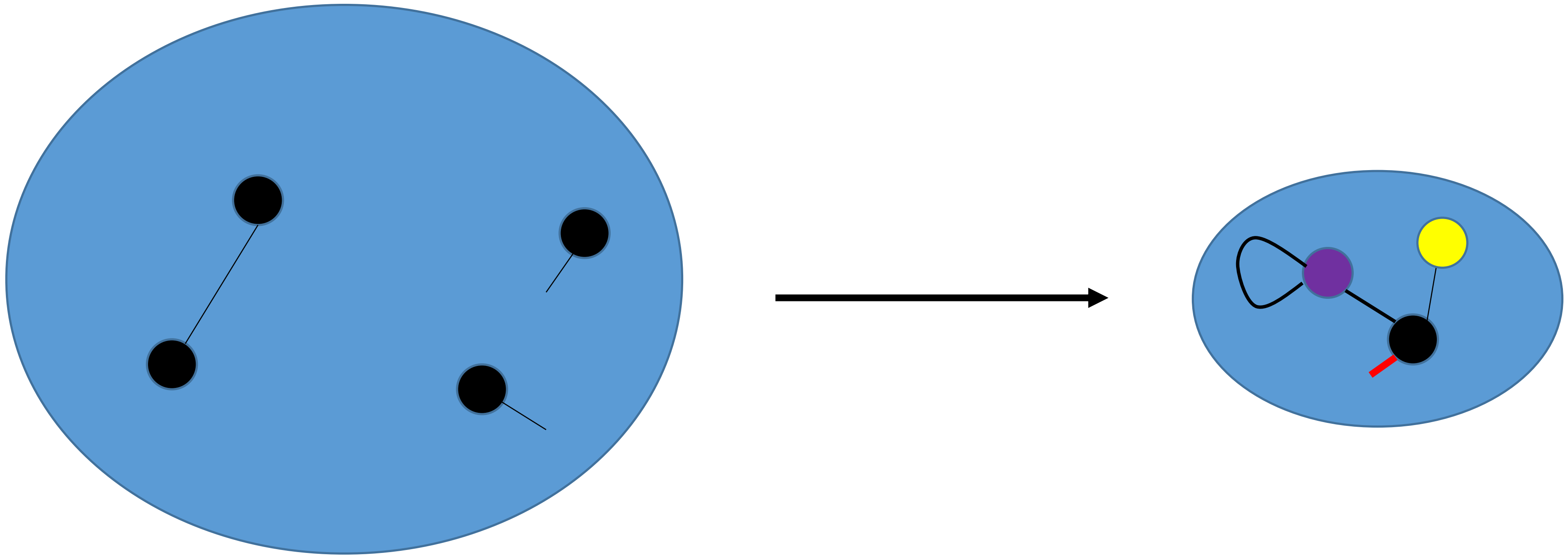

# Some examples

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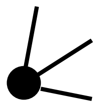

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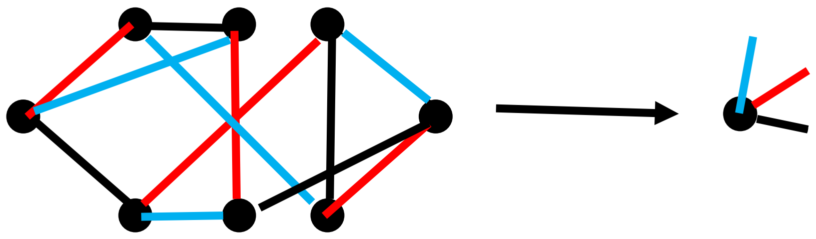
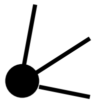
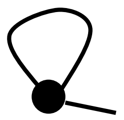


A graph covers  iff it is cubic and 3-edge-colorable.

NP-complete

# Some examples



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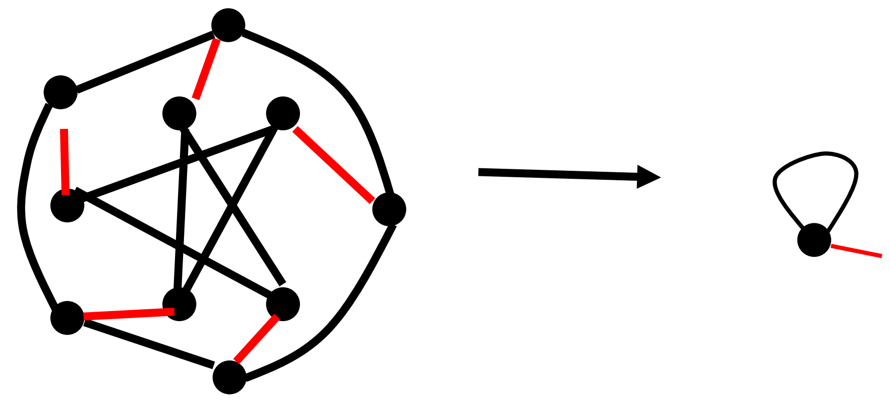

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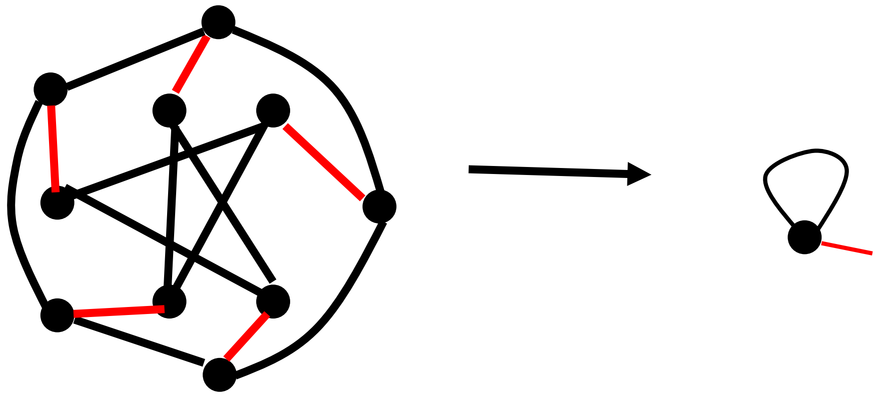
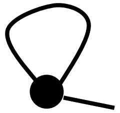
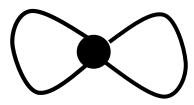


A graph covers  iff it is cubic and has a perfect matching.

Poly time

# Some examples



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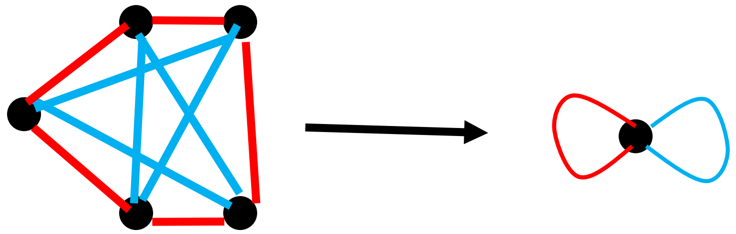
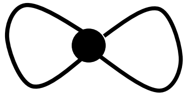
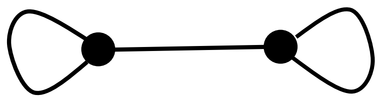


A graph covers  iff it is 4-regular (Petersen/Konig-Hall thm).

Poly time

# Some examples



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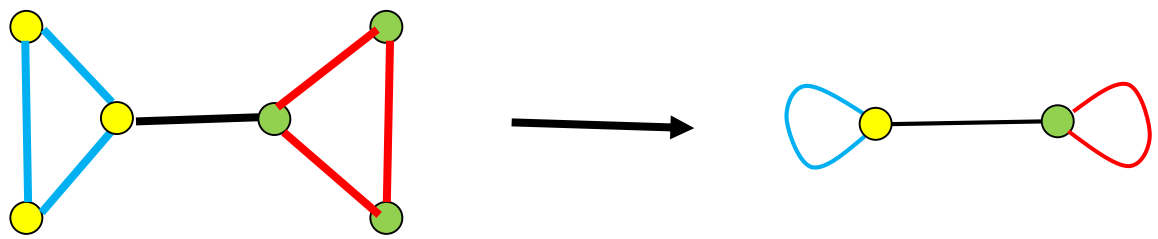

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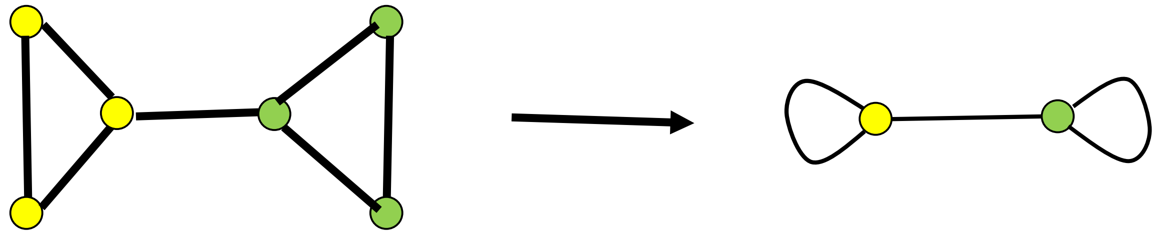
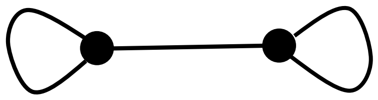


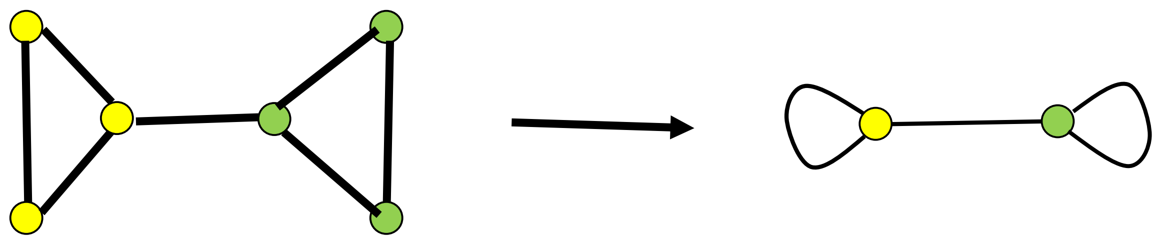
A graph covers  iff it is cubic and its vertices can be 2-colored so that every vertex has two neighbors of its own color and one neighbor of the other color.

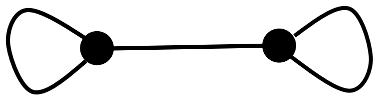
# Some examples

NP-complete 1991 Abello et al (loops on input)

2011 Bilka et al (simple graphs)

2021 Bok et al (simple bipartite graphs)



A graph covers  iff it is cubic and its vertices can be 2-colored so that every vertex has two neighbors of its own color and one neighbor of the other color.

# Strong Dichotomy Conjecture

2021 Bok et al: For every fixed graph H, the H-COVER problem is either polynomial time solvable for arbitrary input graphs (loops, multiple edges, semi-edges allowed), or NP-complete for simple input graphs.

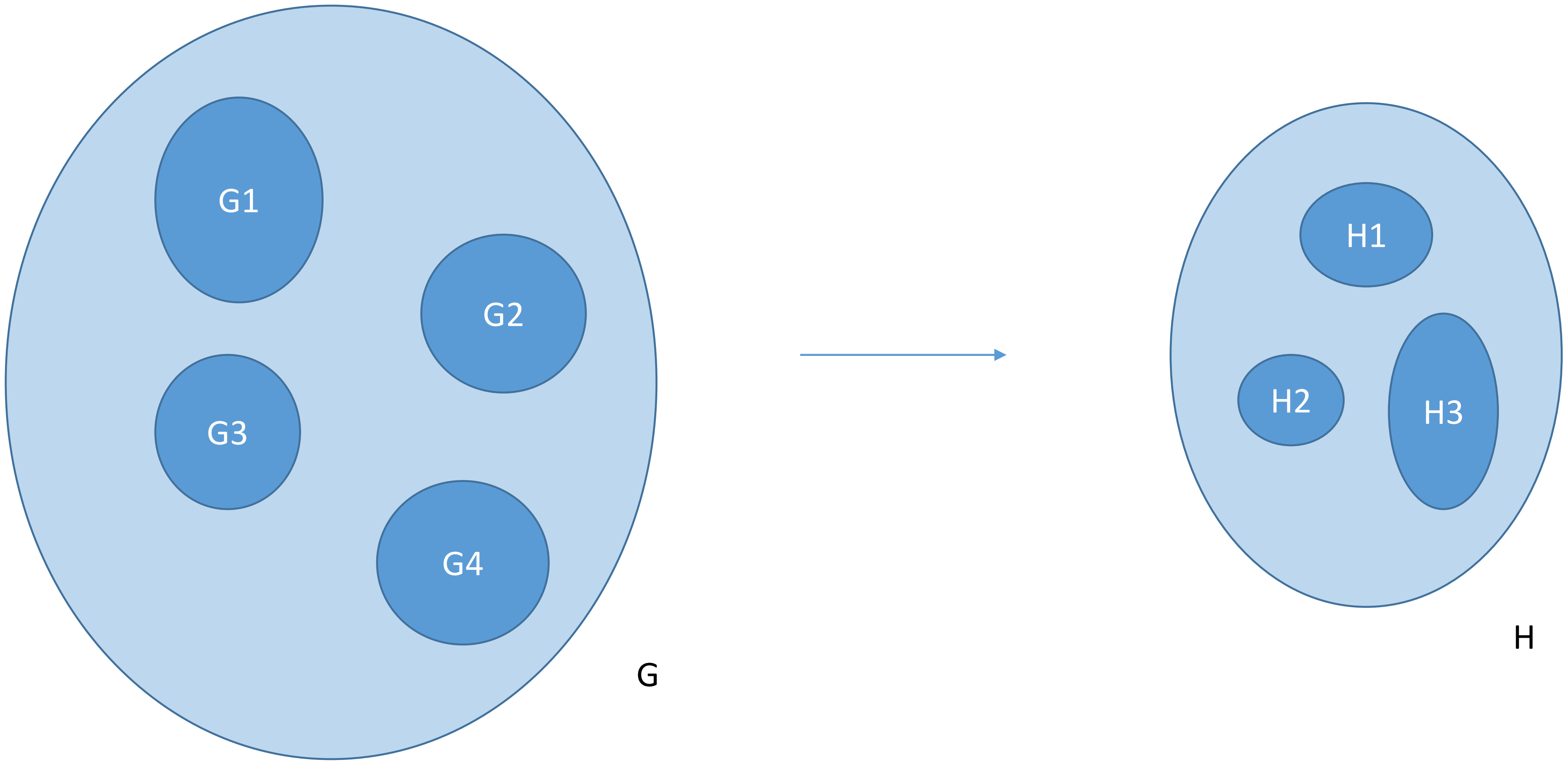
# Covers of disconnected graphs

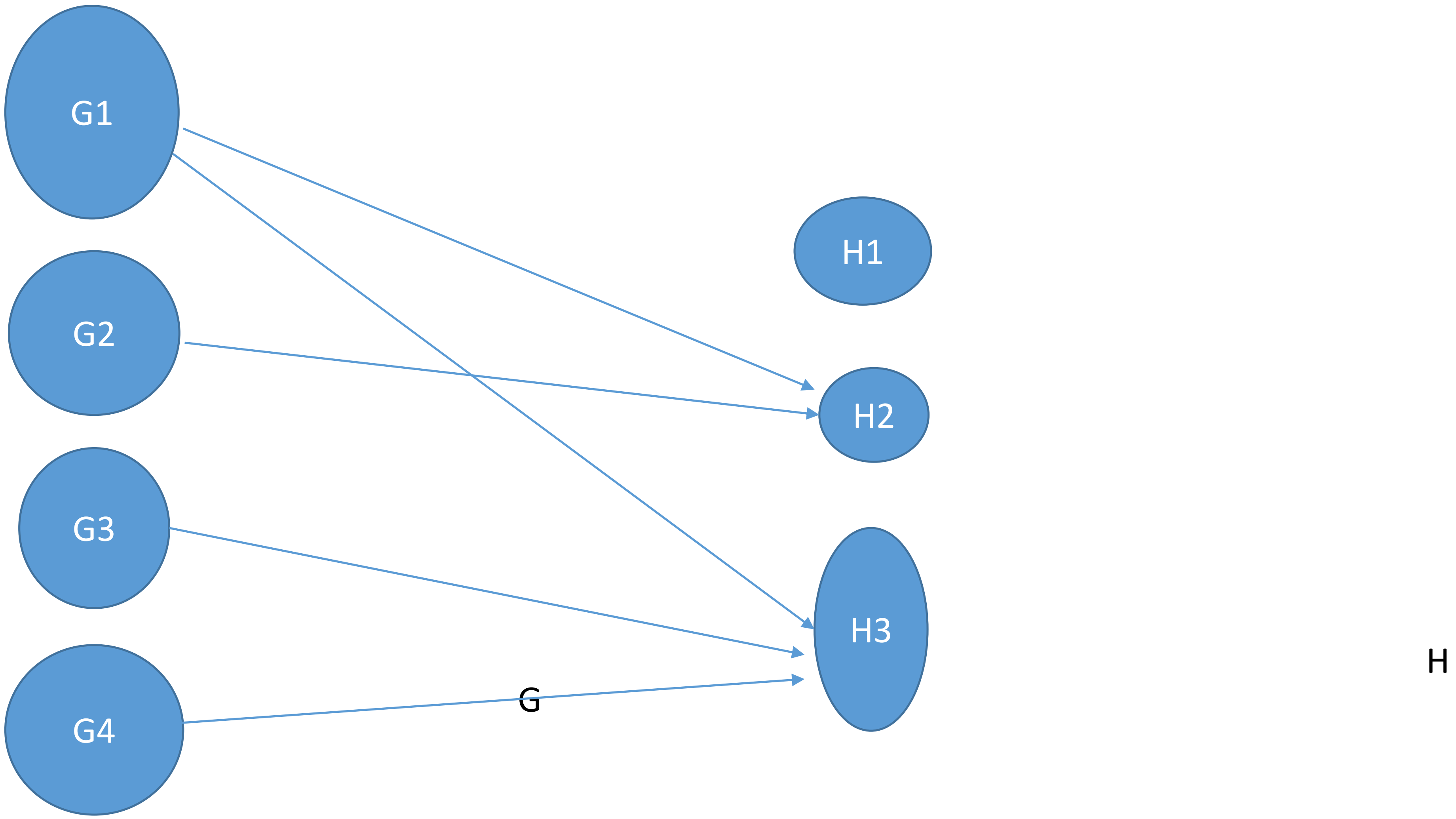
## Complexity of Graph Covering Problems

Jan Kratochvíl[1], Andrzej Proskurowski[2] and Jan Arne Telle[2]

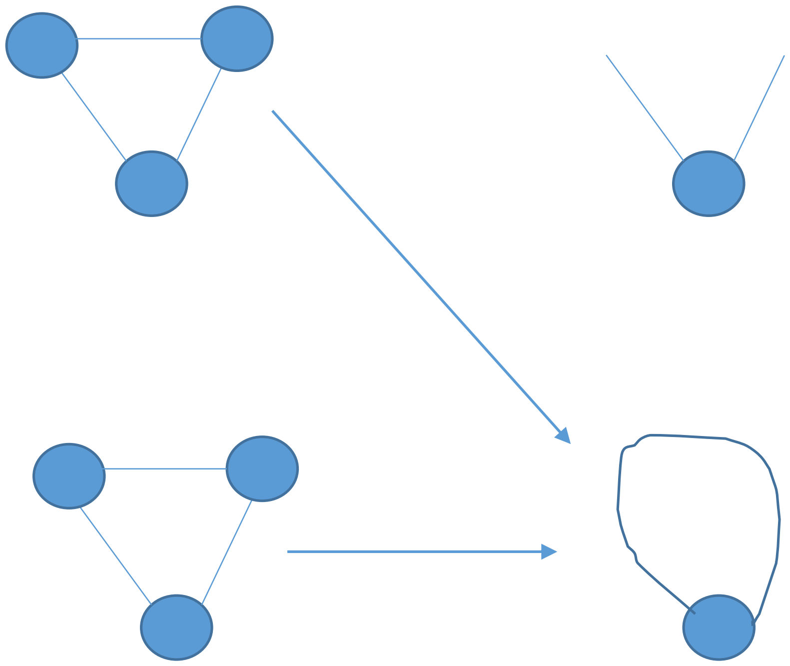
[1] Charles University, Prague, Czech Republic
[2] University of Oregon, Eugene, Oregon

**Abstract.** Given a fixed graph $H$, the $H$-cover problem asks whether an input graph $G$ allows a degree preserving mapping $f : V(G) \to V(H)$ such that for every $v \in V(G)$, $f(N_G(v)) = N_H(f(v))$. In this paper, we design efficient algorithms for certain graph covering problems according to two basic techniques. The first one is a reduction to the 2-SAT problem. The second technique exploits necessary and sufficient conditions for the existence of regular factors in graphs. For other infinite classes of graph covering problems we derive $\mathcal{NP}$-completeness results by reductions from graph coloring problems. We illustrate this methodology by classifying all graph covering problems defined by simple graphs with at most 6 vertices.


G1
G2
G3
G4
G
H1
H2
H3
H

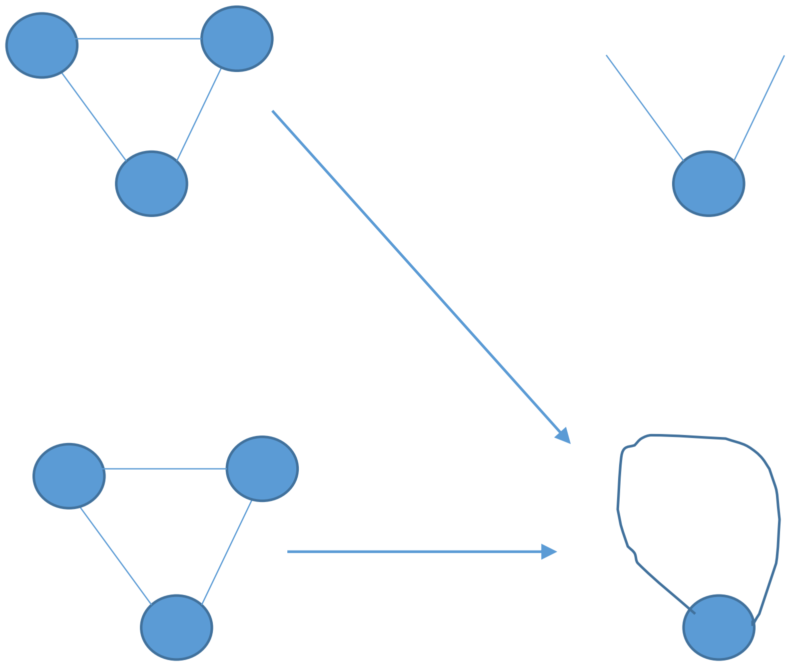

G1
G2
G3
G4
H1
H2
H3
G
H

# Locally bijective homomorphism

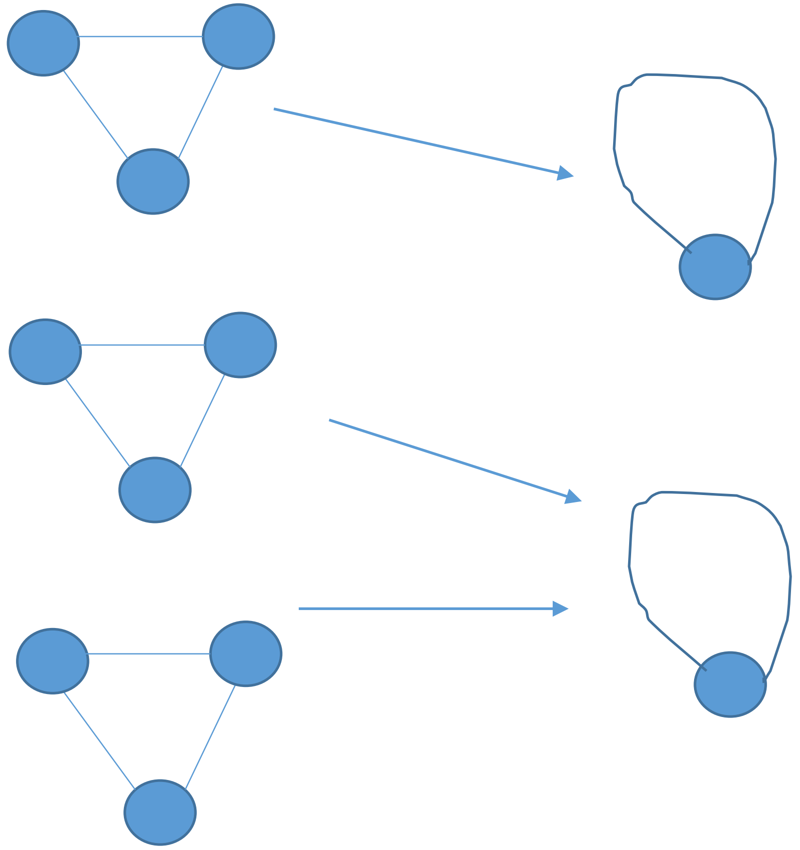


# Covers of disconnected graphs

## Locally bijective homomorphism

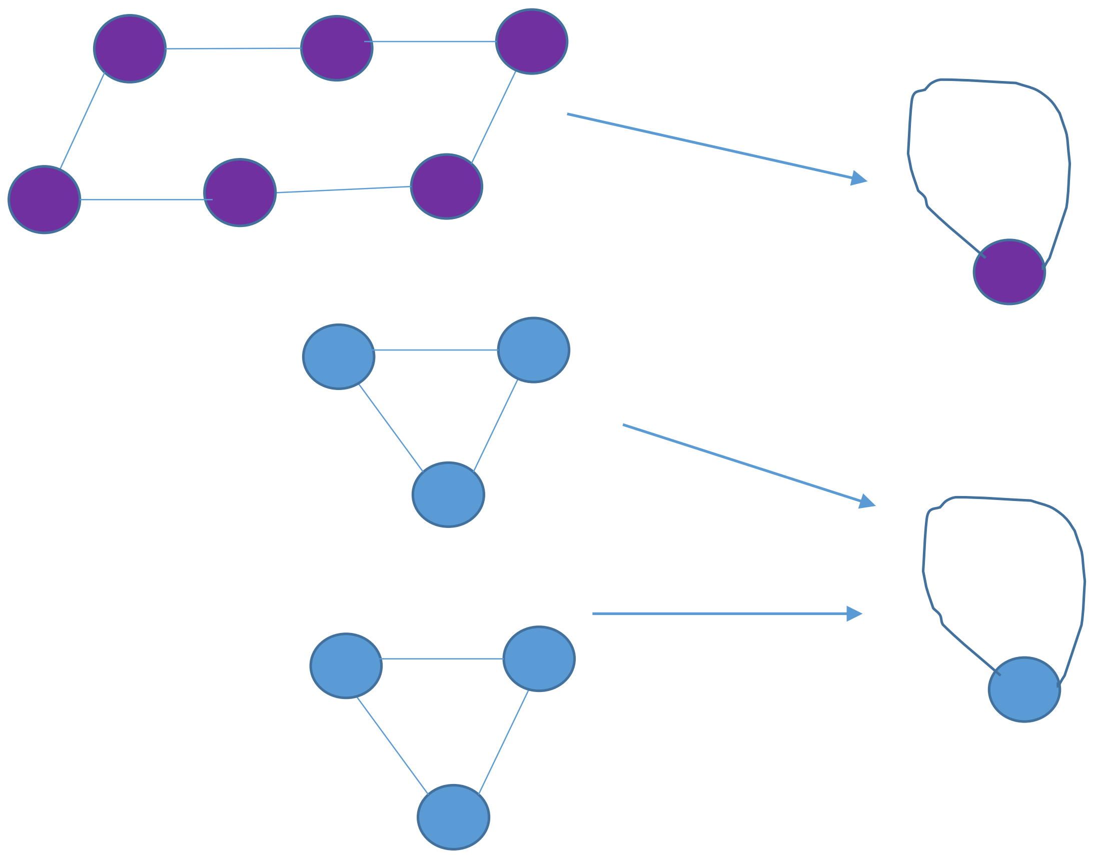


## Surjective cover



# Covers of disconnected graphs

Equitable cover



# Computational complexity of covering disconnected graphs

Thm (Bok, Fiala, Jedlickova, Kratochvil, Seifertova FCT2021):
For a disconnected graph H,
- both the H-SURJECTIVE-COVER and H-EQUITABLE-COVER problems are polynomially solvable if the $H_i$-COVER problem is polynomially solvable for every connected component $H_i$ of H, and
- both the H-SURJECTIVE-COVER and H-EQUITABLE-COVER problems are NP-complete for simple input graphs if the $H_i$-COVER problem is NP-complete for simple input graphs for some connected component $H_i$ of H.

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Proof of “the H-SURJECTIVE-COVER problem is NP-complete for simple input graphs if the $H_i$-COVER problem is NP-complete for simple input graphs for some connected component $H_i$ of H.“

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Proof of “the H-SURJECTIVE-COVER problem is NP-complete for simple input graphs if the $H_i$-COVER problem is NP-complete for simple input graphs for some connected component $H_i$ of H.“

Let H=H1+H2+…+Hk. Suppose that H1-COVER is NP-complete for simple input graphs, and let G1 be a simple graph whose covering of H1 is to be tested. For each j=2,3,…,k, fix a simple graph Gj such that Gj covers Hj, and moreover Gj does not cover H1, unless Hj is such that every simple graph that covers Hj also covers H1.

Then G=G1+G2+…+Gk surjectively covers H if and only if G1 covers H1.

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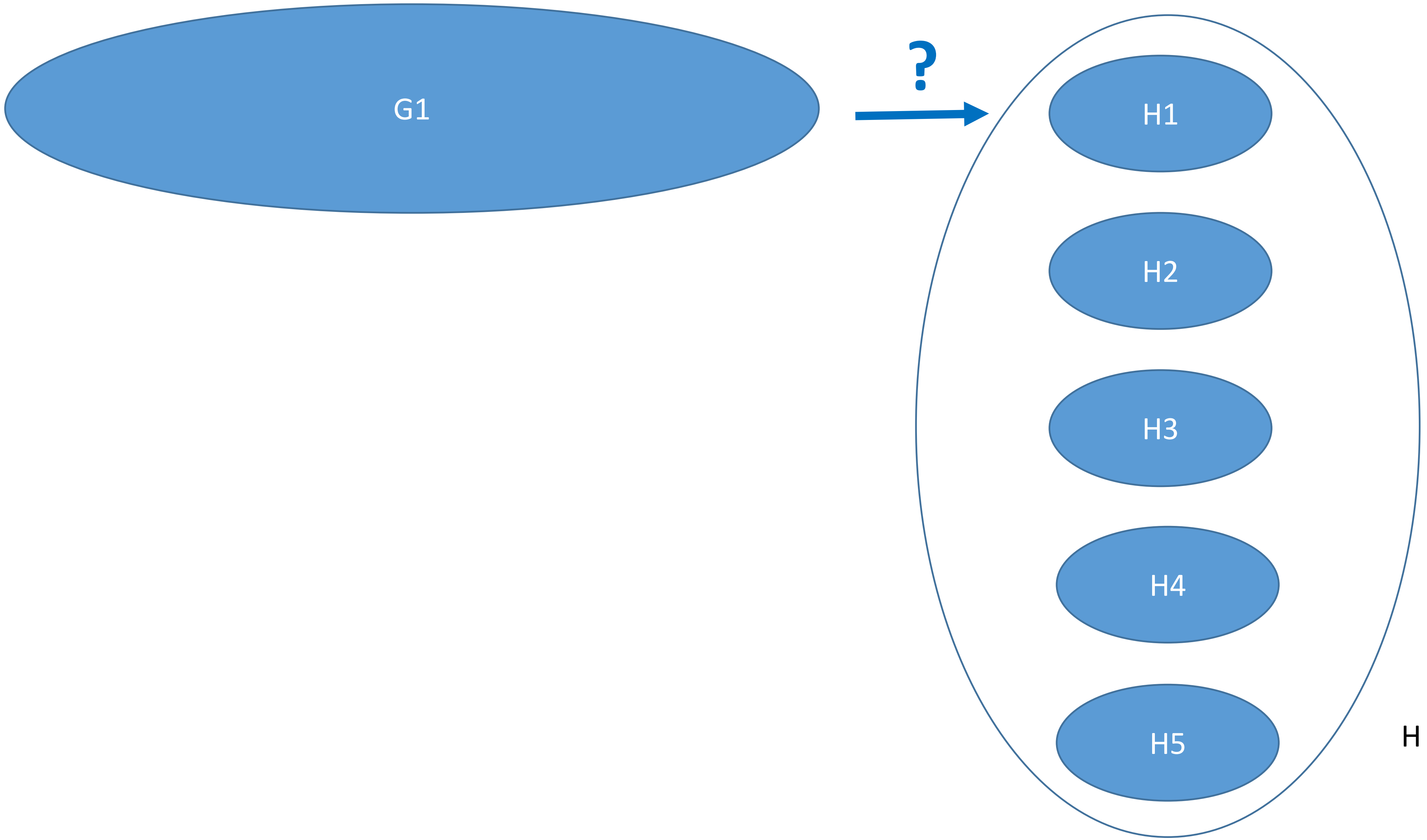

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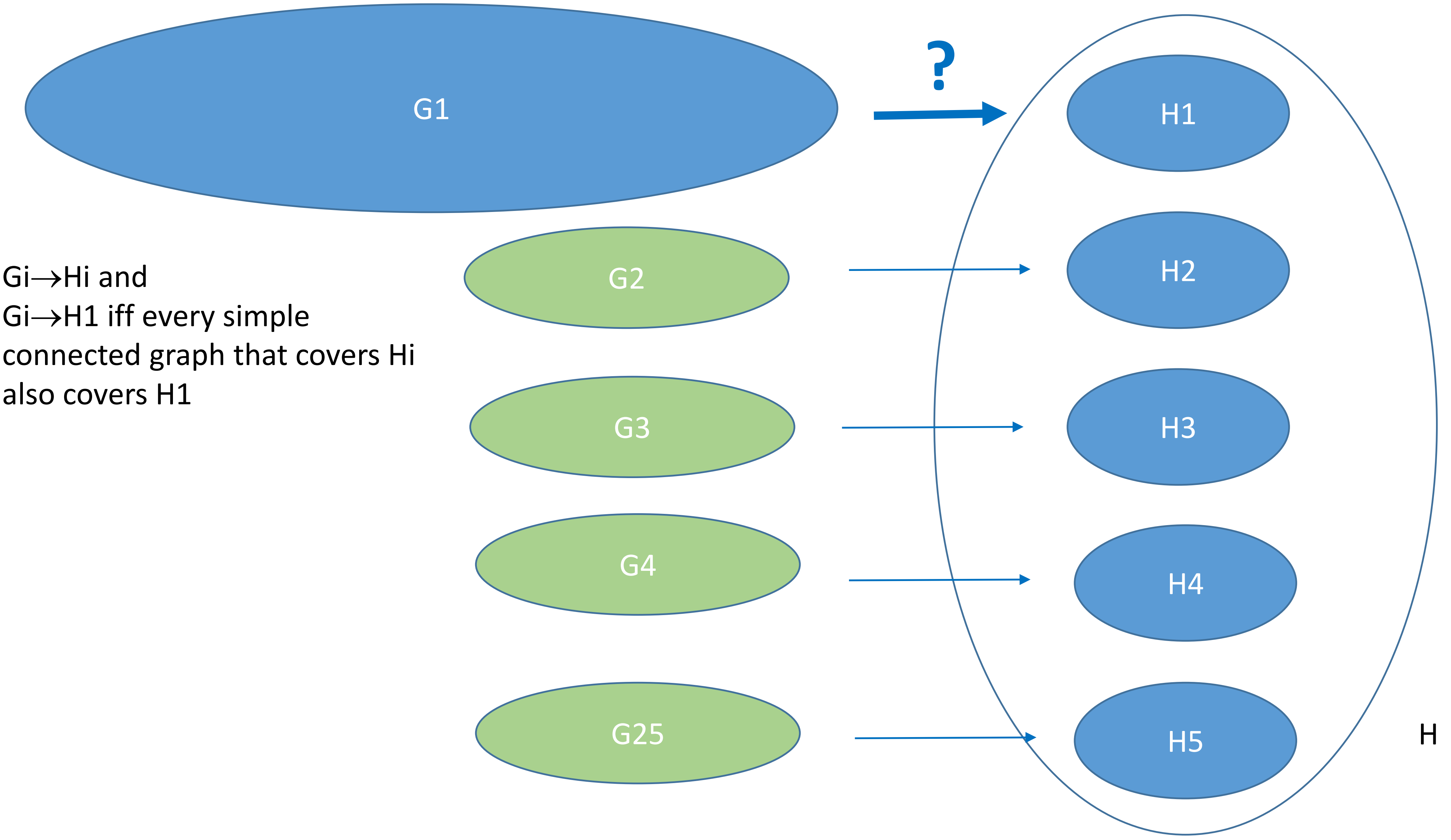


Gi→Hi and
Gi→H1 iff every simple connected graph that covers Hi also covers H1

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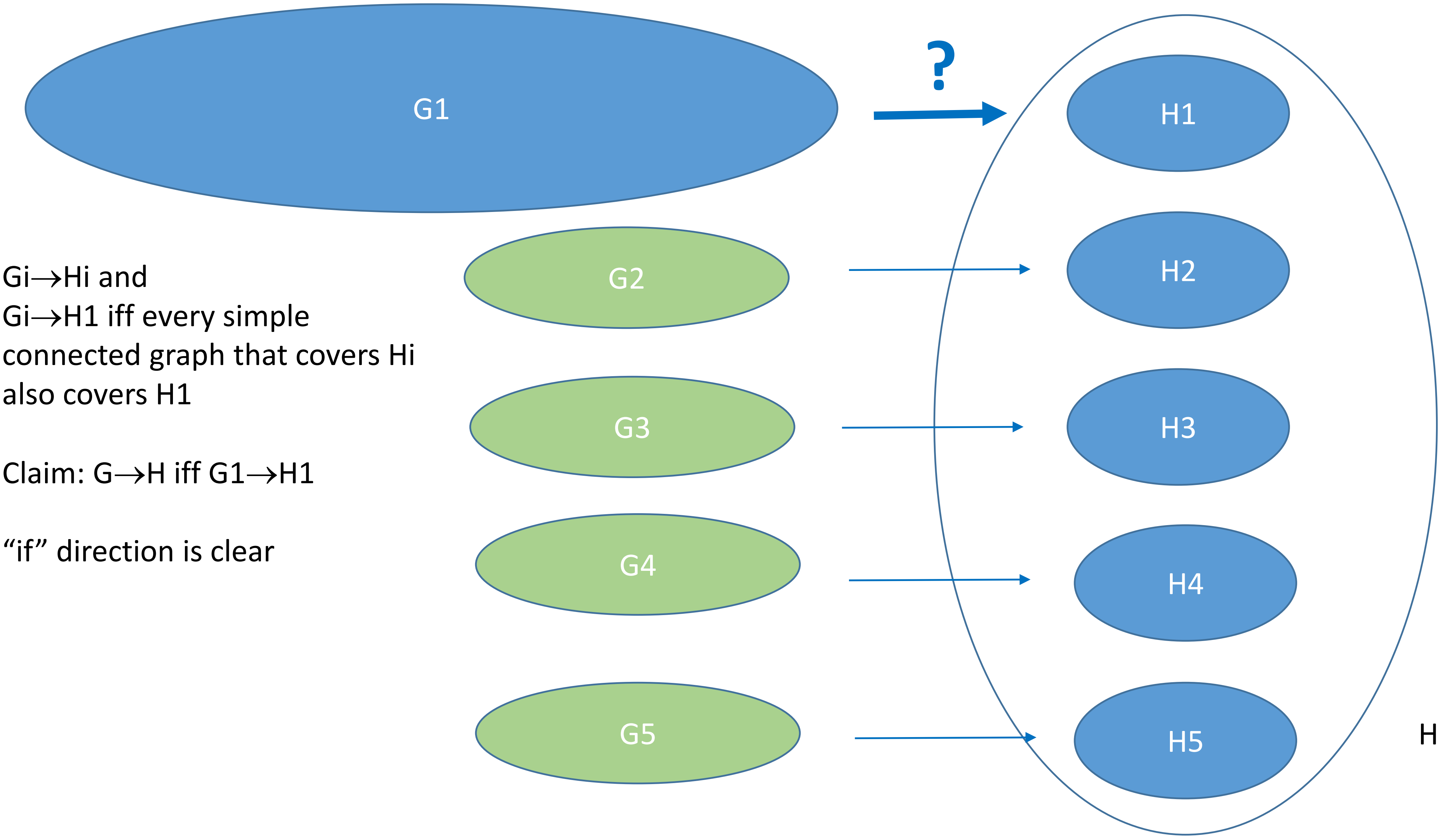


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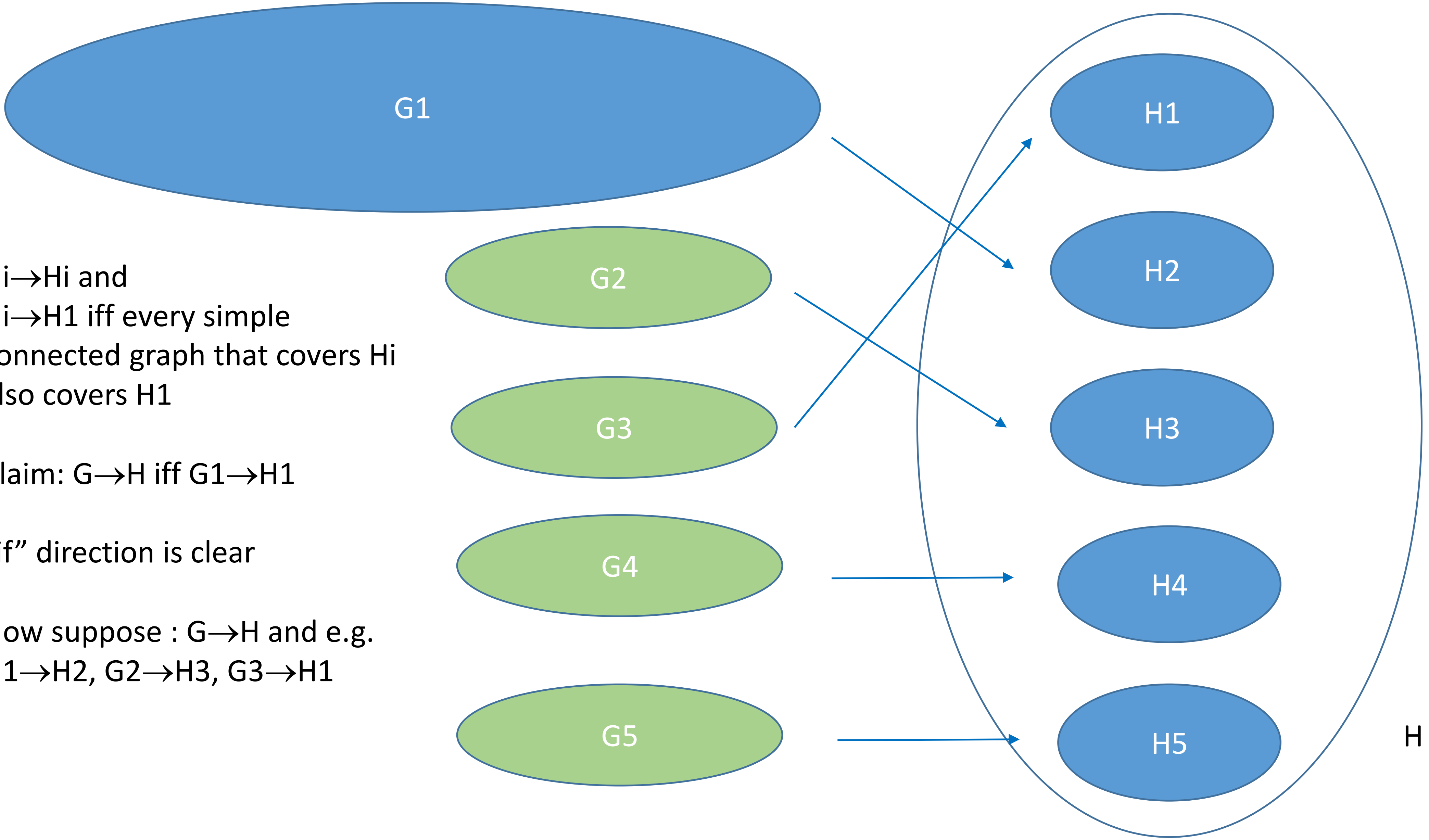


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G1→H2, G2→H3, G3→H1

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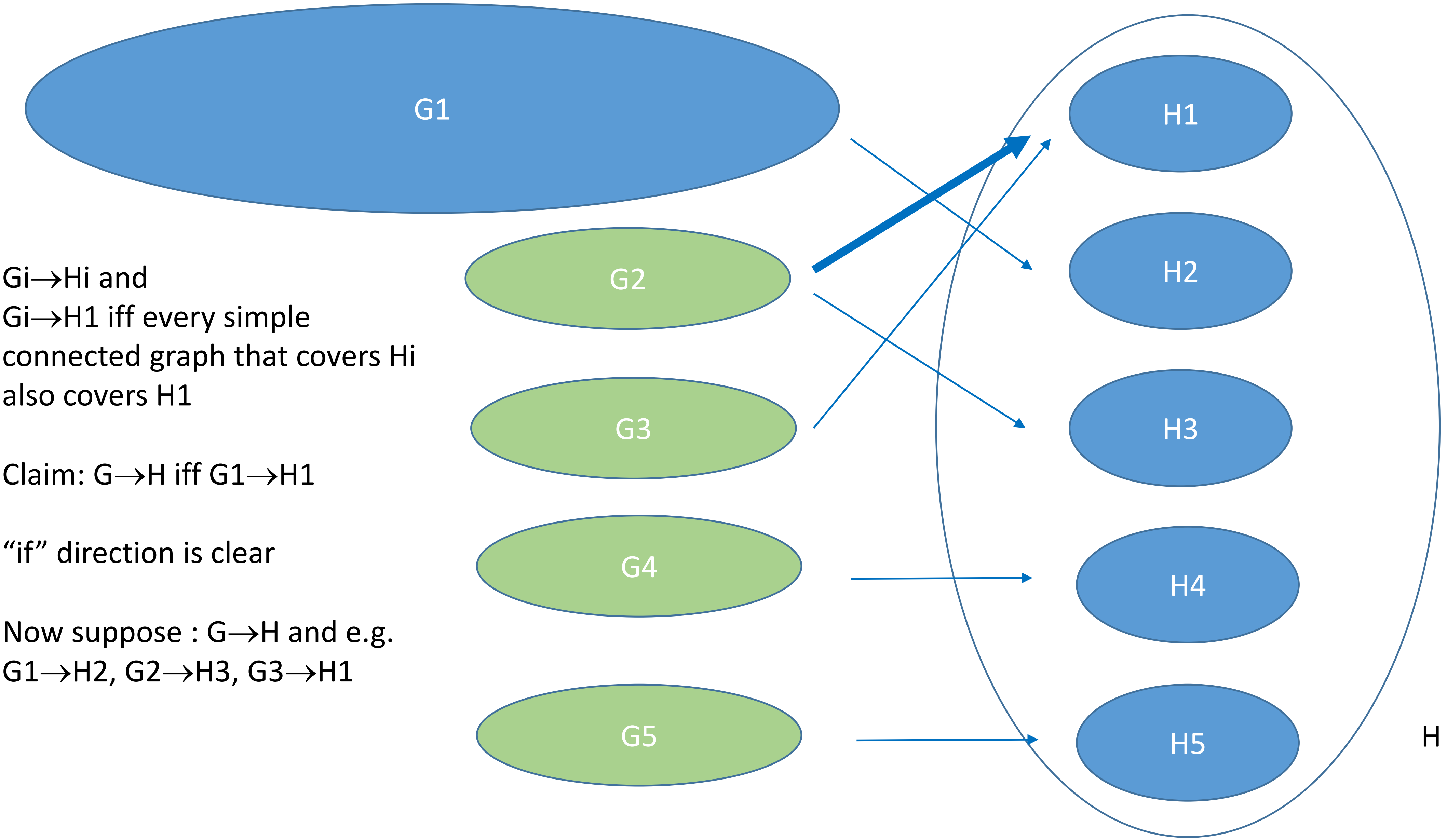


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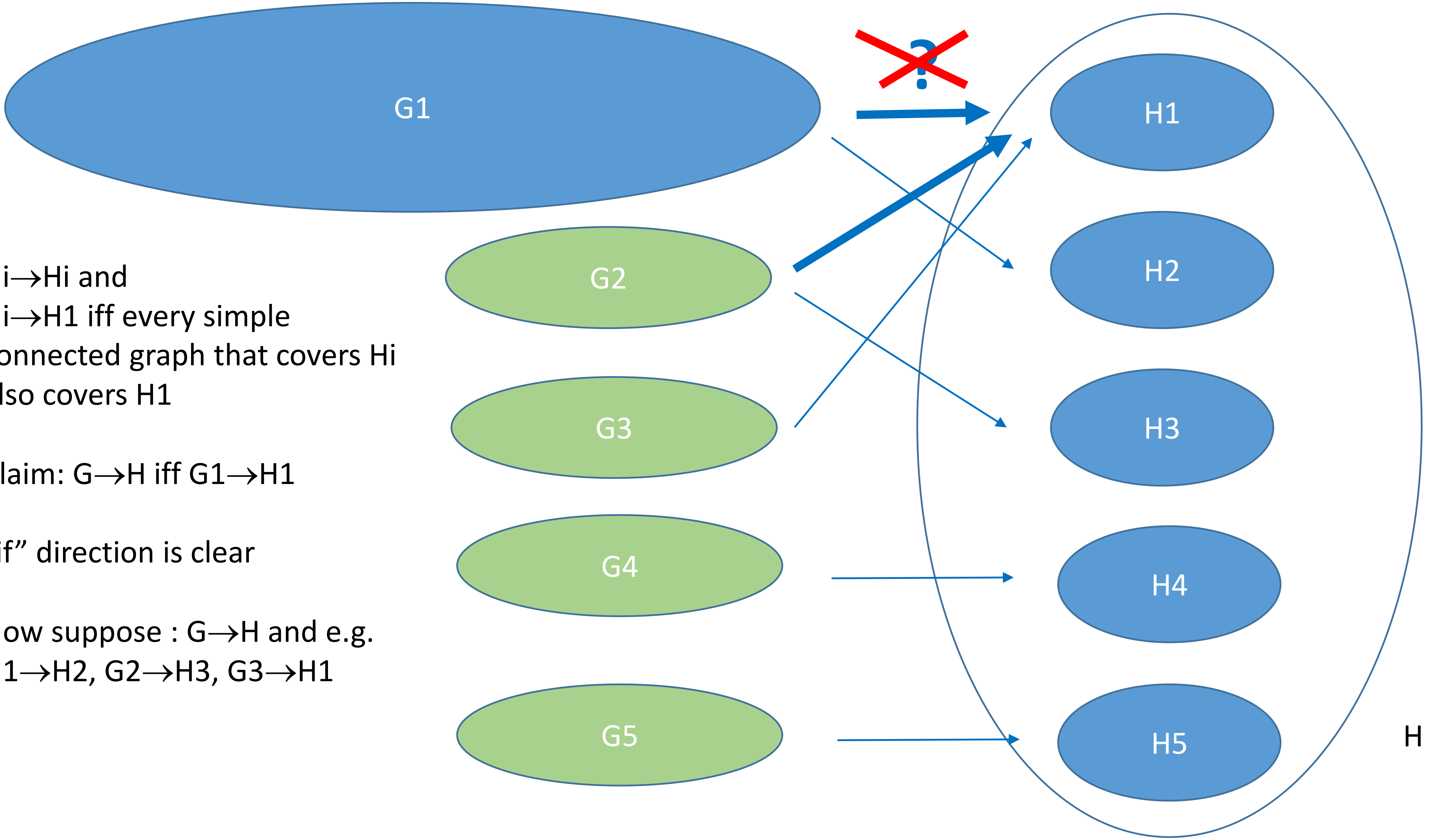


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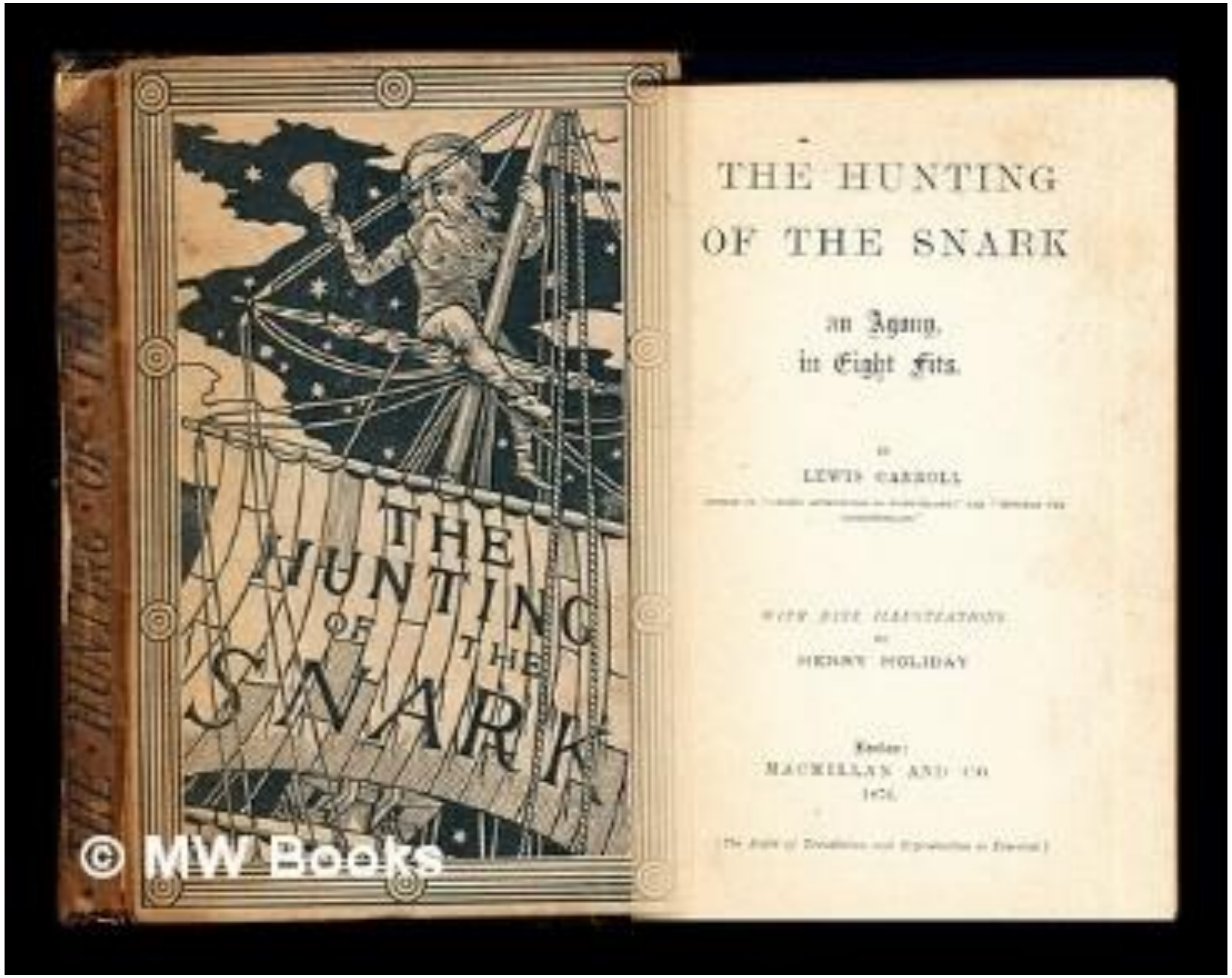
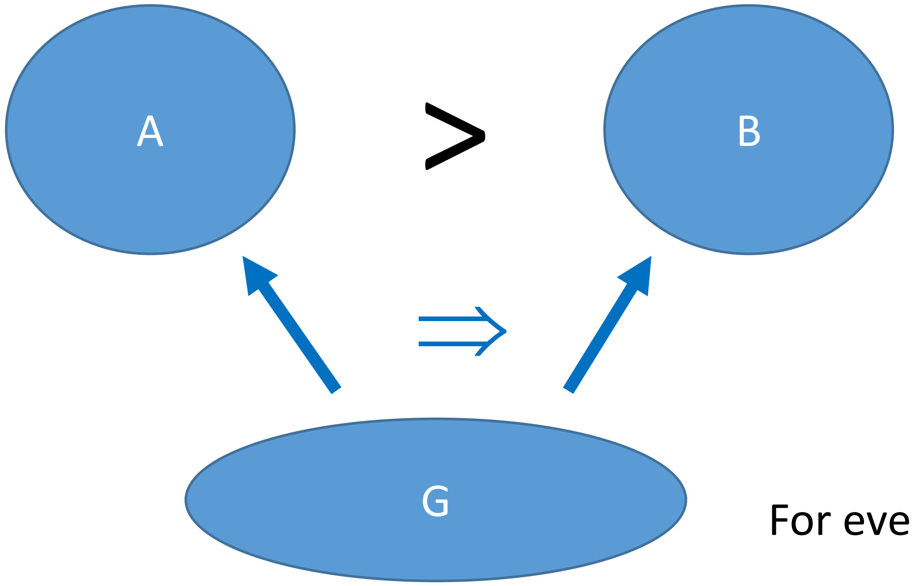
# Hunting for Snarks



# > relation on connected graphs

Definition: Given connected graphs A and B, we say that A > B if for every simple graph G, it is true that G covers B whenever G covers A.




For every simple connected G

# $>$ relation on connected graphs

Definition: Given connected graphs A and B, we say that $A > B$ if for every simple graph G, it is true that G covers B whenever G covers A.

Example 1: If $A \to B$, then $A > B$.

# > relation on connected graphs

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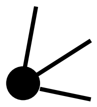
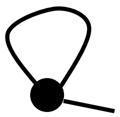
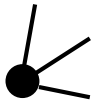
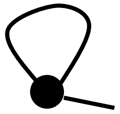
Example 1: If A $\rightarrow$ B, then A > B.

Example 2:  > 

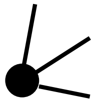
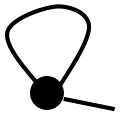
# > relation on connected graphs

Open problem: Describe all pairs of connected graphs A and B such that A > B and A does not cover B.

Conjecture (Bok et al. 2022): If A has no semi-edges, then
A > B if and only if A covers B.

JK, Nedela (2023+): True for B =  and B =  and arbitrary A.

Final comment: If $\neg$(A>B), then there is a witness G (a simple graph) such that G covers A but G does not cover B. How big would such a witness be? Can such a witness be constructed easily?

We know that $\neg$ (  >  ). 2-connected witneses are **snarks**.

謝謝