The quest for optimality in geometric intersection graphs

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Episode 1: exact algorithms

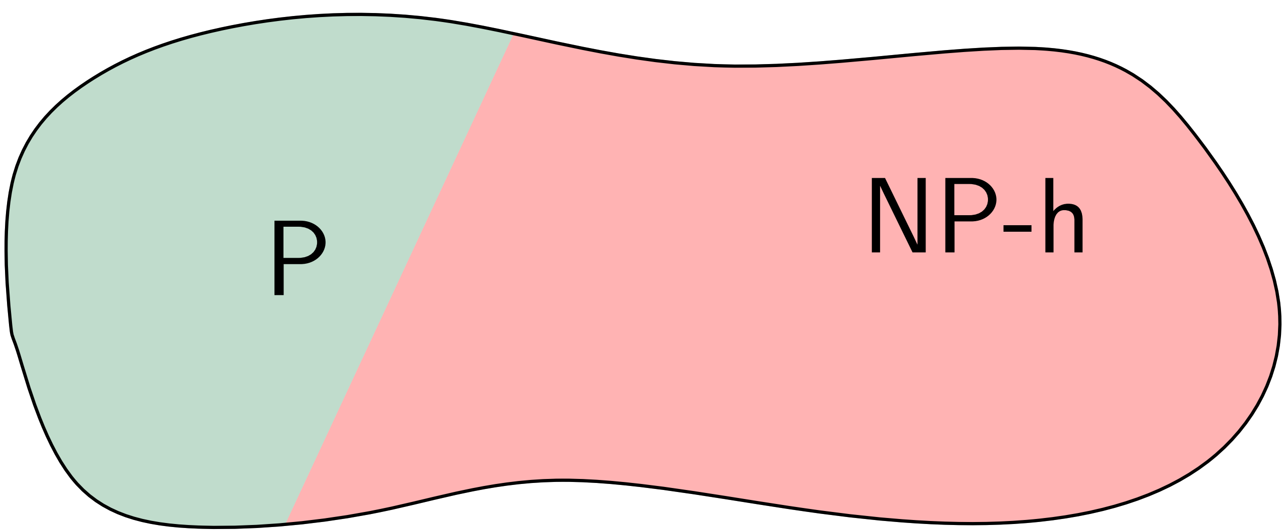
Fine-grained complexity and the Exponential-Time Hypothesis

Classical approach to complexity theory

Assuming P $\neq$ NP, we partition problems into two sets:

- P (solvable in polynomial time)
  proven by presenting an algorithm

- NP-hard (no polynomial algorithm)
  proven by polynomial reductions



Classical approach to complexity theory

Assuming P $\neq$ NP, we partition problems into two sets:

- P (solvable in polynomial time)
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  worth attention,
  how fast can be solve them?

- NP-hard (no polynomial algorithm)
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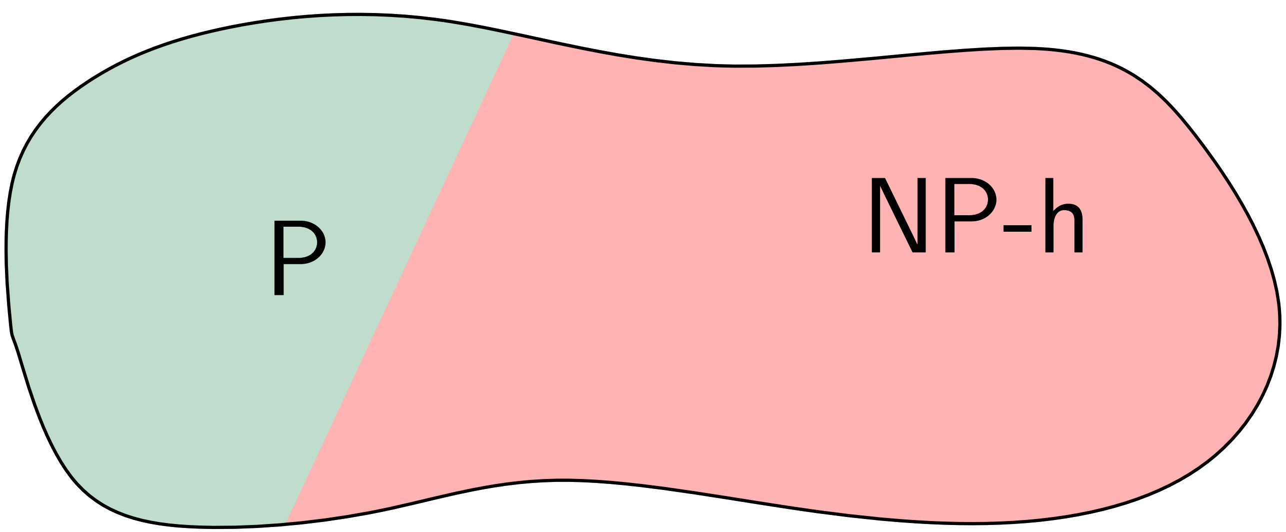
  hopeless, unsolvable



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Exponential Time Hypothesis (ETH) [Impagliazzo, Paturi, 1999]
There is no algorithm solving $3\text{-}\textsc{Sat}$ with n variables and $\mathcal{O}(n)$ clauses in time $2^{o(n)}$.

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subexponential time: $2^{o(n)}$,
e.g. $2^{\mathcal{O}(n^{0.99})}$ or $2^{\mathcal{O}(n/\log n)}$

A closer look

Being a stronger assumption than P $\neq$ NP, ETH allows for a finer analysis:

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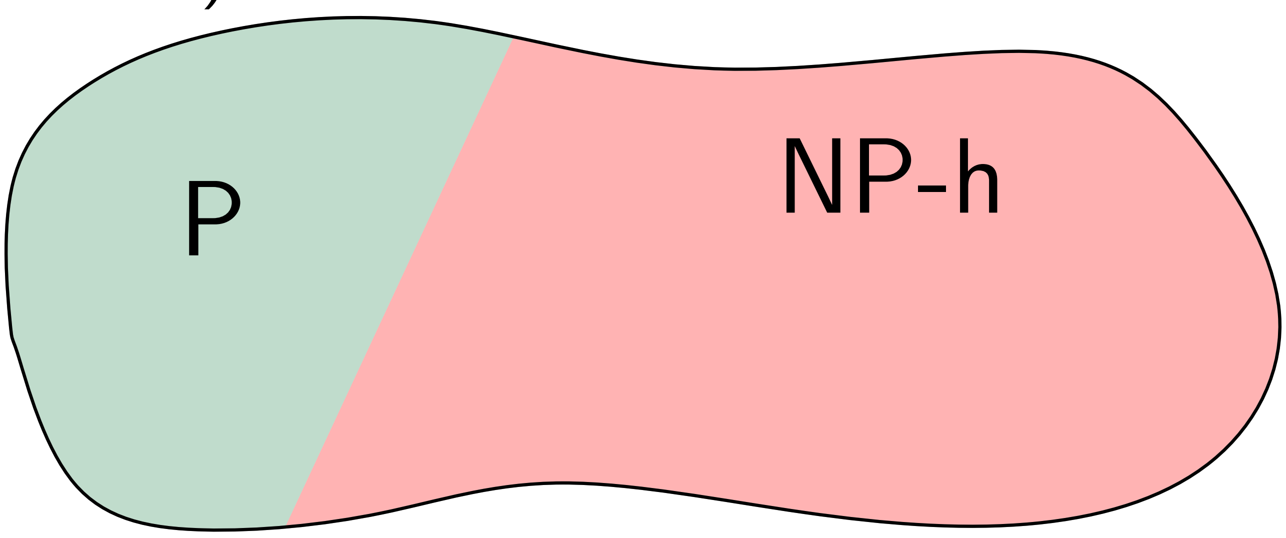

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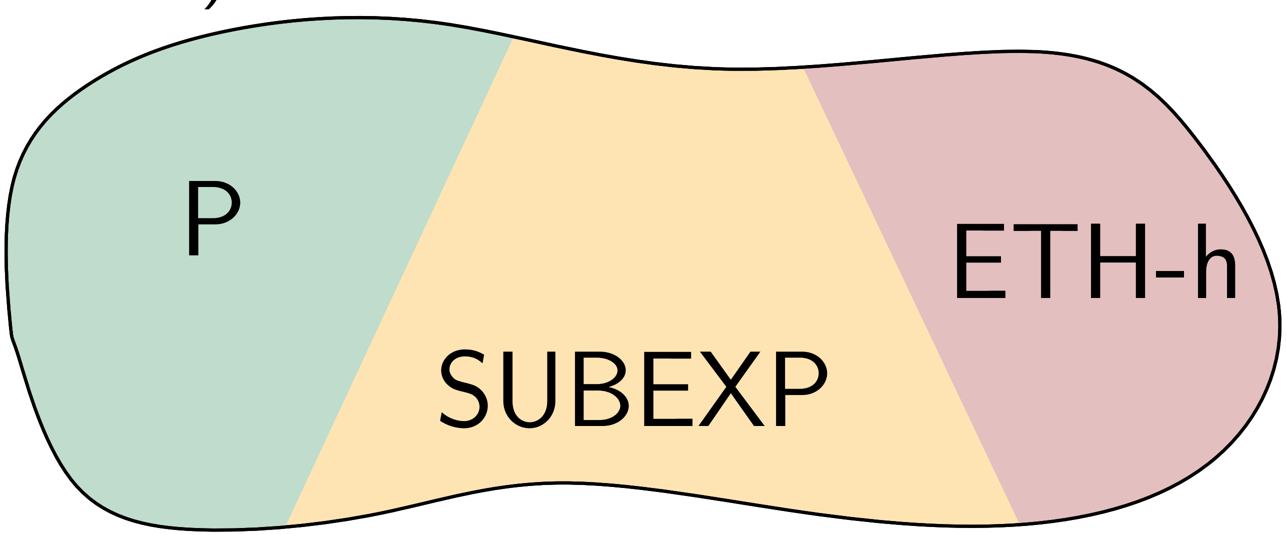
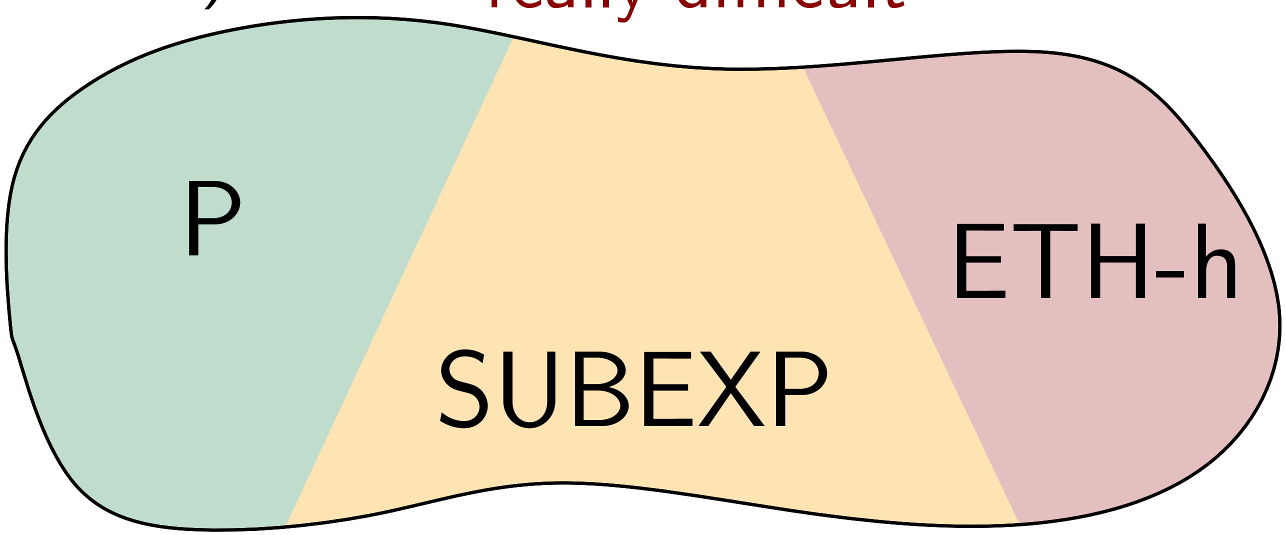


A closer look

Being a stronger assumption than P $\neq$ NP, ETH allows for a finer analysis:

- P (solvable in polynomial time)
  easy
- NP-hard (no polynomial algorithm)
  - SUBEXP (solvable in subexponential time)
    not so super-hard
  - ETH-hard (no subexponential algorithm)
    really difficult



Lower bounds

- hardness is proven via reductions
- start from 3-SAT with n variables and $m = \mathcal{O}(n)$ clauses
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$\alpha = 1$ (linear reduction) $\rightarrow$ no $2^{o(n)}$ algorithm

$\alpha = 2$ (quadratic reduction) $\rightarrow$ no $2^{o(\sqrt{n})}$ algorithm

What can we hope for?

- bad news: assuming the ETH, there are no subexponential algorithms for canonical graph problems
  3-Coloring, Independent Set, Clique, Dominating Set, Vertex Cover, Hamiltonian Cycle, Max Cut etc.

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- Square-root phenomenon: for planar graphs, most canonical problems can be solved in time $2^{\mathcal{O}(\sqrt{n})}$

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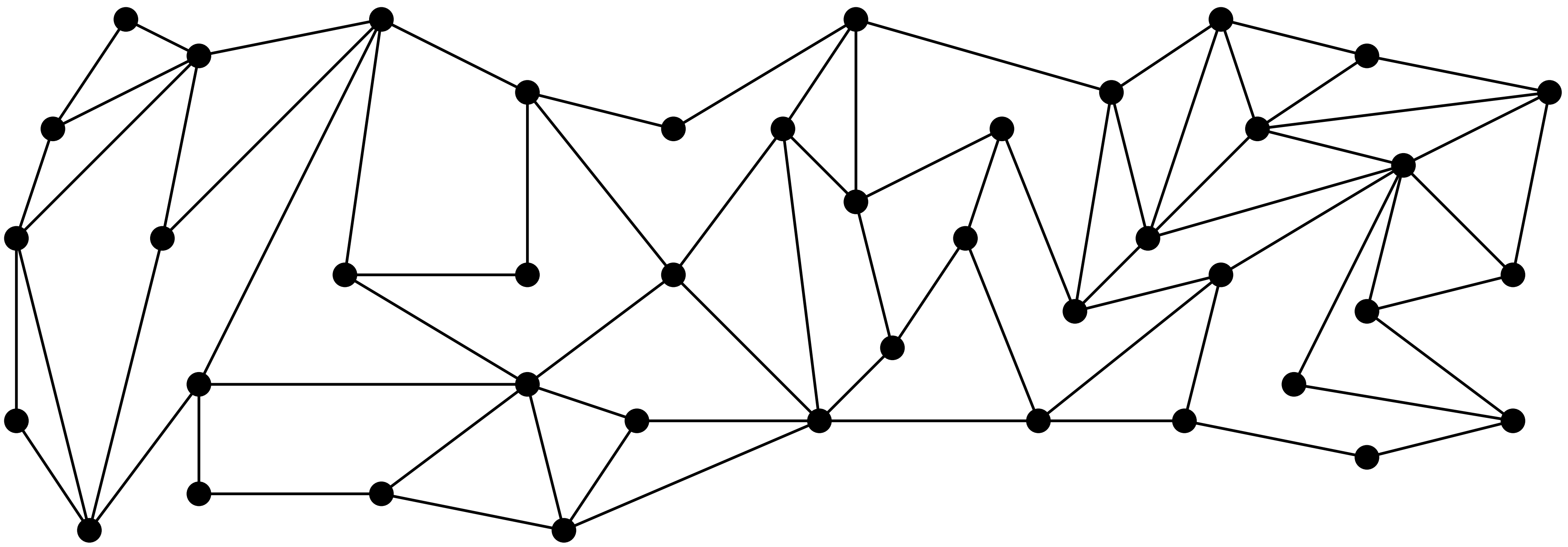
Still boring!

Subexponential algorithms for planar graphs

Planar separator theorem [Lipton, Tarjan, 1979].

Every planar graph has a balanced separator of size $\mathcal{O}(\sqrt{n})$.

- also specialized versions, e.g. the separator is a cycle



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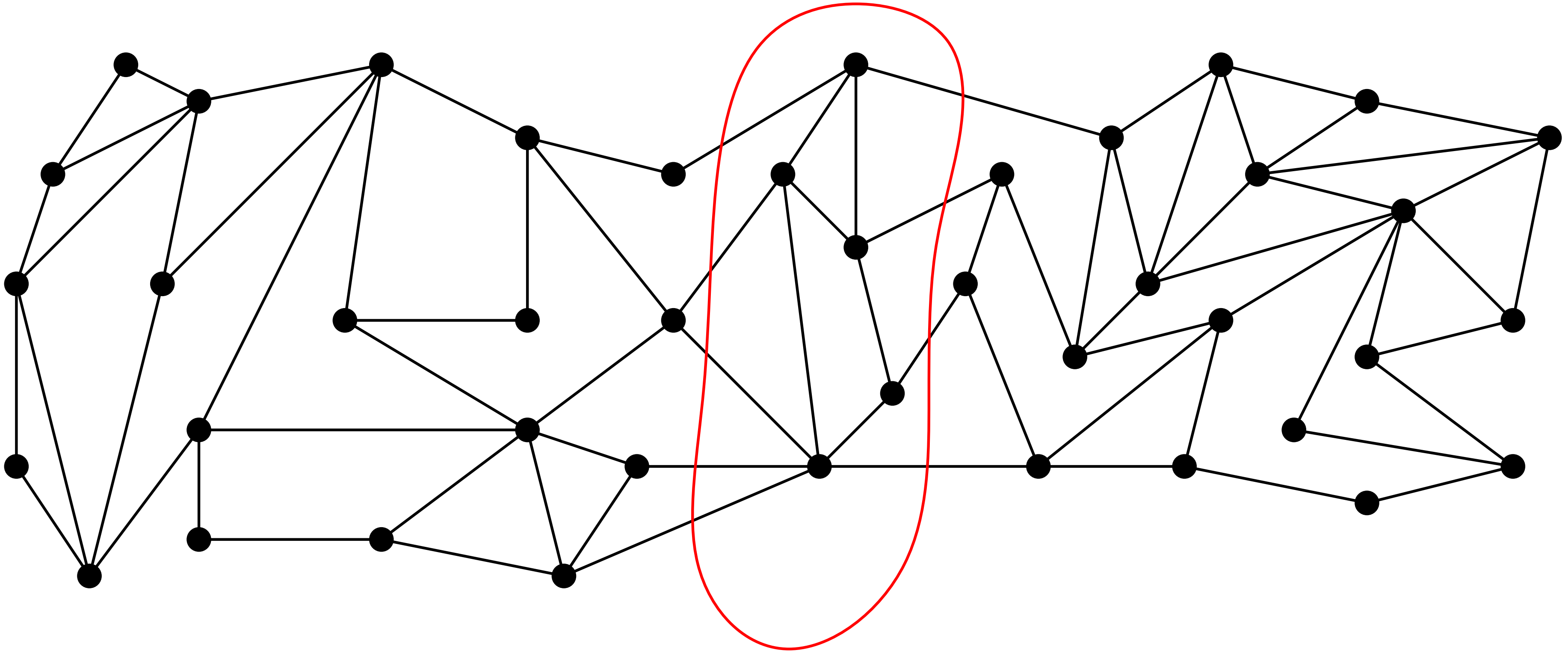

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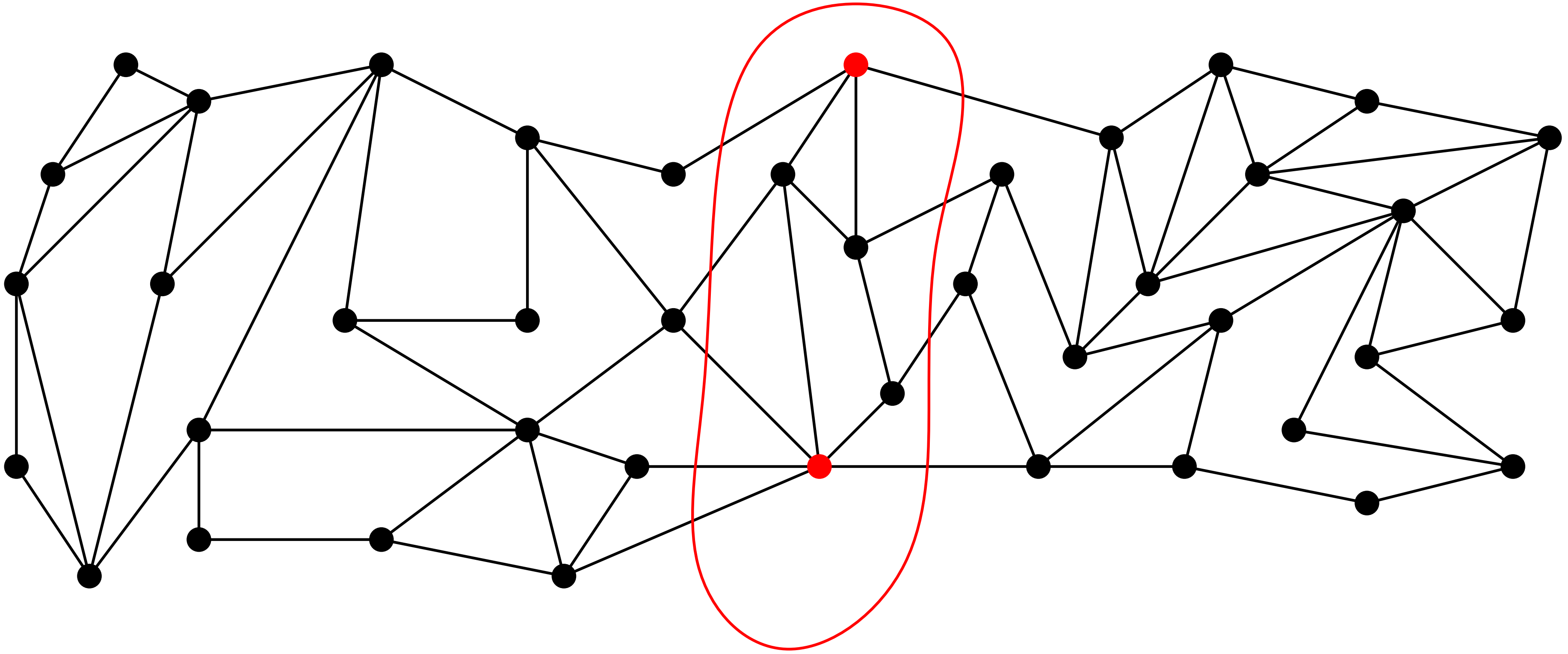

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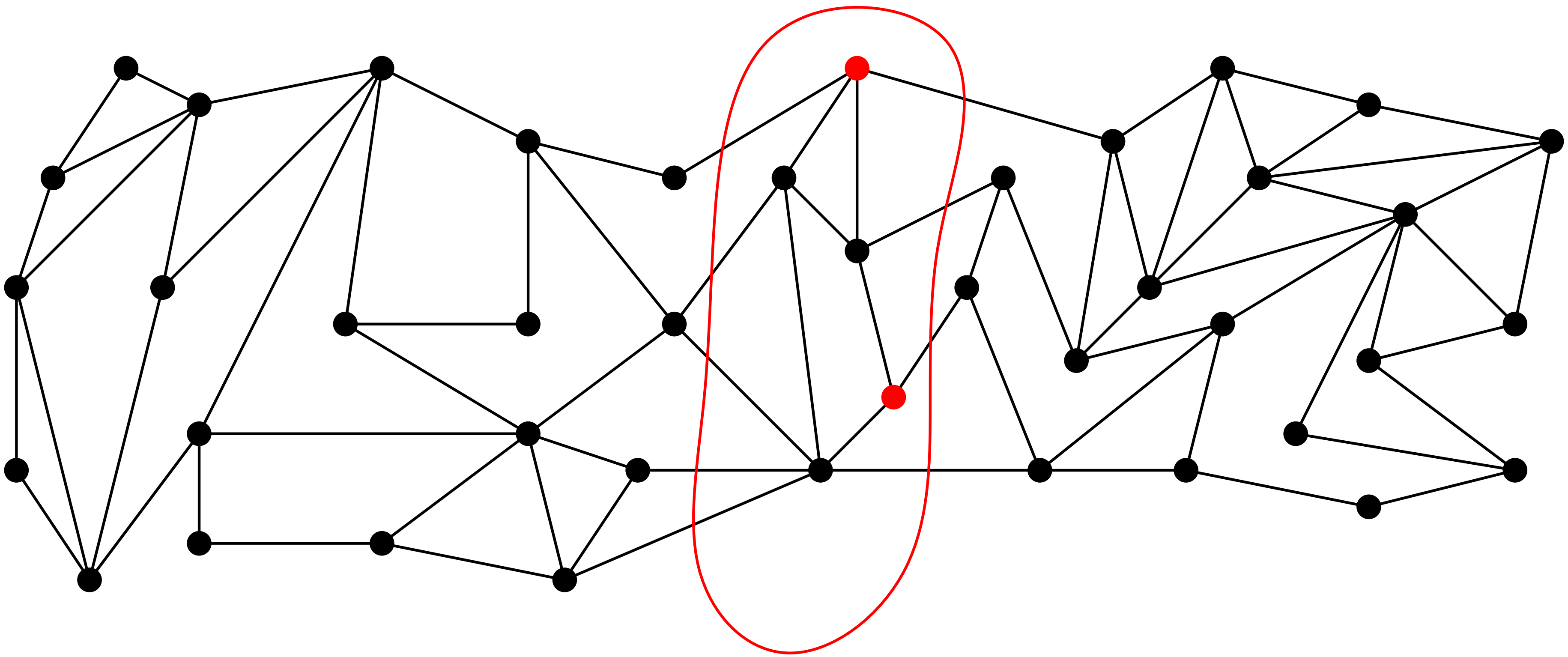

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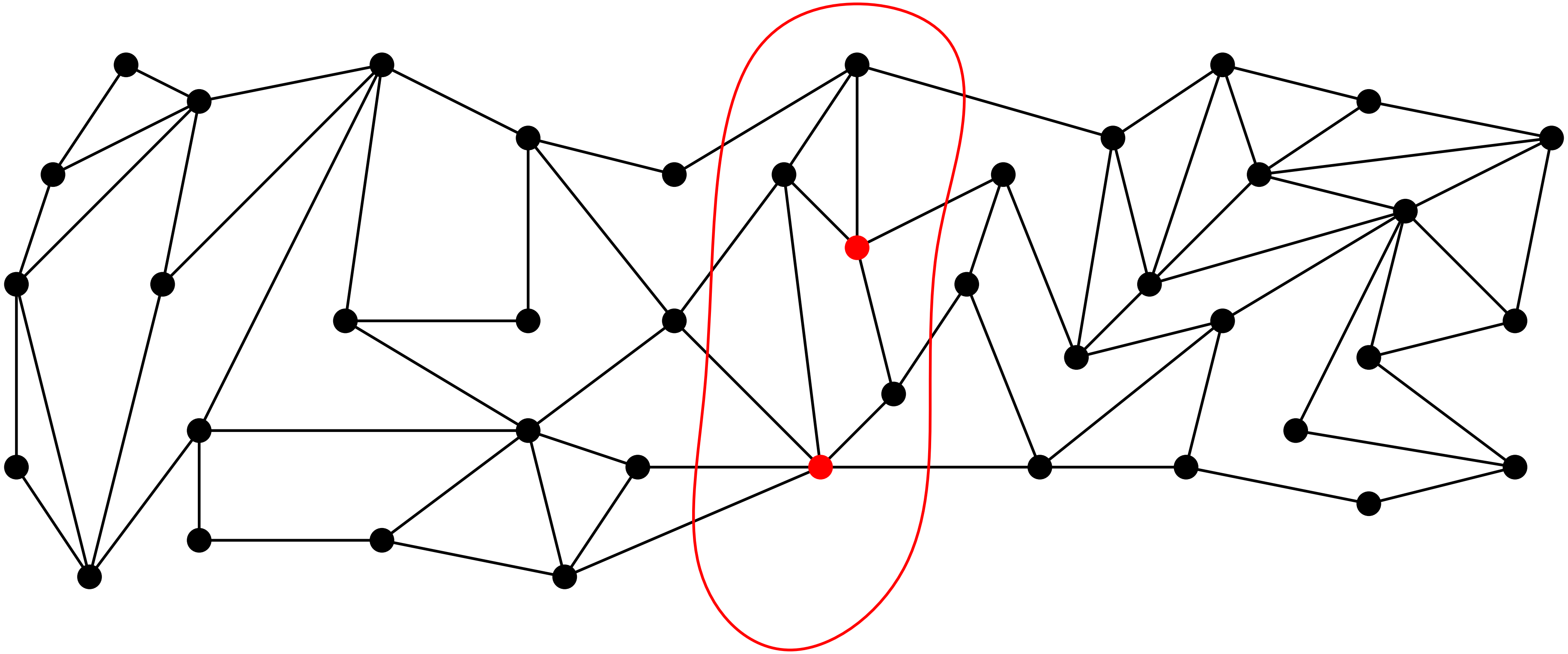

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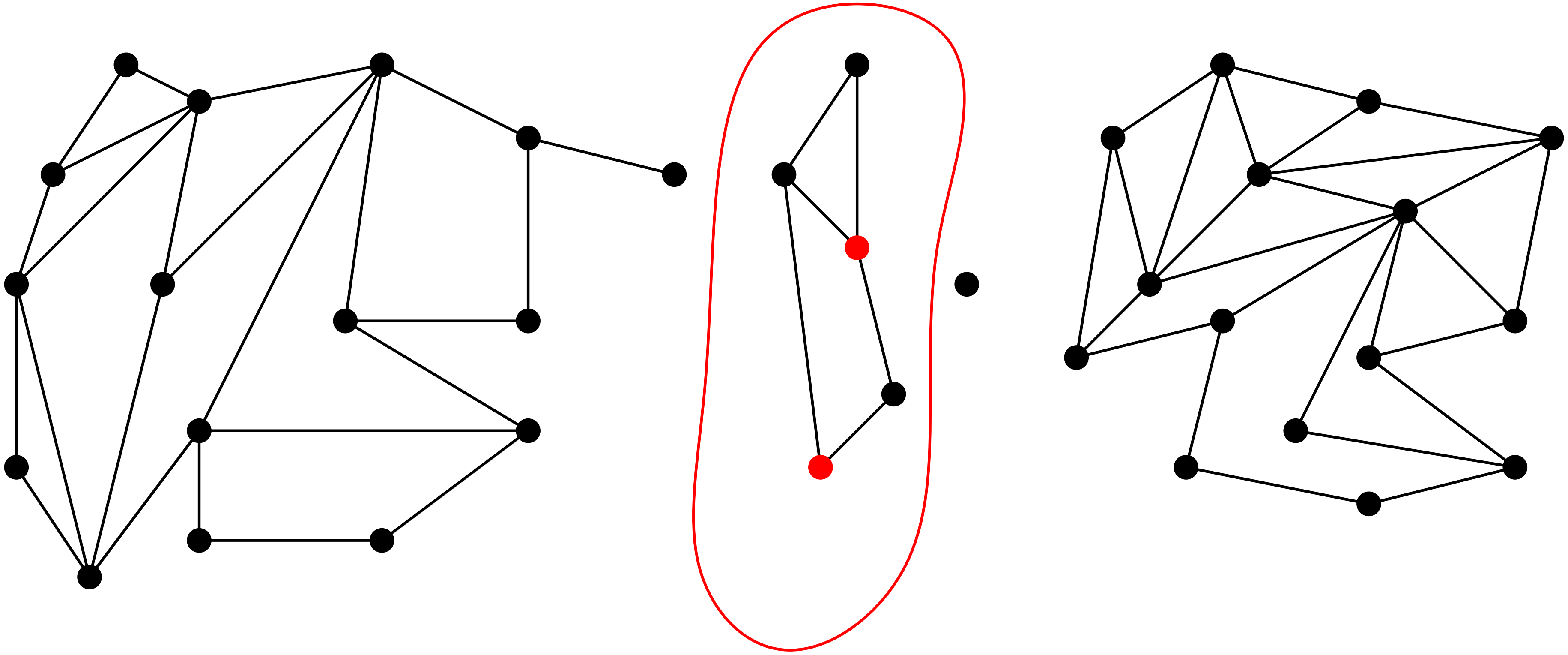

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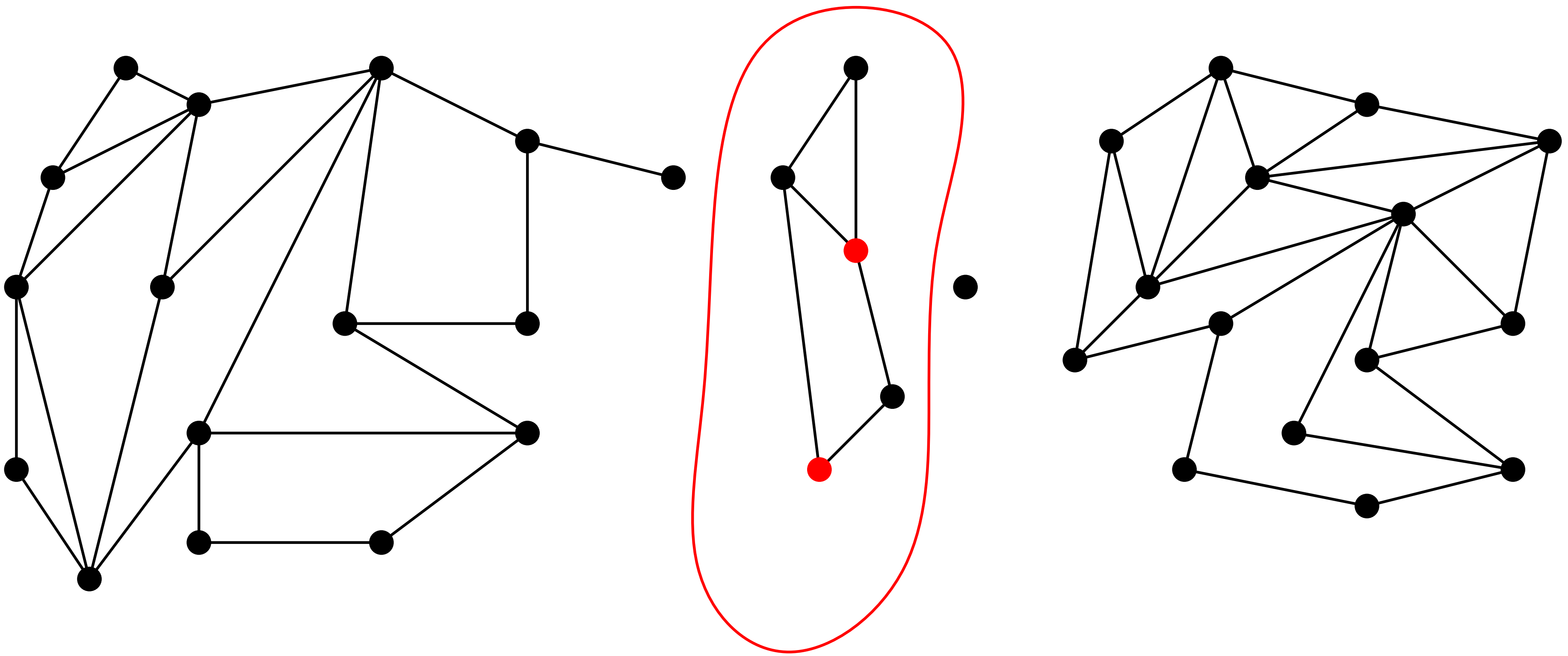
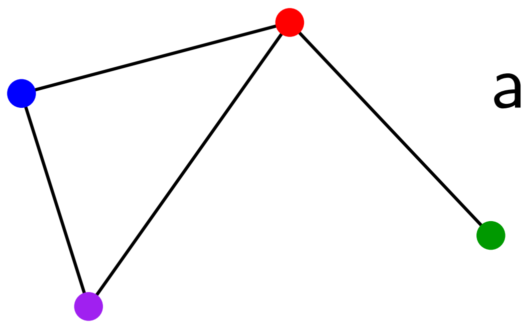


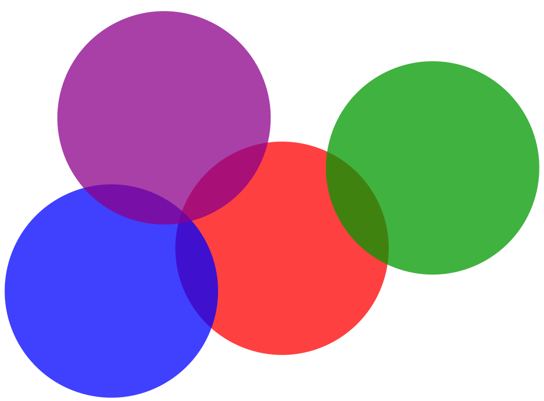
Independent Set

- divide & conquer gives a $2^{\mathcal{O}(\sqrt{n})}$ algorithm

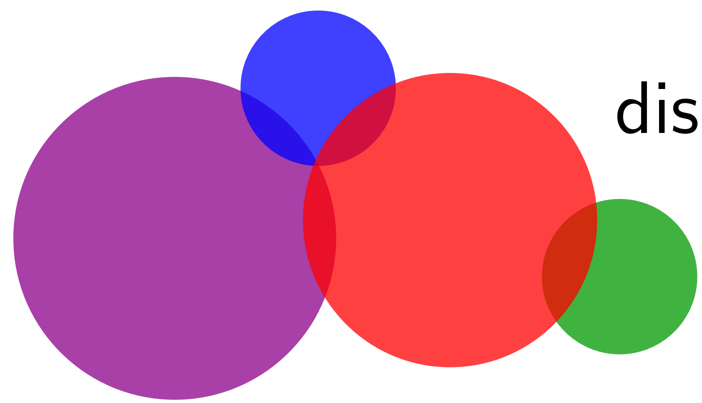
Geometric intersection graphs



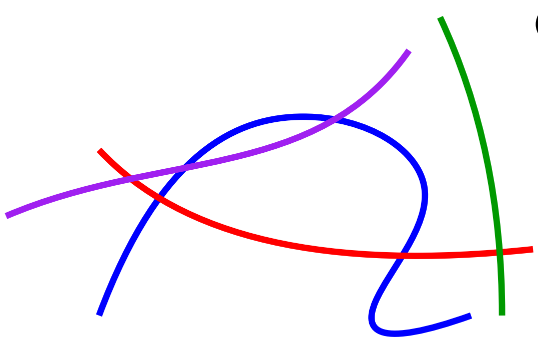
as an intersection graph of:



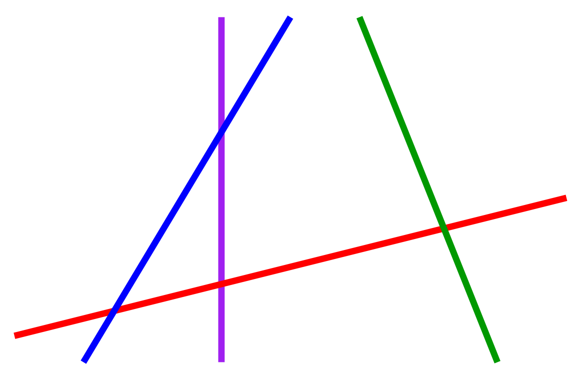
unit disks (UDG)



disks (DG)

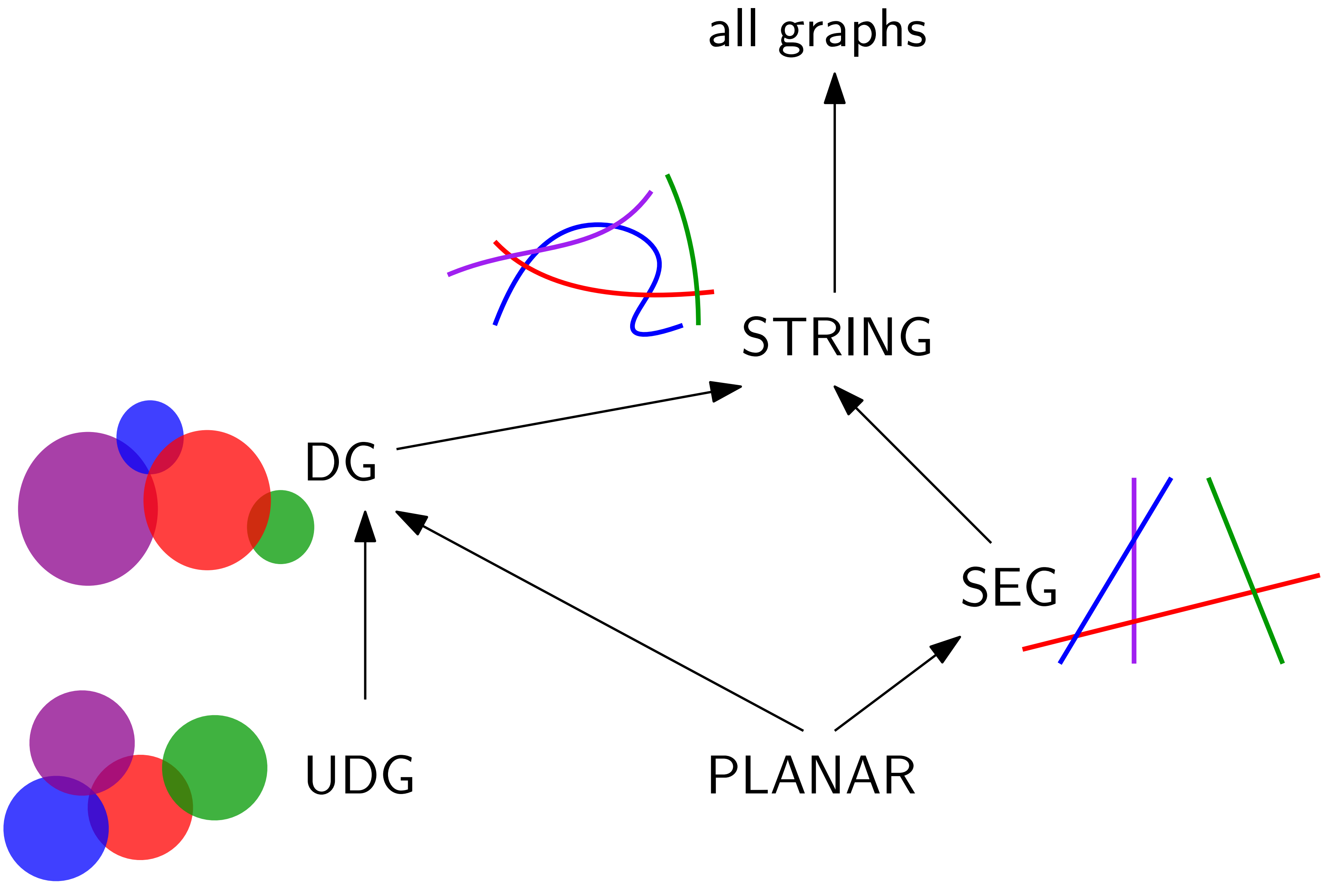


curves (STRING)



segments (SEG)

Relations between classes



Separator-based algorithms for disk intersection graphs

k-Coloring disk graphs

Disk separator theorem [Miller et al., 1997].

Intersection graph of disks with ply at most k has a balanced separator of size $\mathcal{O}(\sqrt{nk})$.

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k-Coloring of disk graphs

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Key observation:

Yes-instances of k-Coloring do not have large cliques.

Independent Set for disk graphs

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- ... but not too much can happen on a clique

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- at most one vertex of Q belongs to the optimal solution
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Independent Set for disk graphs, ctd.

- we have two basic steps:
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Optimality for
segment and string graphs

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   - we either discard v, or choose it to the solution

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   - guess the solution on S and recurse

$$\text{complexity } 2^{\widetilde{\mathcal{O}}(n^{2/3})}$$

3-Coloring

1. there is a vertex v of degree at least $\tau = n^{1/3} \to$???
   - guessing a color for v does not mean we can discard $N(v)$!

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The resulting graph might not be a string graph
$\rightarrow$ we cannot use the separator theorem!

k-Coloring of string graphs

Theorem [Bonnet, Rz., 2018]. k-Coloring for string graphs:

1. for $k = 3$, can be solved in time $2^{\widetilde{\mathcal{O}}(n^{2/3})}$,
2. for $k \geq 4$, cannot be solved in time $2^{o(n)}$ (under the ETH).

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Even though G is dense, almost all its edges are meaningless!

Hardness of List 4-Coloring

- reduce from 3-Sat with n variables and $m = \mathcal{O}(n)$ clauses
- variables: $v_1, v_2 \ldots, v_n$, clauses $C_1, \ldots, C_m$
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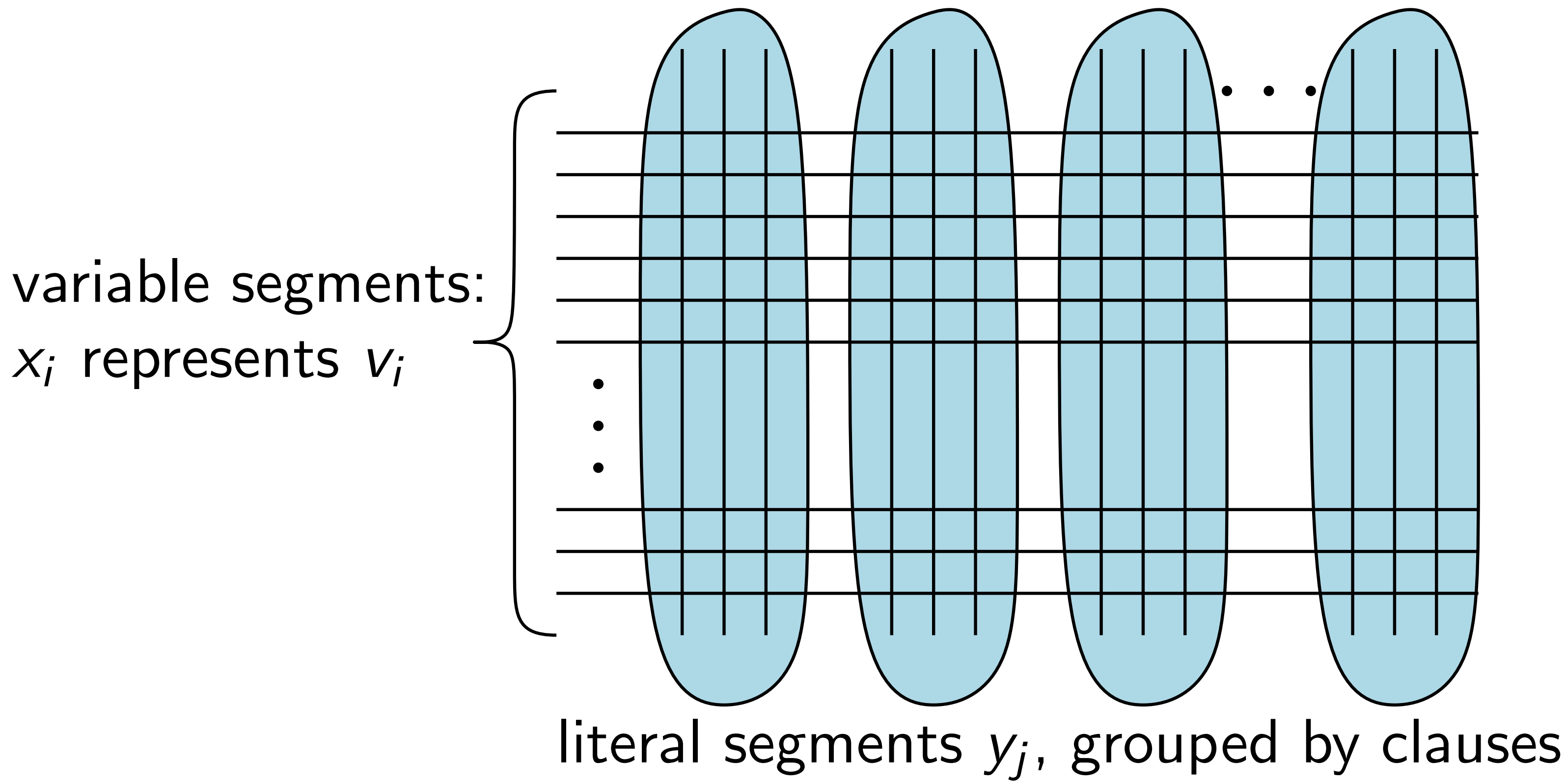

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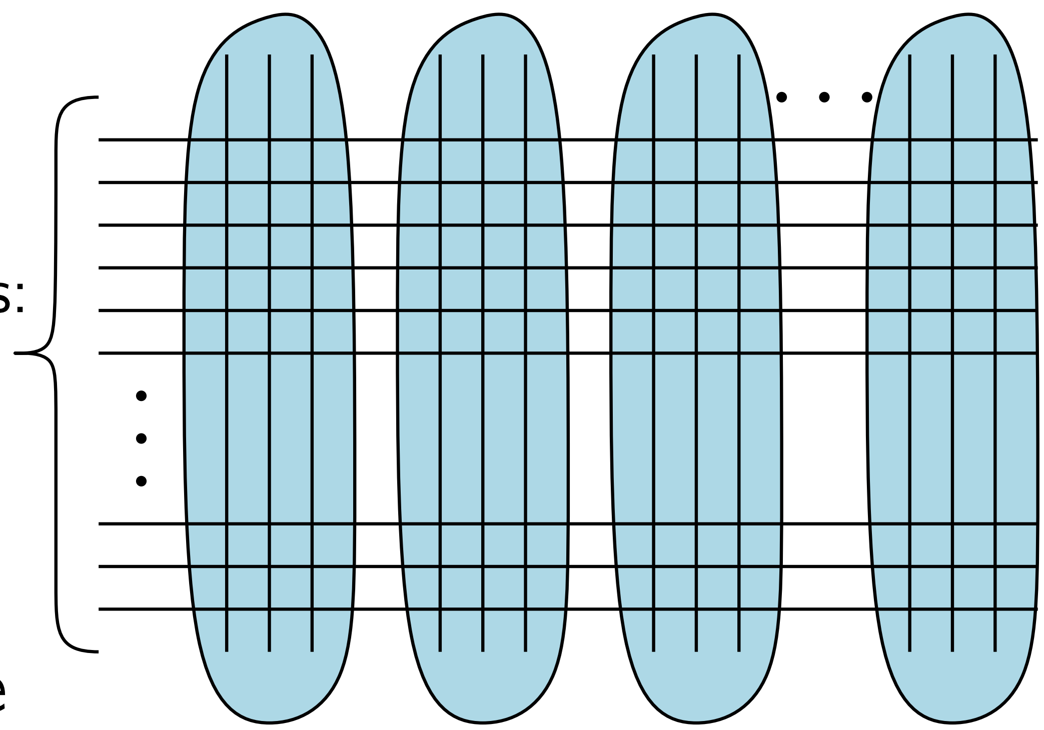
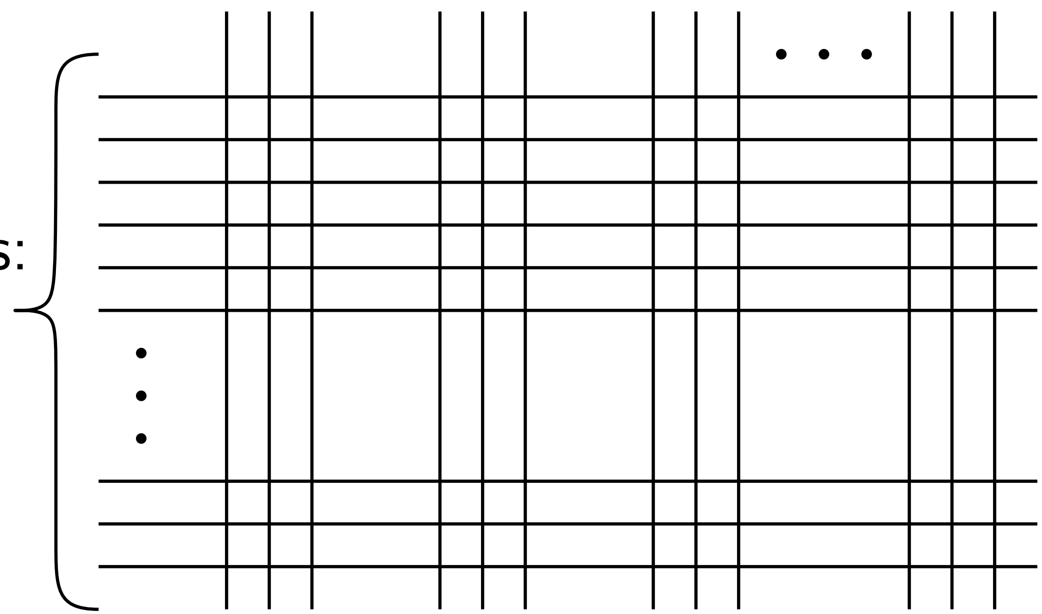


Intended meaning:
1 and 3 correspond to true
2 and 4 correspond to false

Hardness of List 4-Coloring, ctd.

- consistency of colorings segments x_i and segments y_j, that correspond to the same variable



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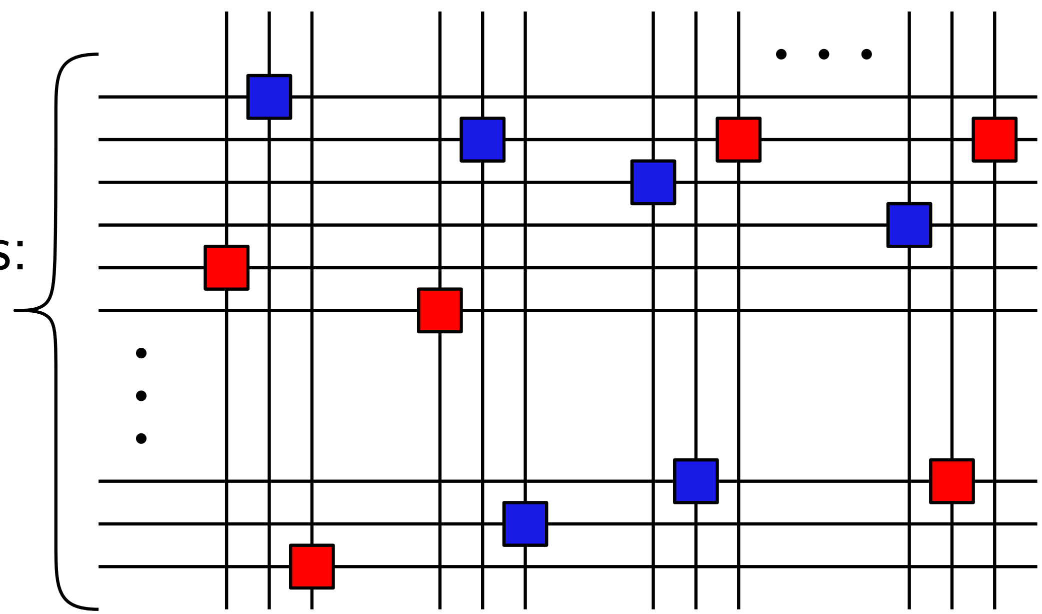

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- satisfiability

at least one of y's must be colored 3

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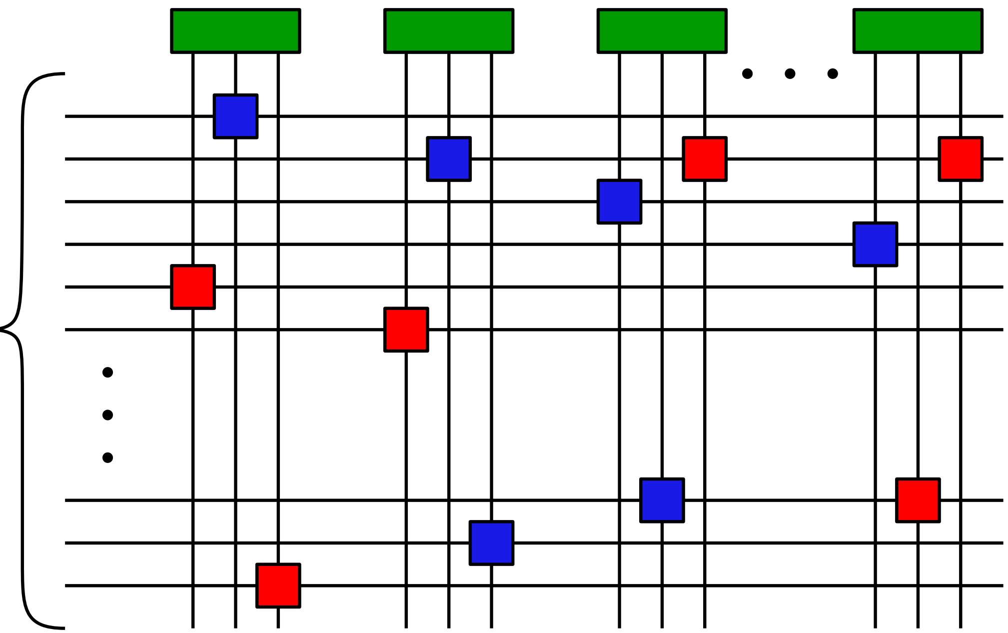

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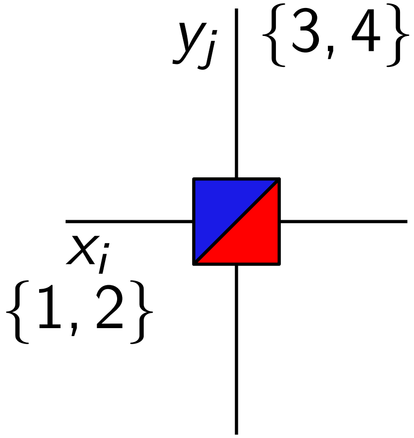
Consistency gadgets



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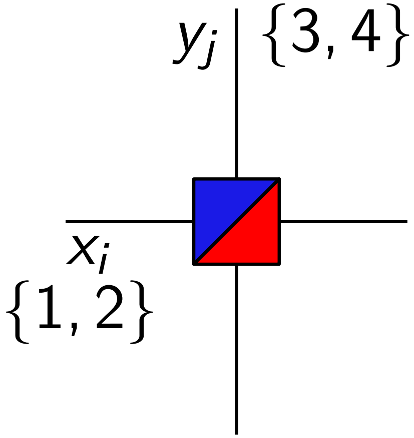

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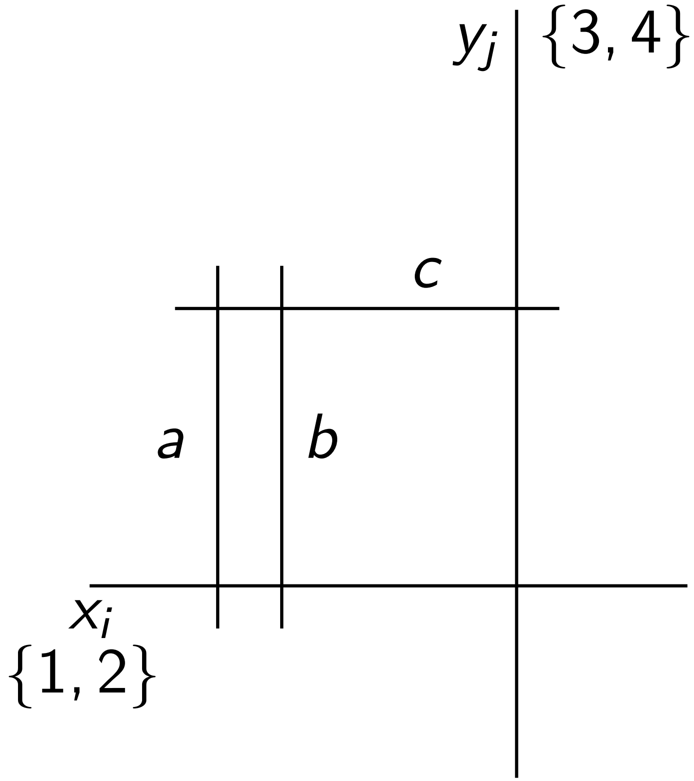
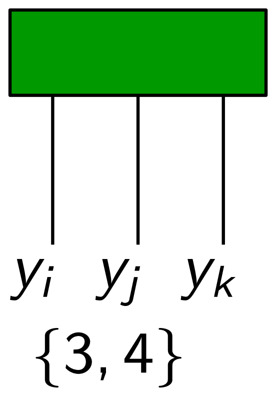


a	1, 3	1, 4
b	2, 4	2, 3
c	3, 4	3, 4

Satisfiability gadget



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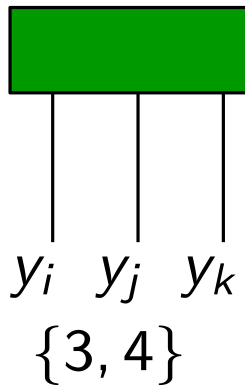

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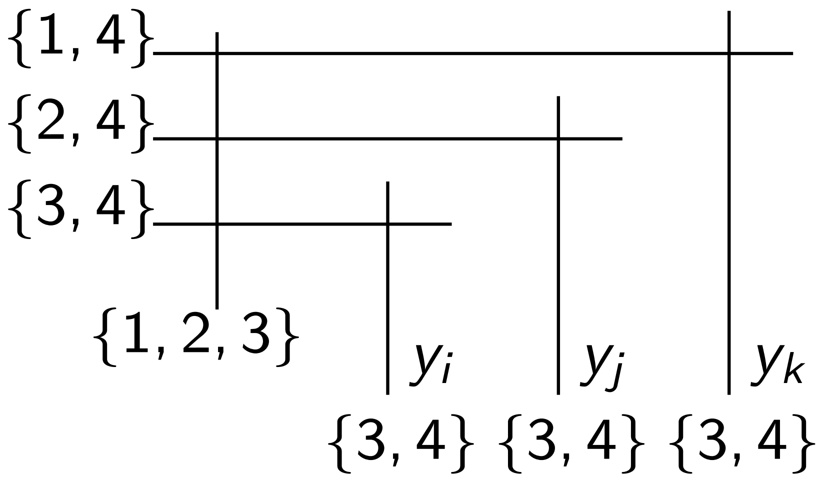
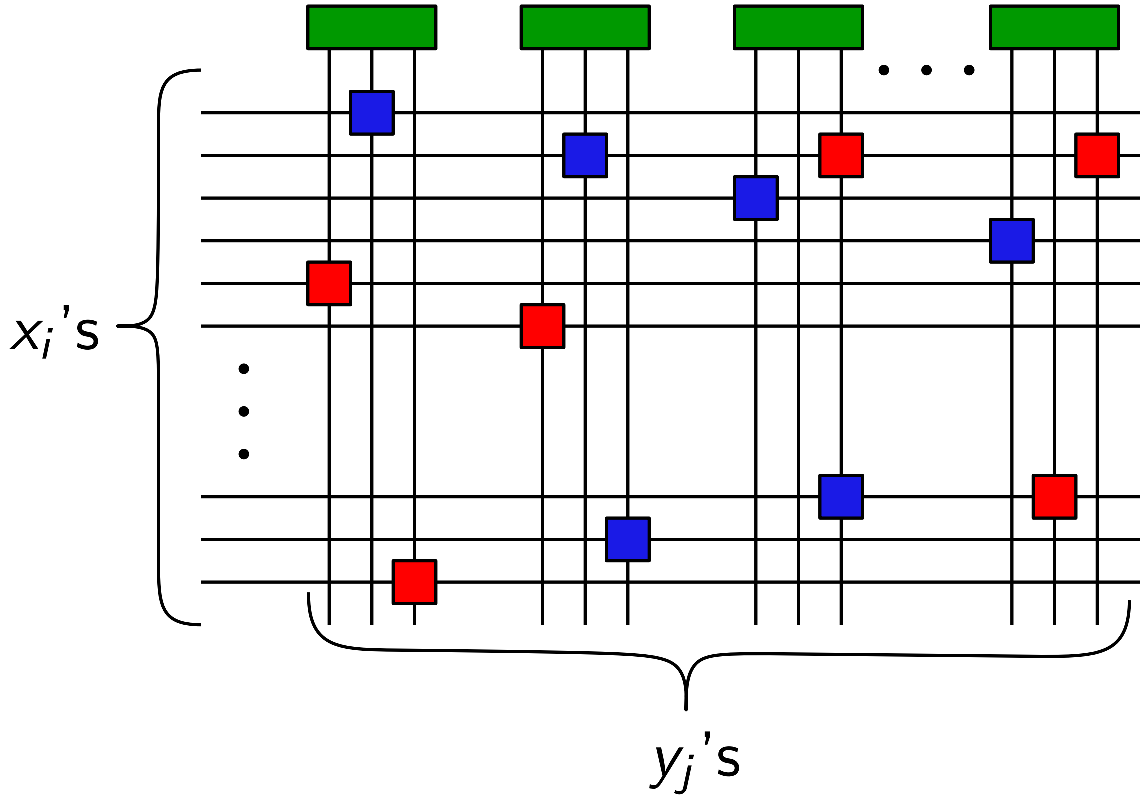
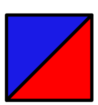


- note segments with three-element lists (if all lists have at most two elements, then the problem is in P)

Wrap-up

- we reduced from 3-SAT with n variables and $m = \mathcal{O}(n)$ clauses
- how many segments do we have?



x_i's	
y_j's	
	
	
tota	

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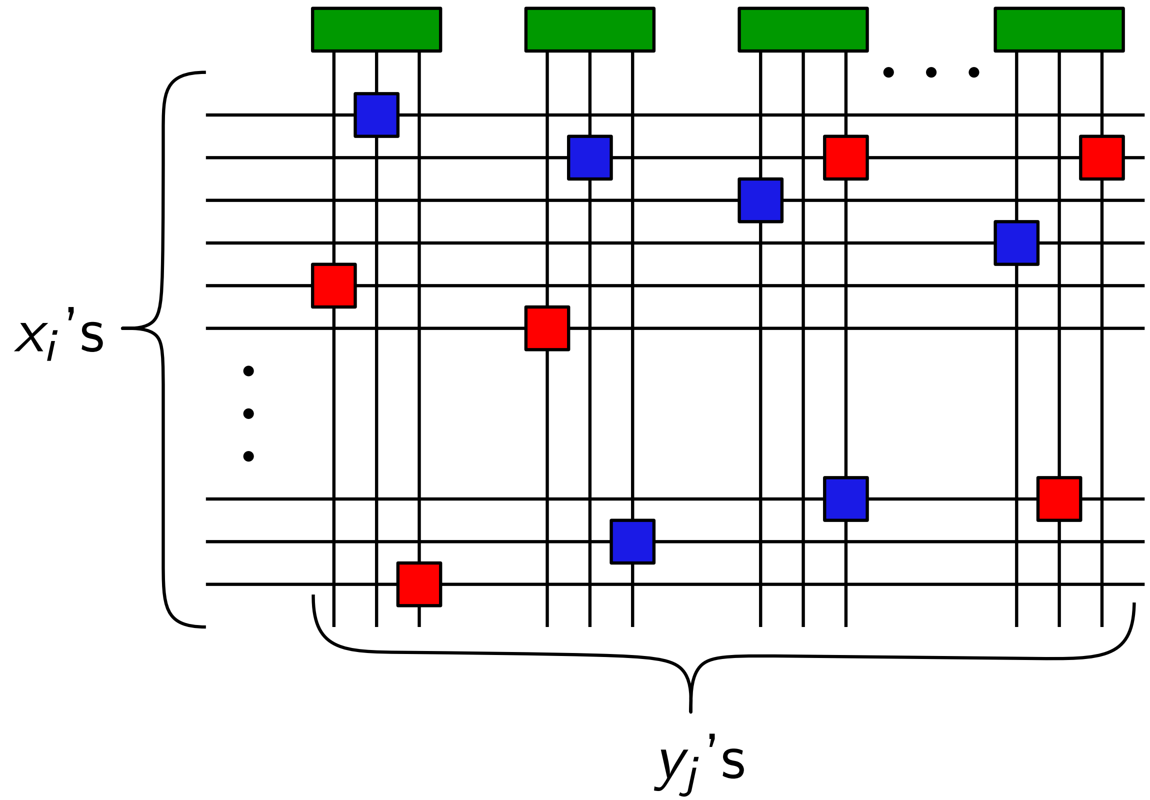
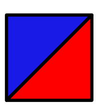


x_i's	n
y_j's	$3m$
	$3m \times 3$
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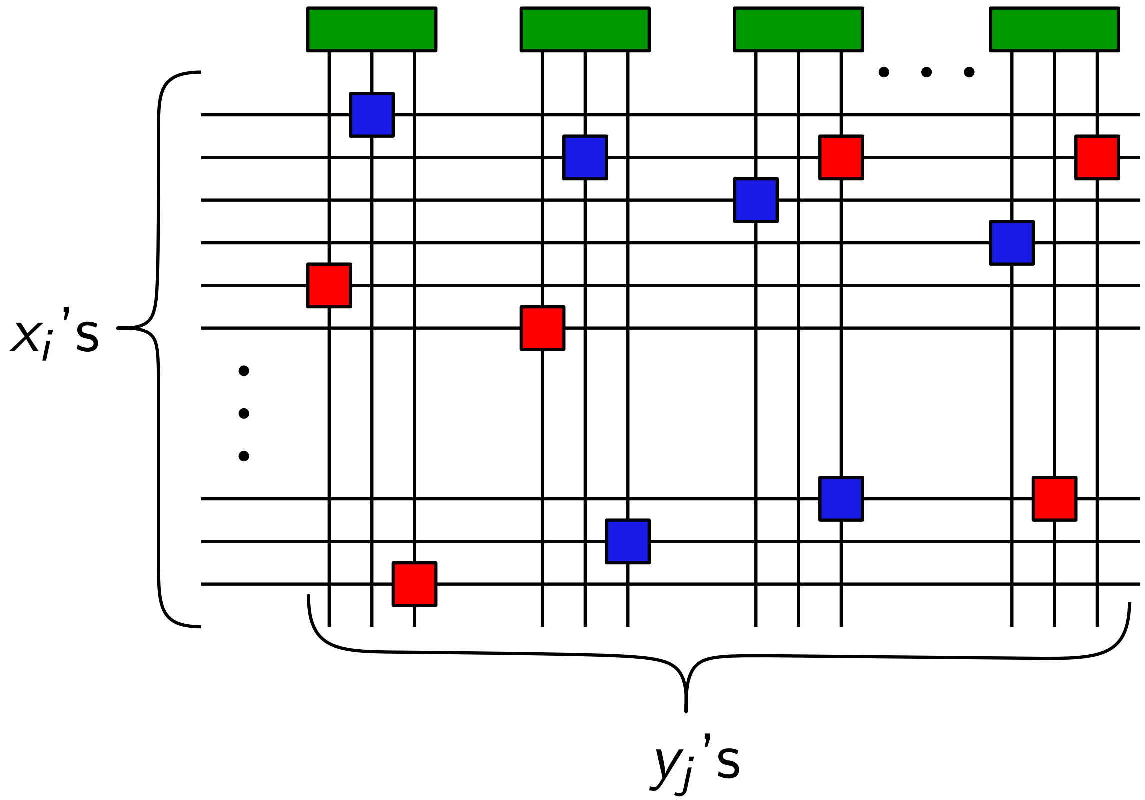

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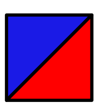	$3m \times 3$
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- solving LIST 4-COLORING in segment graphs with N vertices in time $2^{o(N)}$
  $\rightarrow$ solving 3-SAT in time $2^{o(n)}$
  $\rightarrow$ ETH fails

Feedback Vertex Set in string graphs

- remove the minimum number vertices to destroy all cycles
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- what if we have a vertex of large degree?

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- combining with the separator of size $\mathcal{O}(\sqrt{m})$, we get

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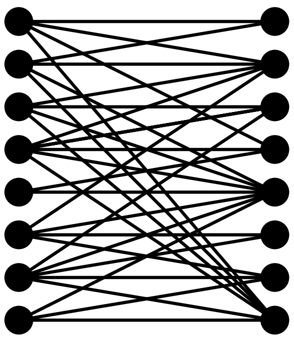

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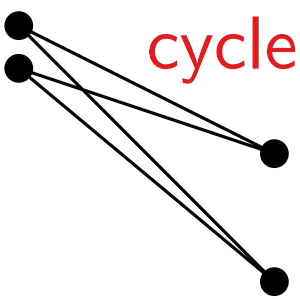

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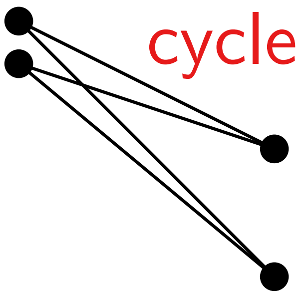

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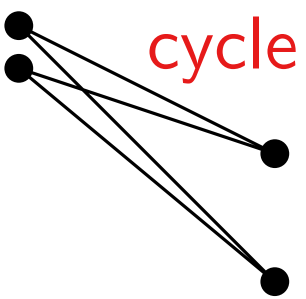

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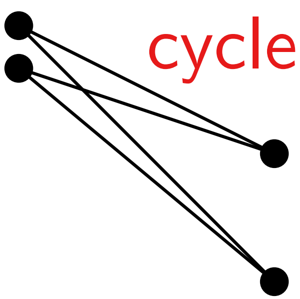

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- But no $2^{o(n)}$ algorithm for Odd Cycle Transversal

A detour: the need of representation and robust algorithms

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Theorem [Schaefer, Sedgewick, Štefankovič, '03].
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x_i's are **boolean**,
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NP $=$ class of problems polynomially equivalent to SAT.	$\exists\mathbb{R}$ – class of problems polynomially equivalent to ETR.
SAT: decide if a formula is **true** $\exists x_1 \exists x_2 \ldots \exists x_n\ \Phi(x_1, \ldots, x_n)$	ETR: decide is a formula is **true** $\exists x_1 \exists x_2 \ldots \exists x_n\ \Phi(x_1, \ldots, x_n)$
x_i's are **boolean**, Φ is quantifier-free and uses $\wedge, \vee, \neg, =, \rightarrow$	x_i's are **reals**, Φ is quantifier-free and uses $\wedge, \vee, \neg, =, \rightarrow, >, +, -, \times$ (in $\mathbb{R}$)

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Theorem [Schaefer, Štefankovič, '17].
Recognizing segment graphs is in $\exists\mathbb{R}$-complete.

NP $=$ class of problems polynomially equivalent to SAT.	$\exists\mathbb{R}$ – class of problems polynomially equivalent to ETR.
SAT: decide if a formula is **true** $\exists x_1 \exists x_2 \dots \exists x_n\ \Phi(x_1, \dots, x_n)$	ETR: decide is a formula is **true** $\exists x_1 \exists x_2 \dots \exists x_n\ \Phi(x_1, \dots, x_n)$
x_i's are **boolean**, Φ is quantifier-free and uses $\wedge, \vee, \neg, =, \rightarrow$	x_i's are **reals**, Φ is quantifier-free and uses $\wedge, \vee, \neg, =, \rightarrow, >, +, -, \times$ (in $\mathbb{R}$)

- a strong indication that the problem is not in NP!
- similar for unit disk graphs [Kang, Müller, '12]

What about our algorithms?

Independent Set in disk graphs

1. ply $> n^{1/3} \rightarrow$ a clique of size $> n^{1/3}$, branch
2. ply $\leq n^{1/3} \rightarrow$ a balanced separator S of size $\mathcal{O}(n^{2/3})$
3. guess the solution on S
4. recurse using divide & conquer

Total running time: $2^{\widetilde{\mathcal{O}}(n^{2/3})}$.

- where do we need a representation?

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What about our algorithms?

Independent Set in disk graphs

1. if we find a clique of size $> n^{1/3}$, branch
2. otherwise, find a balanced separator S of size $\mathcal{O}(n^{2/3})$
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Total running time: $2^{\widetilde{\mathcal{O}}(n^{2/3})} + 2^{\widetilde{\mathcal{O}}(n^{2/3})} = 2^{\widetilde{\mathcal{O}}(n^{2/3})}$.

- where do we need a representation?
- enumerating all possibilities takes time $n^{n^{2/3}} = 2^{\widetilde{\mathcal{O}}(n^{2/3})}$
- we do not really need a representation!

Robust algorithms

- An algorithm is robust, if it either
    - computes the correct solution, or
    - correctly concludes that the input does not belong to the right class (here: disk graphs)
- notion introduced by Spinrad

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- disk graphs $\subseteq \mathcal{X}$
- on the other hand, our hardness results hold even if a geometric representation is given

When large cliques do not help

Clique in disk graphs

- Clique is polynomially solvable in UDG [Clark et al., 1990]
- the complexity for DG is open

- the existence of a large clique does not make the problem any easier!

CLIQUE in disk graphs

- CLIQUE is polynomially solvable in UDG [Clark et al., 1990]
- the complexity for DG is open
- the existence of a large clique does not make the problem any easier!
- we need to make our hands dirty and look at the properties of geometric representations
- by some epsilon-perturbation we can assume that no three centers are aligned

Notation: vertex v_i is represented by a disk with the center c_i

C_4's in disk graphs

Simple observation.

In any disk representation of of C_4 with vertices v_1, v_2, v_3, v_4:
the line $\ell(c_2 c_4)$ crosses the segment $c_1 c_3$, or
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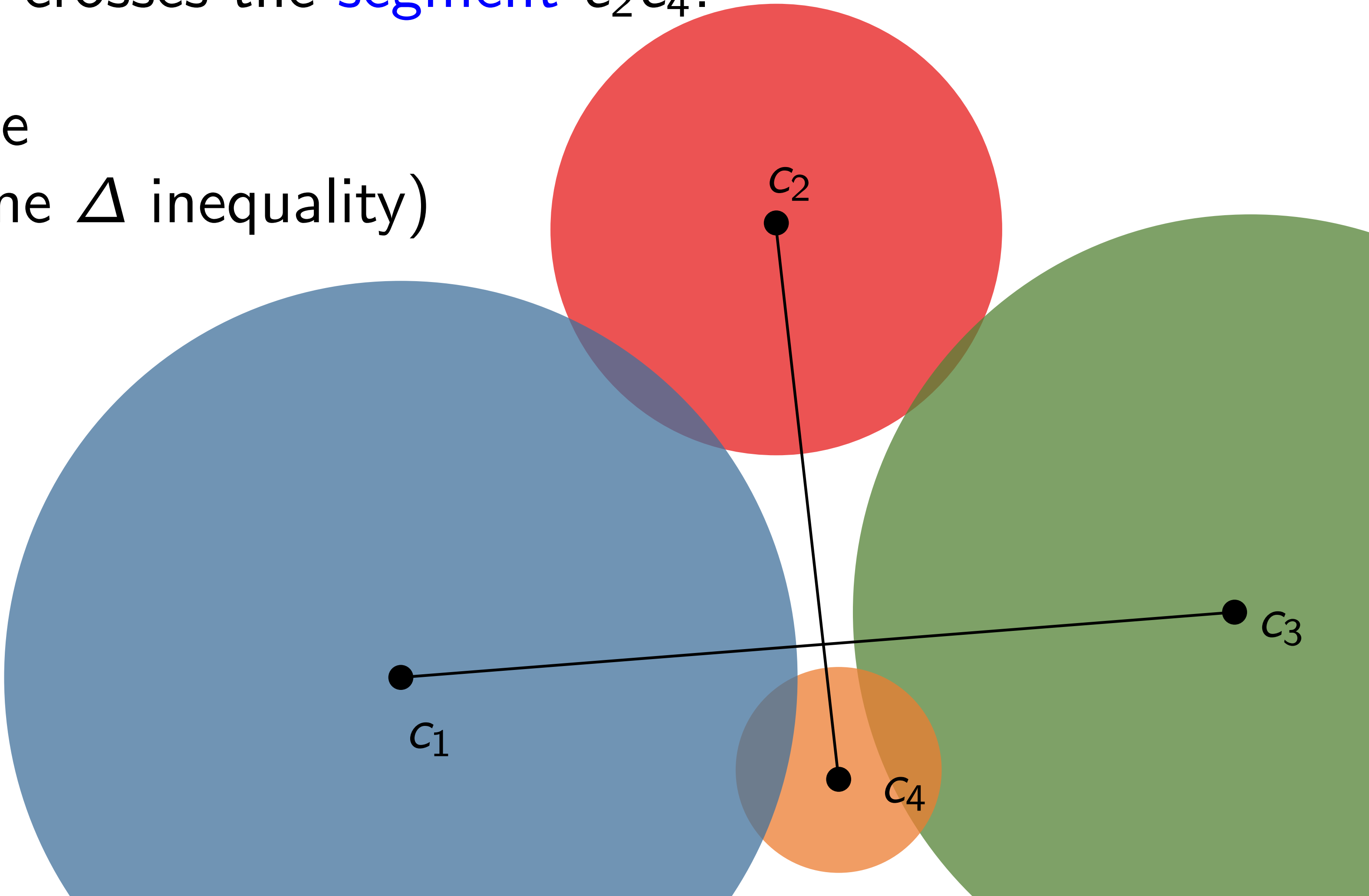
Proof by picture
(follows from the Δ inequality)



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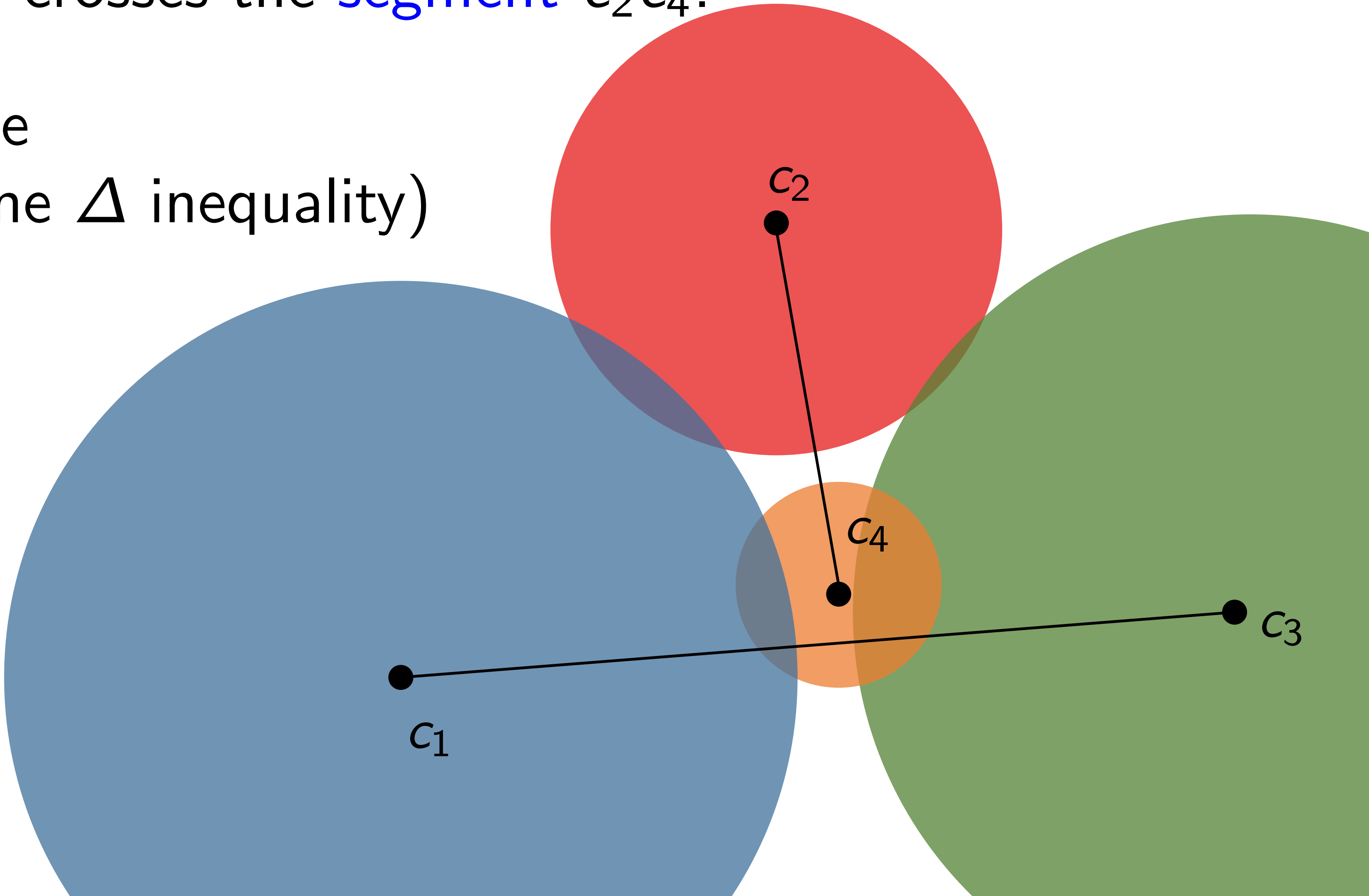

Non-disk graphs

Observation [Bonnet, Giannopoulos, Kim, Rz. Sikora, 2018].
For odd p, q, the graph $G = \overline{C_p + C_q}$ is not a disk graph.

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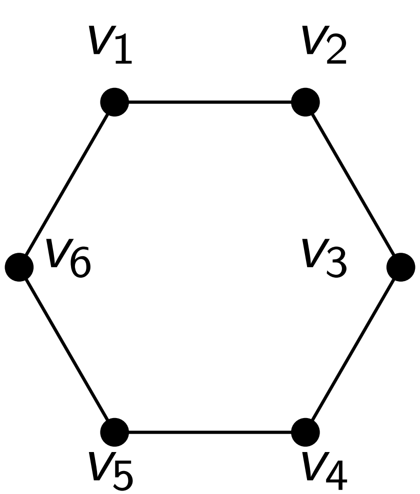

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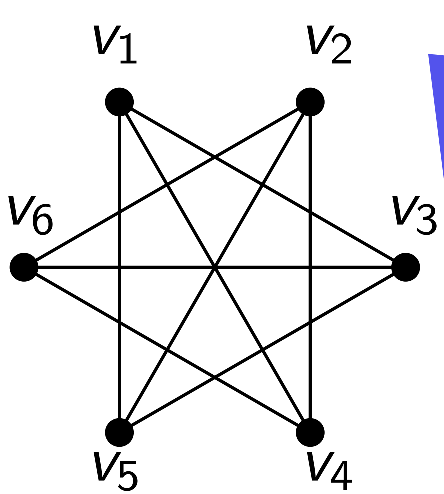

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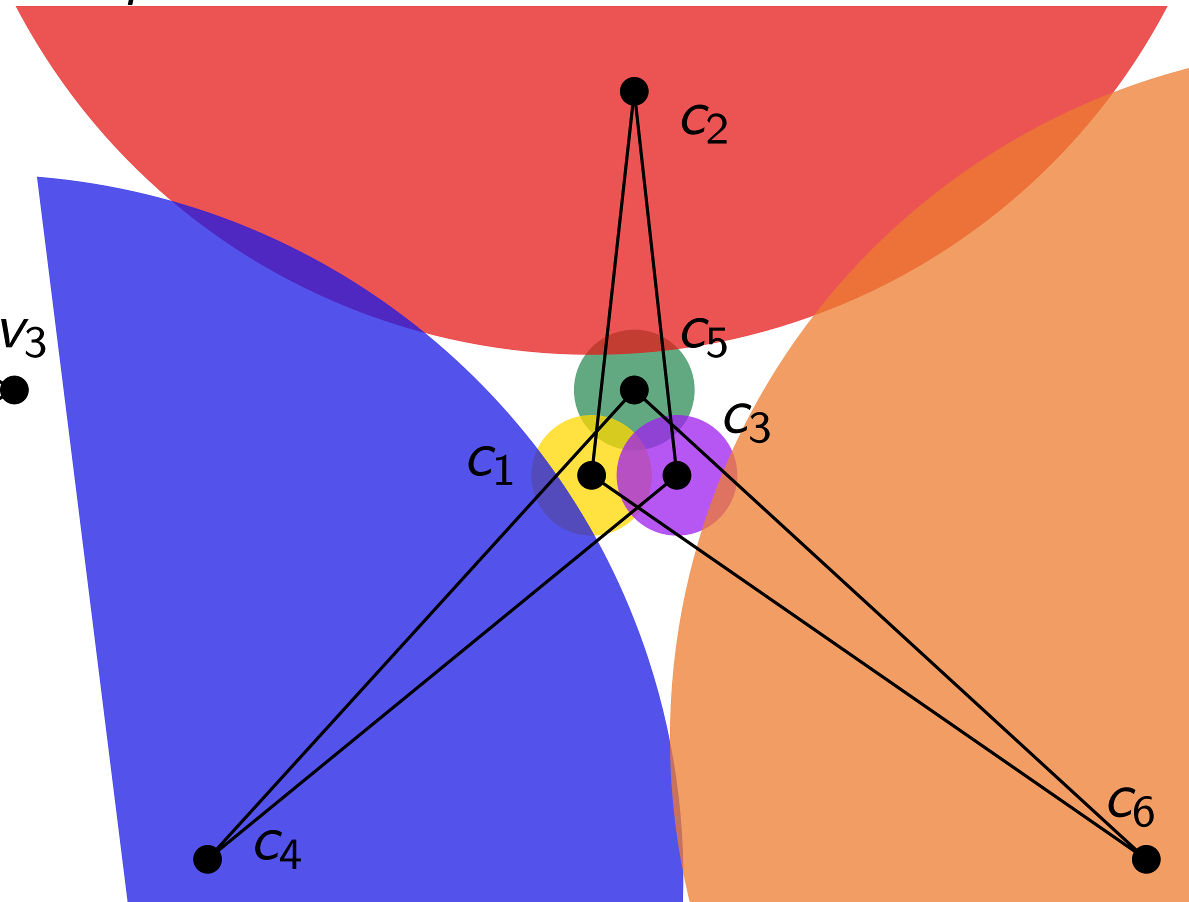

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- suppose there is a representation
- let $S_1, \ldots, S_p$ and $S'_1, \ldots, S'_q$ be segments of the co-cycles
- every S_i and every S'_j correspond to $2K_2$ in $\overline{G}$
  $\rightarrow$ their endpoints induce a C_4 in G
  $\rightarrow$ $\ell(S_i)$ crosses S_j or $\ell(S_j)$ crosses S_j

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$\sum_{i=1}^{p}(a_i + b_i - c_i) =$ number of pairs i, j satisfying $(\star)$ $= pq$

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- $\sum_{i=1}^{p}(a_i + b_i - c_i) = pq$ is even $\rightarrow$ contradiction $\square$

$\textsc{Clique}$ for disk graphs

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Independent Set in a co-disk graph:

1. vertex of degree at least $n^{1/3} \rightarrow$ branching
2. no odd cycle of length $< n^{1/3} \rightarrow$
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3. odd C of length $\leq n^{1/3}$ and $\Delta \leq n^{1/3} \rightarrow$
   $|N[C]| \leq n^{2/3}$ and $G - N[C]$ is bipartite

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$2^{\widetilde{\mathcal{O}}(n^{2/3})}$

guess the solution on X or $N[C]$ and finish in poly time

Theorem [BGKRzS '18].
Clique in disk graphs can be solved in time $2^{\widetilde{\mathcal{O}}(n^{2/3})}$.

Open problem: Max Cut in disk graphs

- partition vertices into two sets, to maximize the number of crossing edges
- NP-hard on unit disk graphs, reduction is quadratic $\rightarrow$ no $2^{o(\sqrt{n})}$algorithm
- is there a subexponential algorithm?

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- is there a subexponential algorithm?
- Warning: edge-weighted version has no subexponential algorithm on complete graphs!
- complexity even unclear for (unit) interval graphs

Episode 2: parameterized algorithms

Geometric separators

k-Independent Set in unit disk graphs

- is there an independent set of size at least k?
- are there k disjoint disks?

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- large area implies that a greedy algorithm works:
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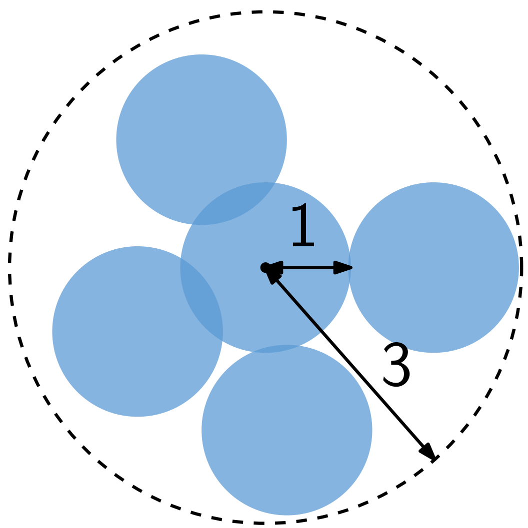

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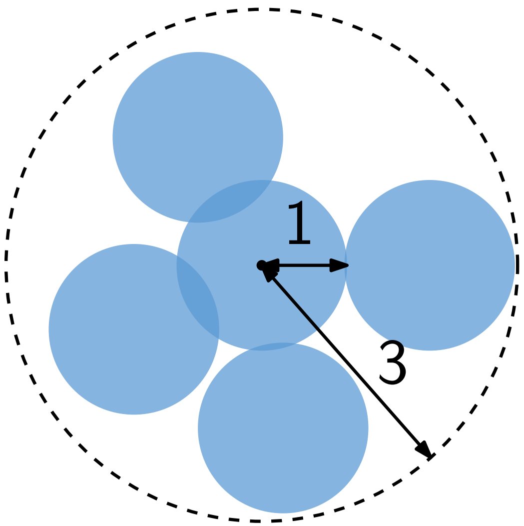

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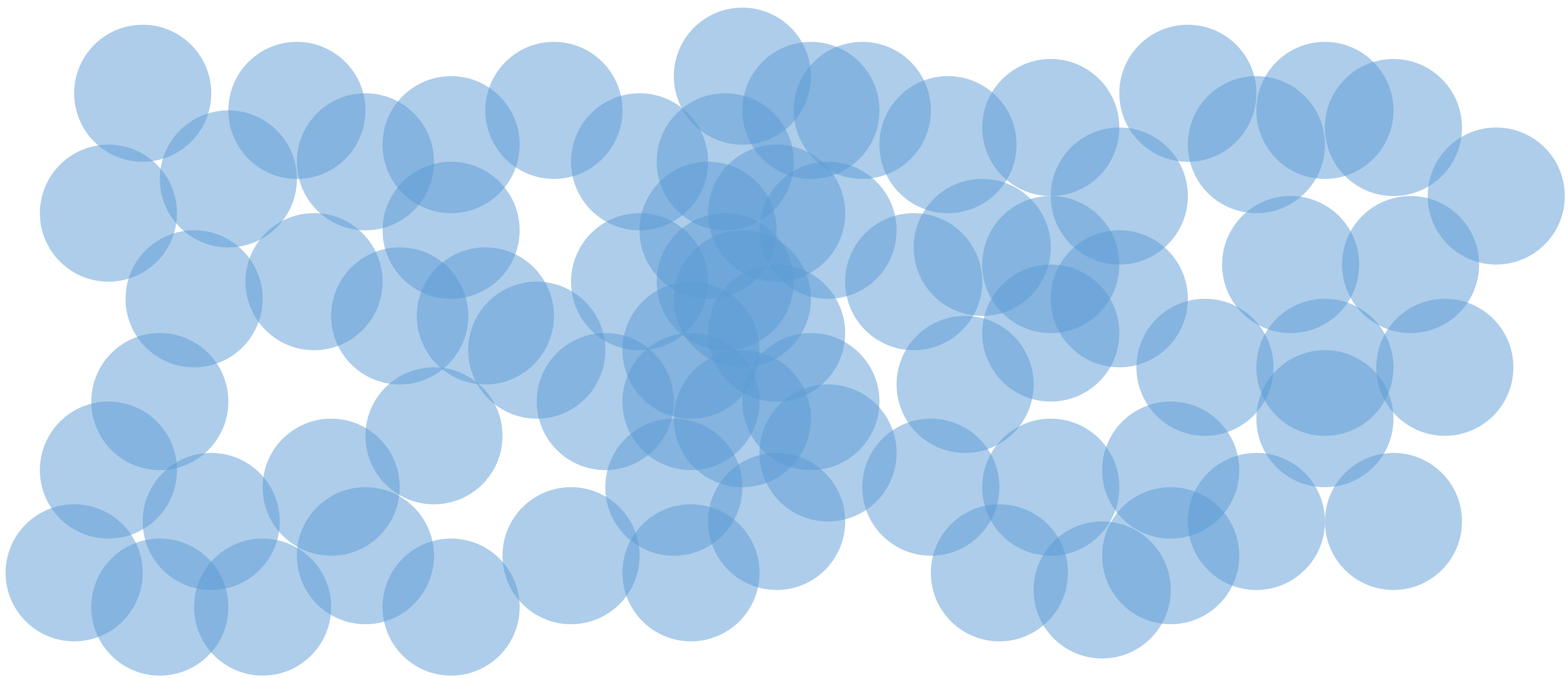
- assume that $\pi \cdot k \leq$ total area $\leq 9\pi \cdot k$

Geometric separator theorem for unit disks

Geometric separator theorem [Alber, Fiala, '04].

Given a collection of unit disks with total area A, there exists a set S of disks, such that:

- total area of disks in S is $\mathcal{O}(\sqrt{A})$,
- removing S gives connected parts of roughly equal area.



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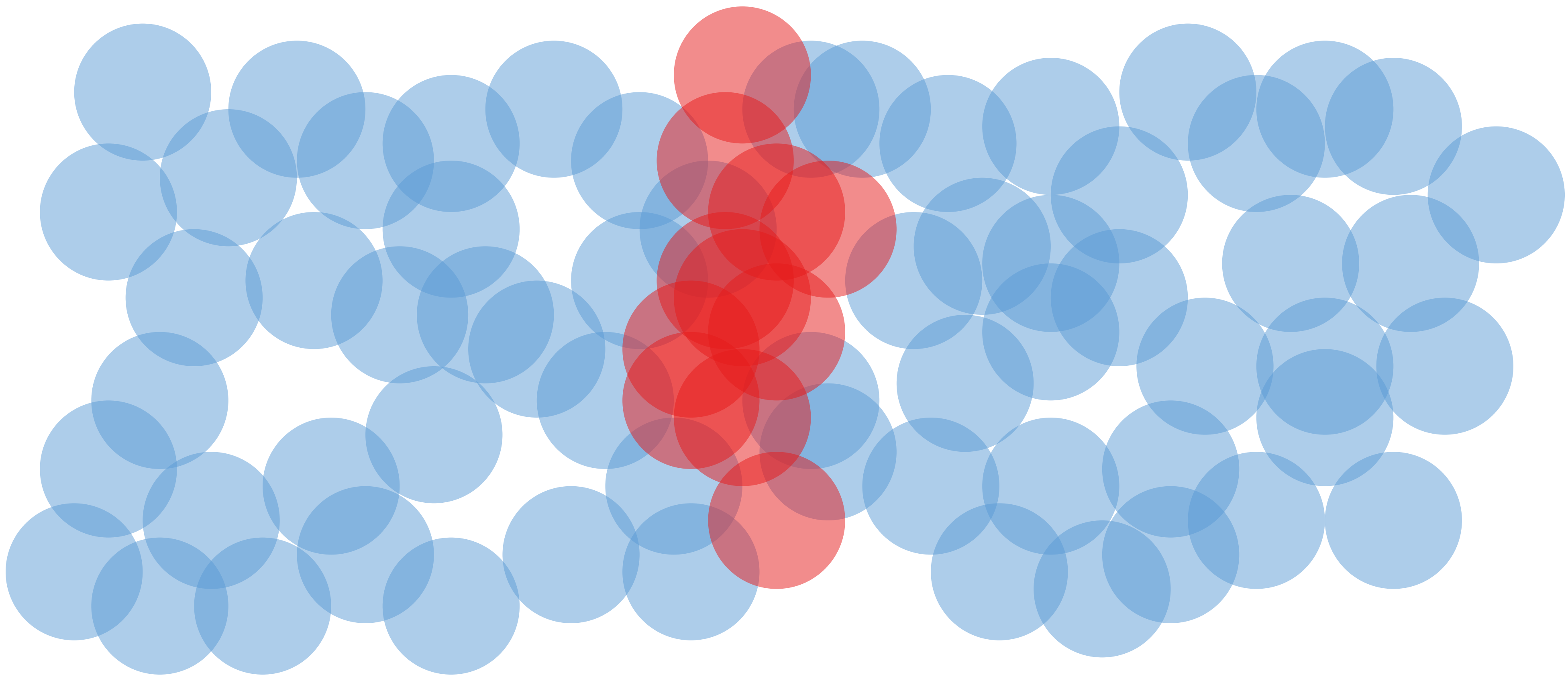

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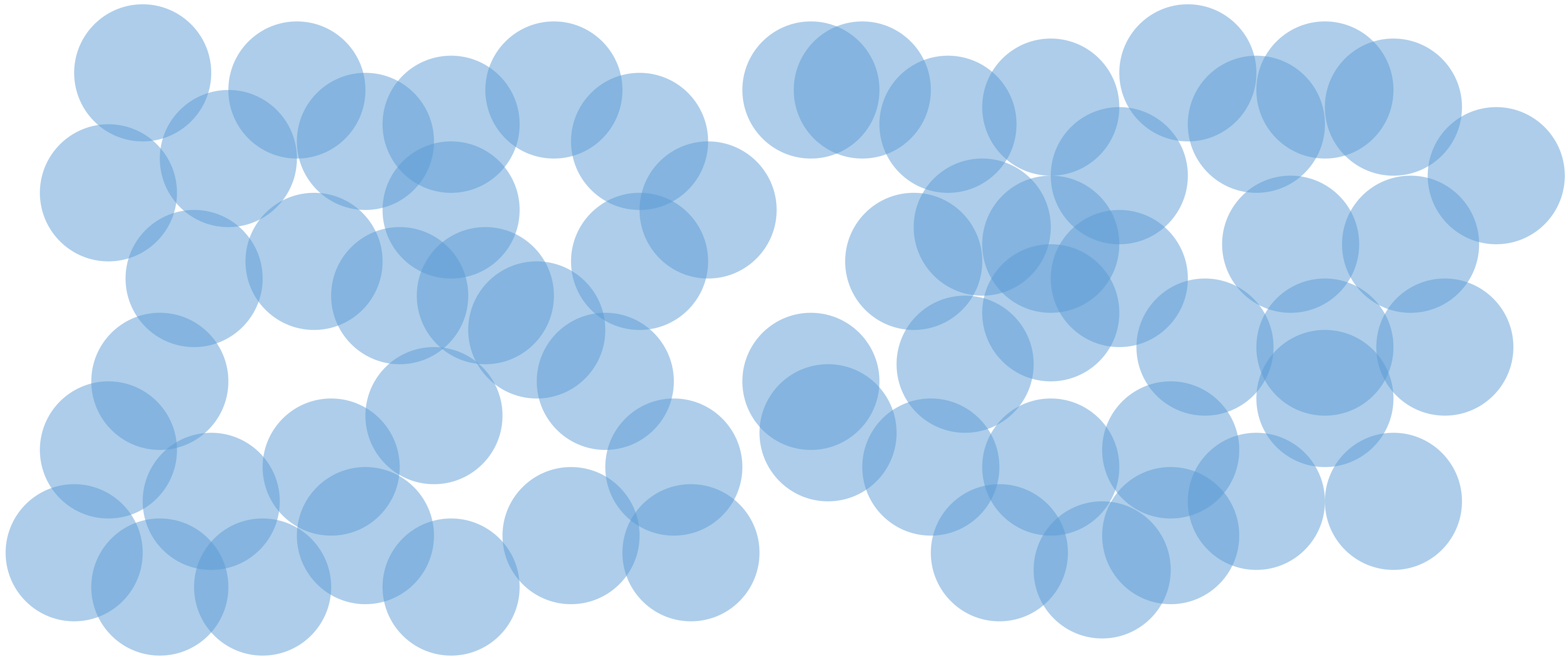

Divide & conquer using geometric separators

Algorithm [Alber, Fiala, '04].

1. $A =$ total area
2. if $A < \pi \cdot k$, return NO
3. if $A > 9\pi \cdot k$, return YES
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$$\sum_{i=0}^{\mathcal{O}(\sqrt{k})} \binom{n}{i} = n^{\mathcal{O}(\sqrt{k})}$$

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- overall complexity is $n^{\mathcal{O}(\sqrt{k})}$

Evaluation

Strengths

- simple
- parameterized
- faster than what we had in the classical setting: $\sum_{k=1}^{n} n^{\mathcal{O}(\sqrt{k})} = 2^{\widetilde{\mathcal{O}}(\sqrt{n})}$, compared to $2^{\widetilde{\mathcal{O}}(n^{2/3})}$
- optimal (under ETH)
- works also for disks and other shapes with bounded area

Weaknesses

- doesn't work for general disk graphs, not to say about segment/string graphs
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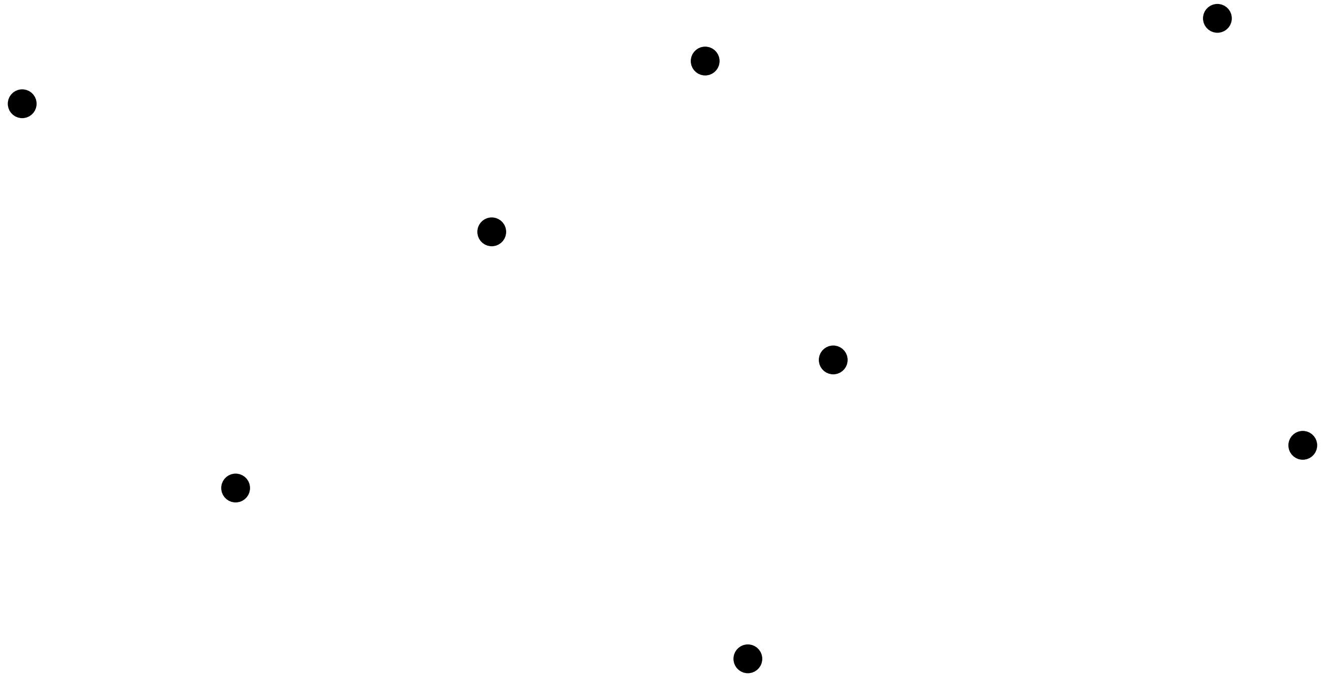
- in the remainder of this part we will learn how to address the first weakness, using a different approach

Voronoi-diagram approach

Voronoi diagrams

- we are given n points in the plane (objects)
- each point of the plane is assigned to the closest object



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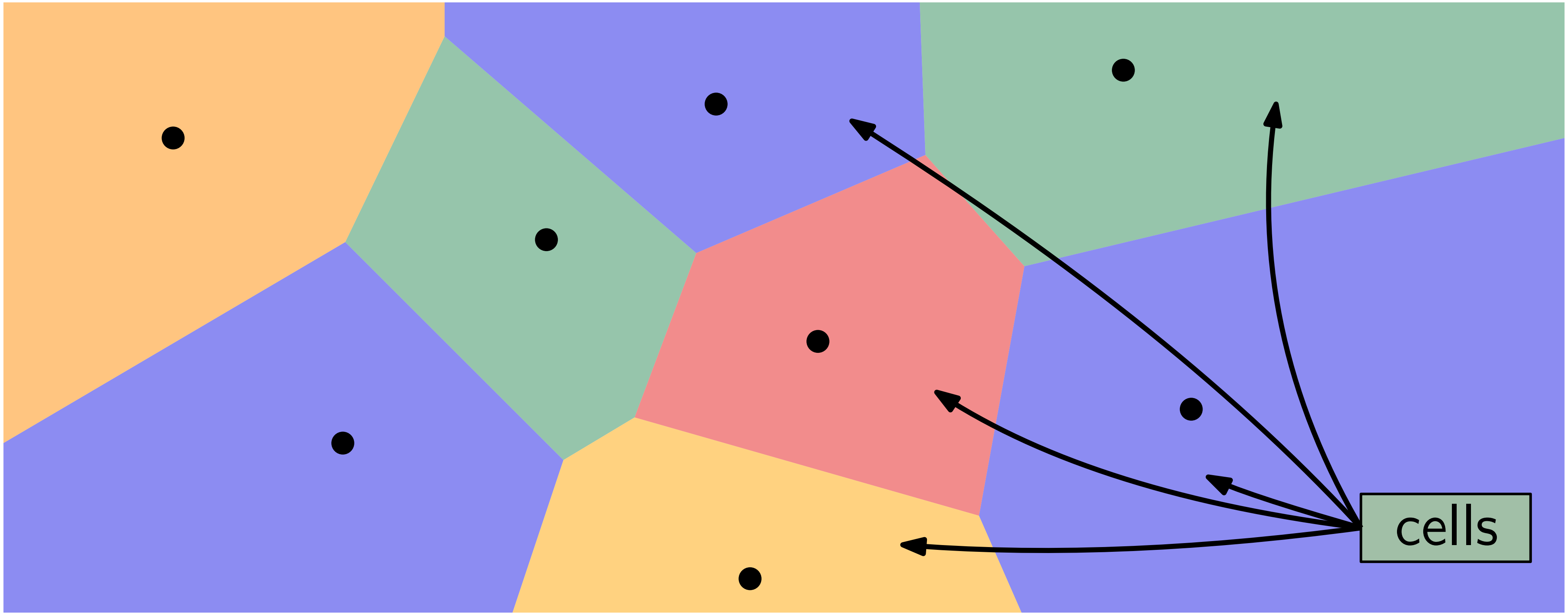

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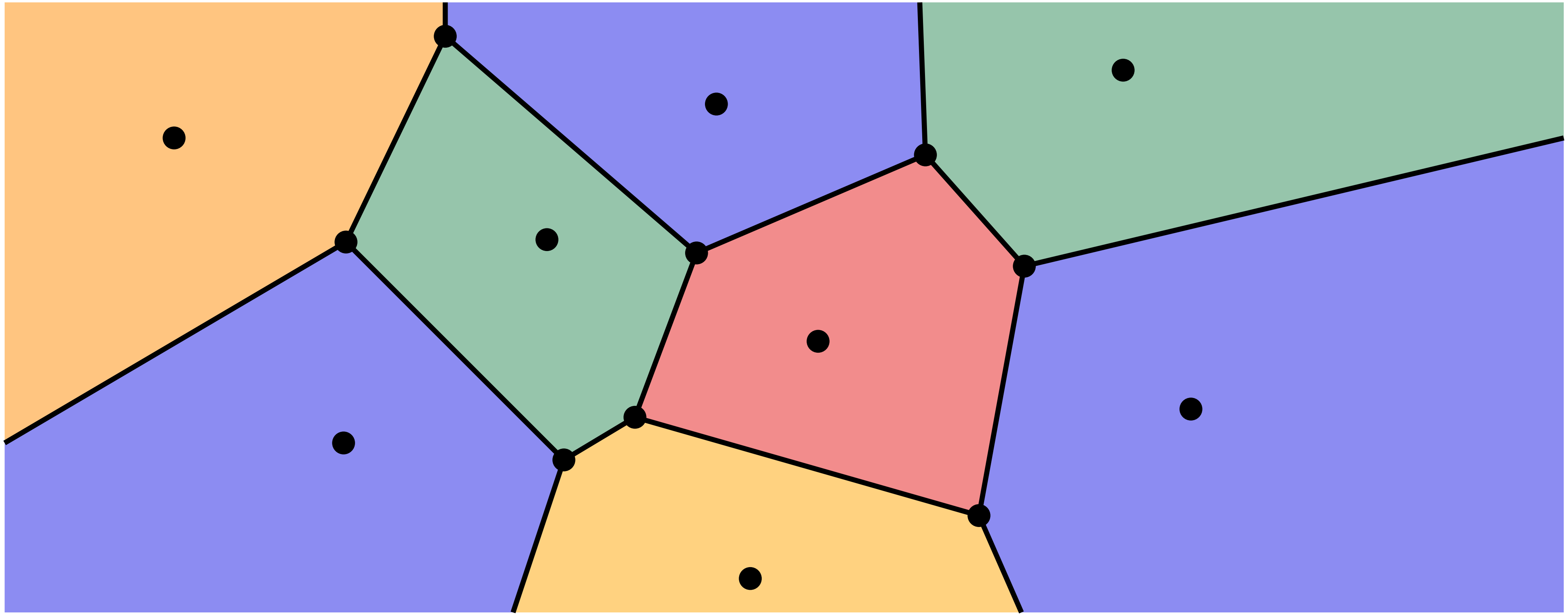

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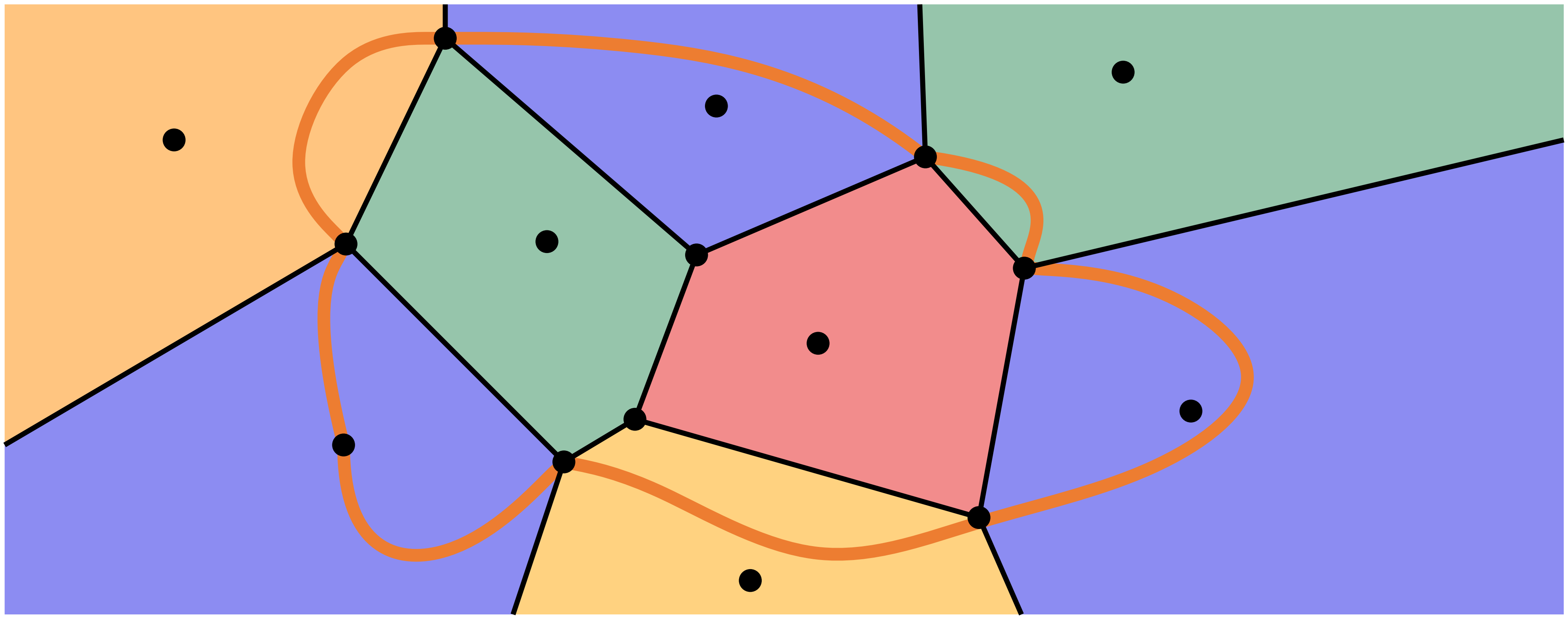

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Theorem [Marx, Pilipczuk '15]. Each graph like this has a balanced noose separator of size $\mathcal{O}(\sqrt{n})$.

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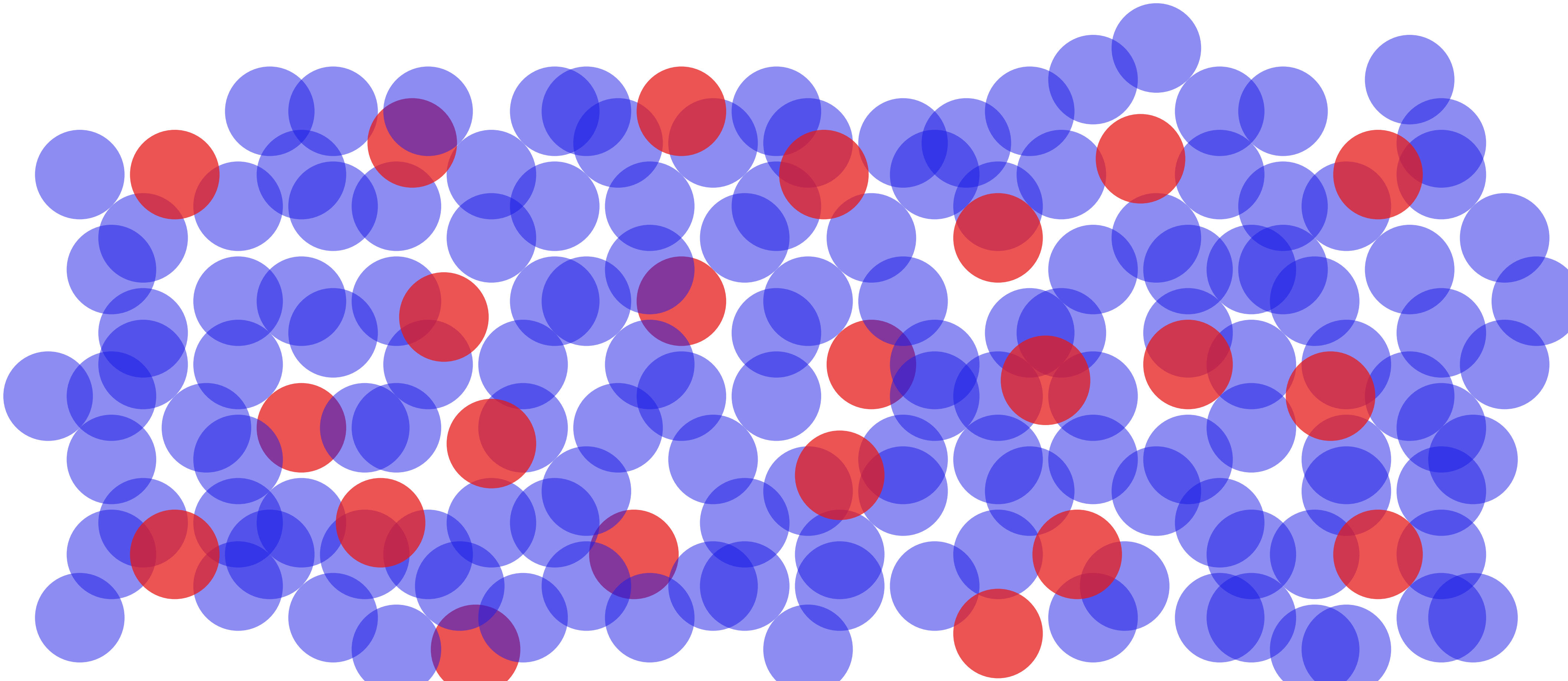

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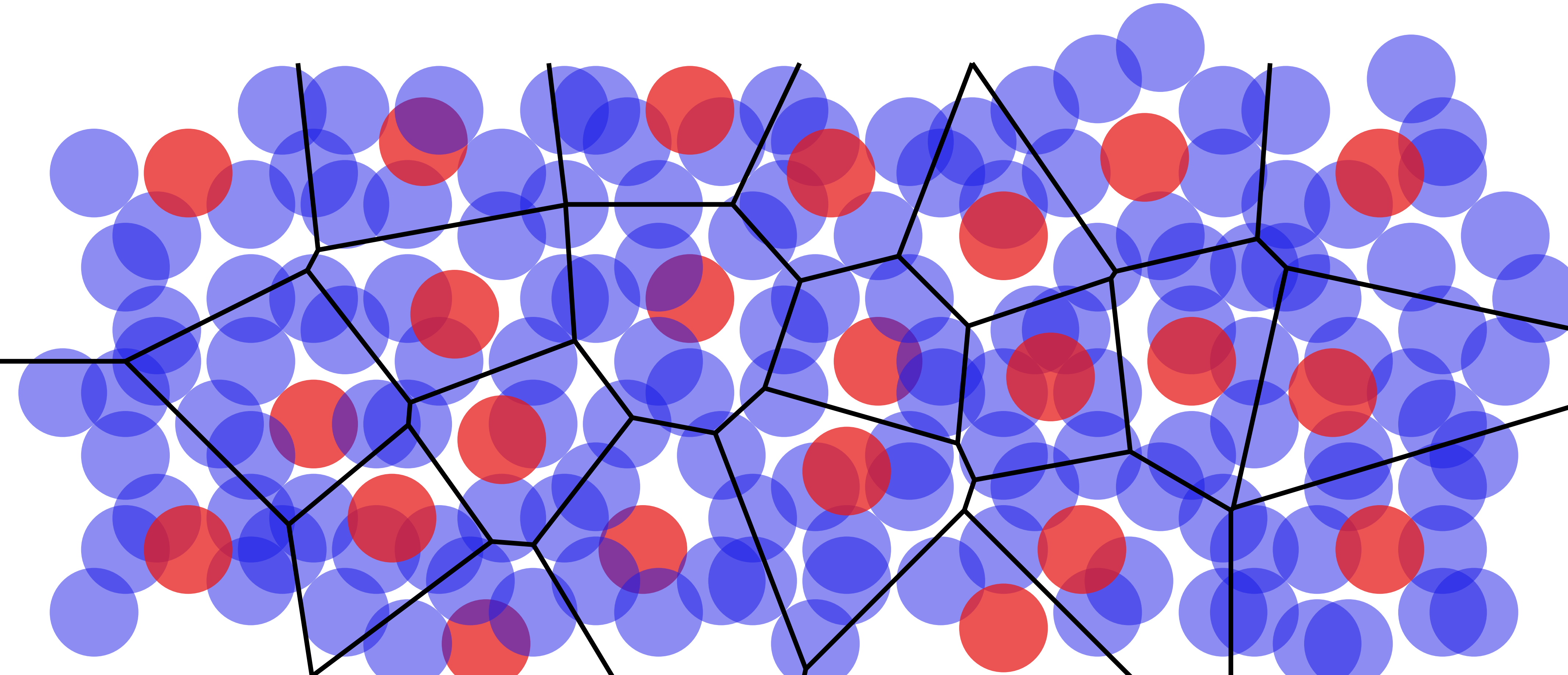

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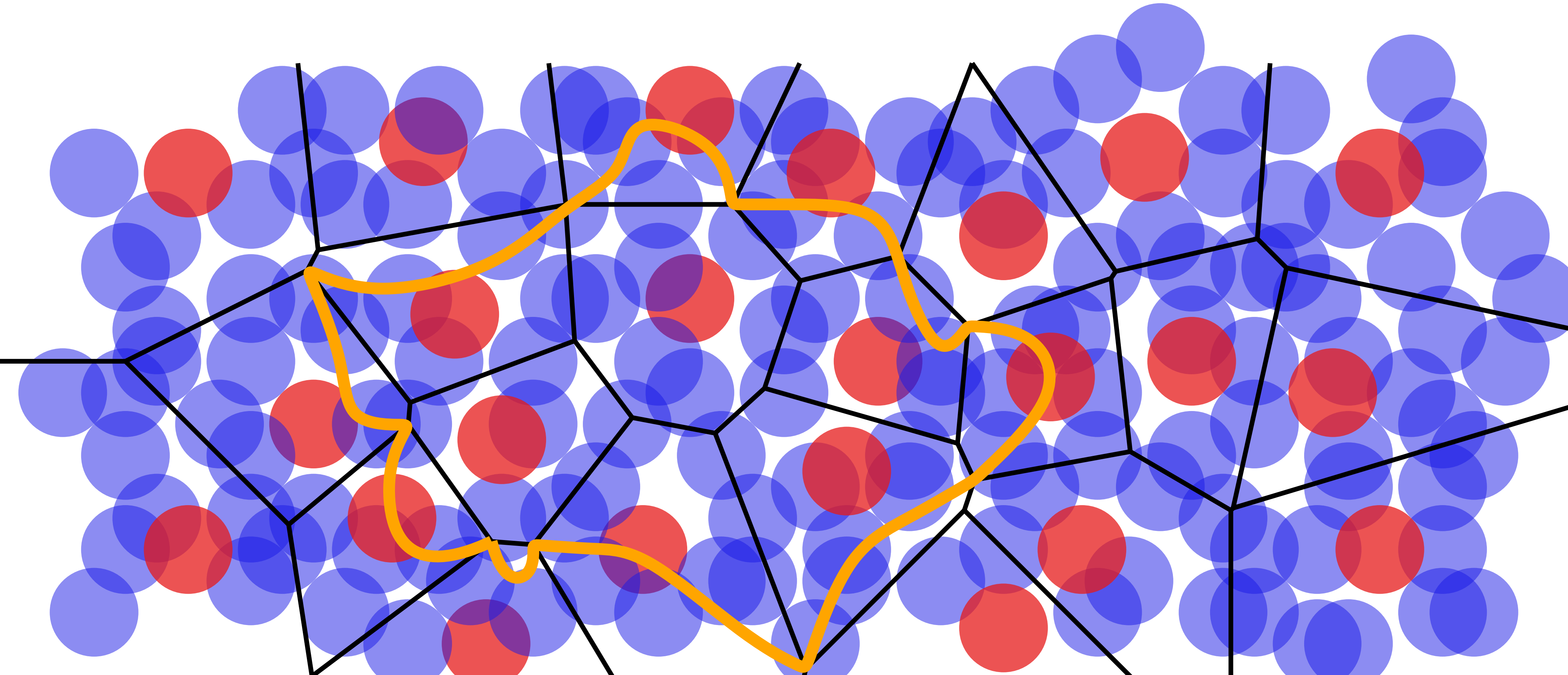

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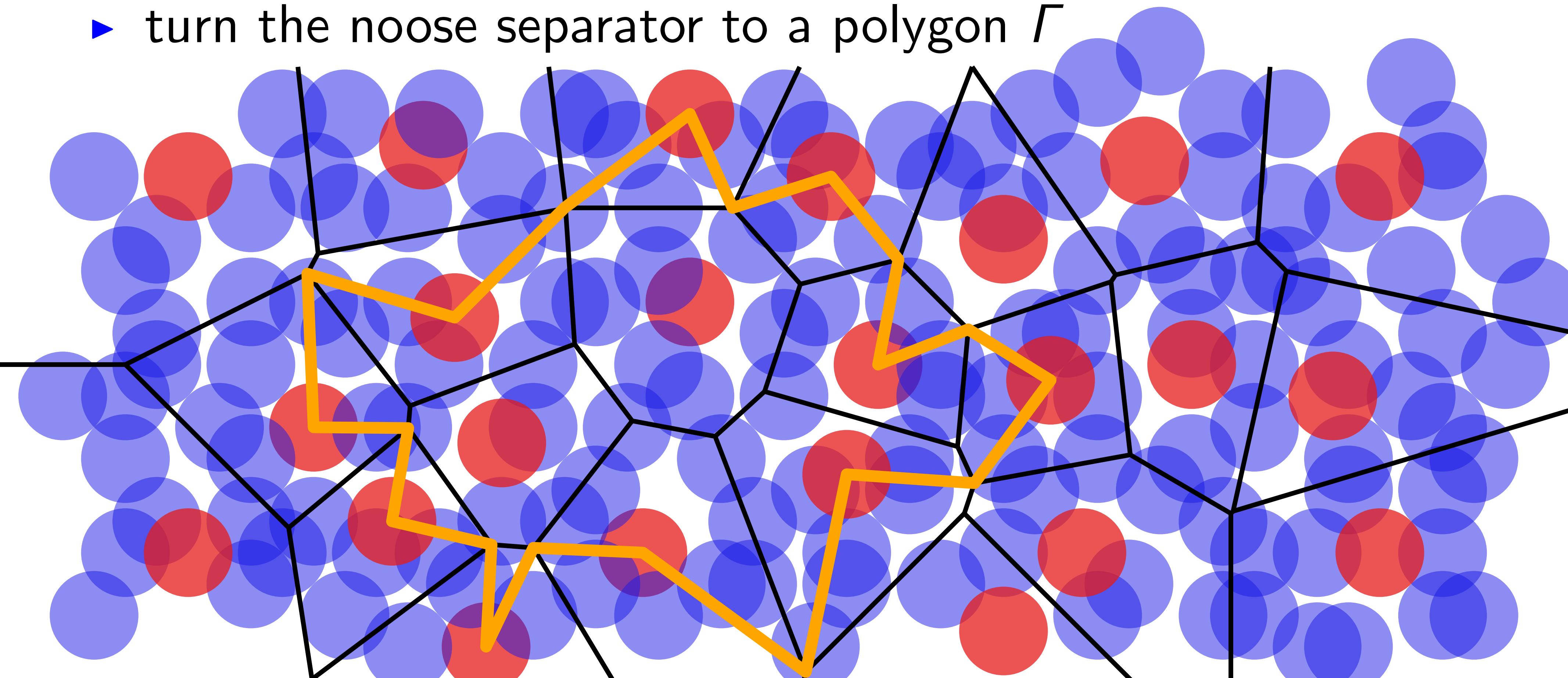

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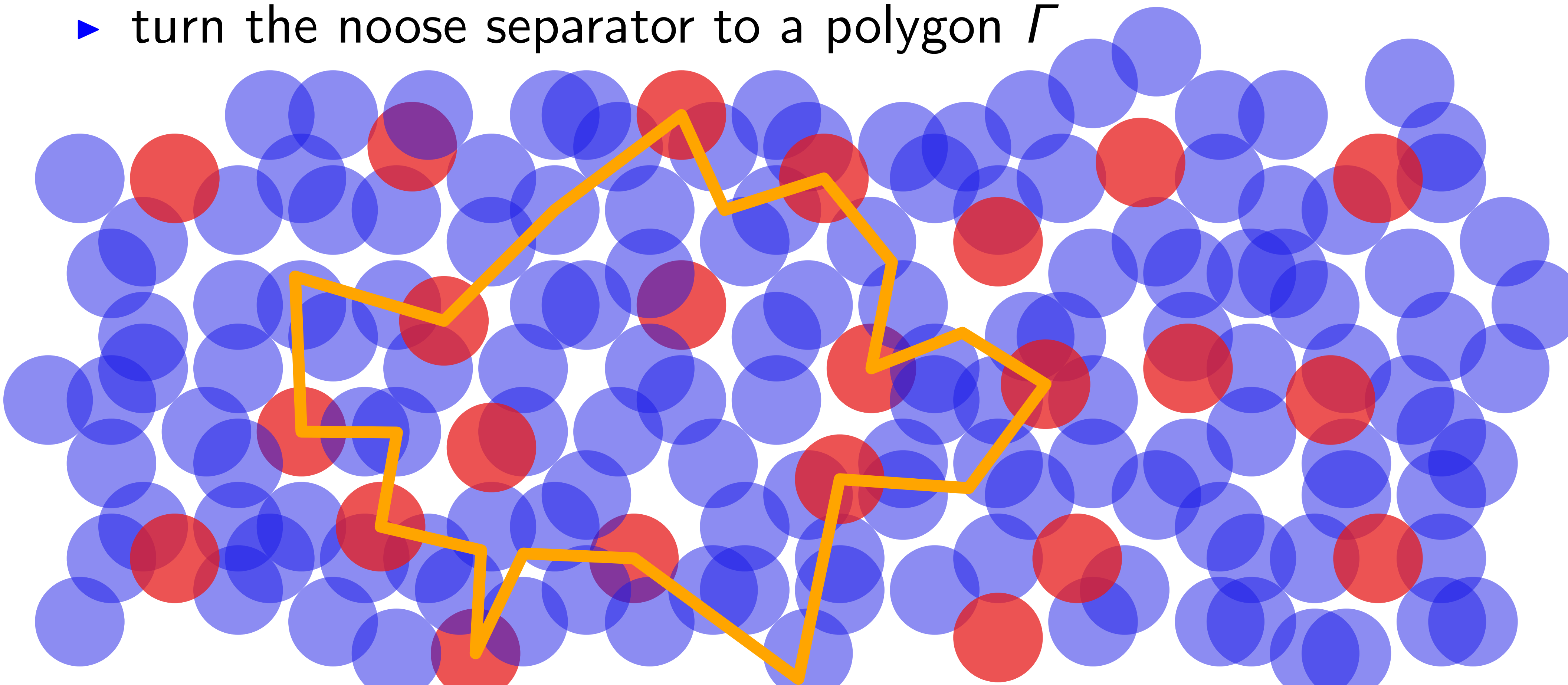

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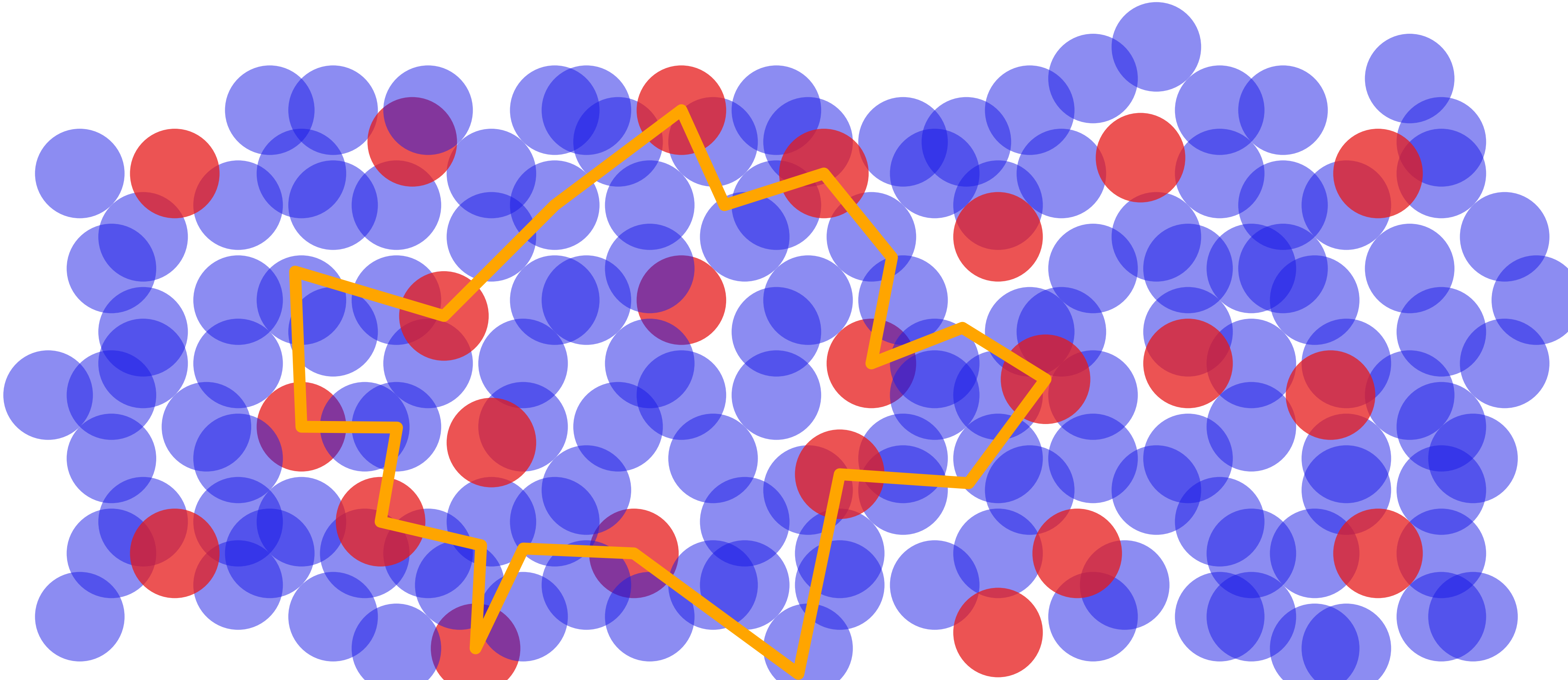

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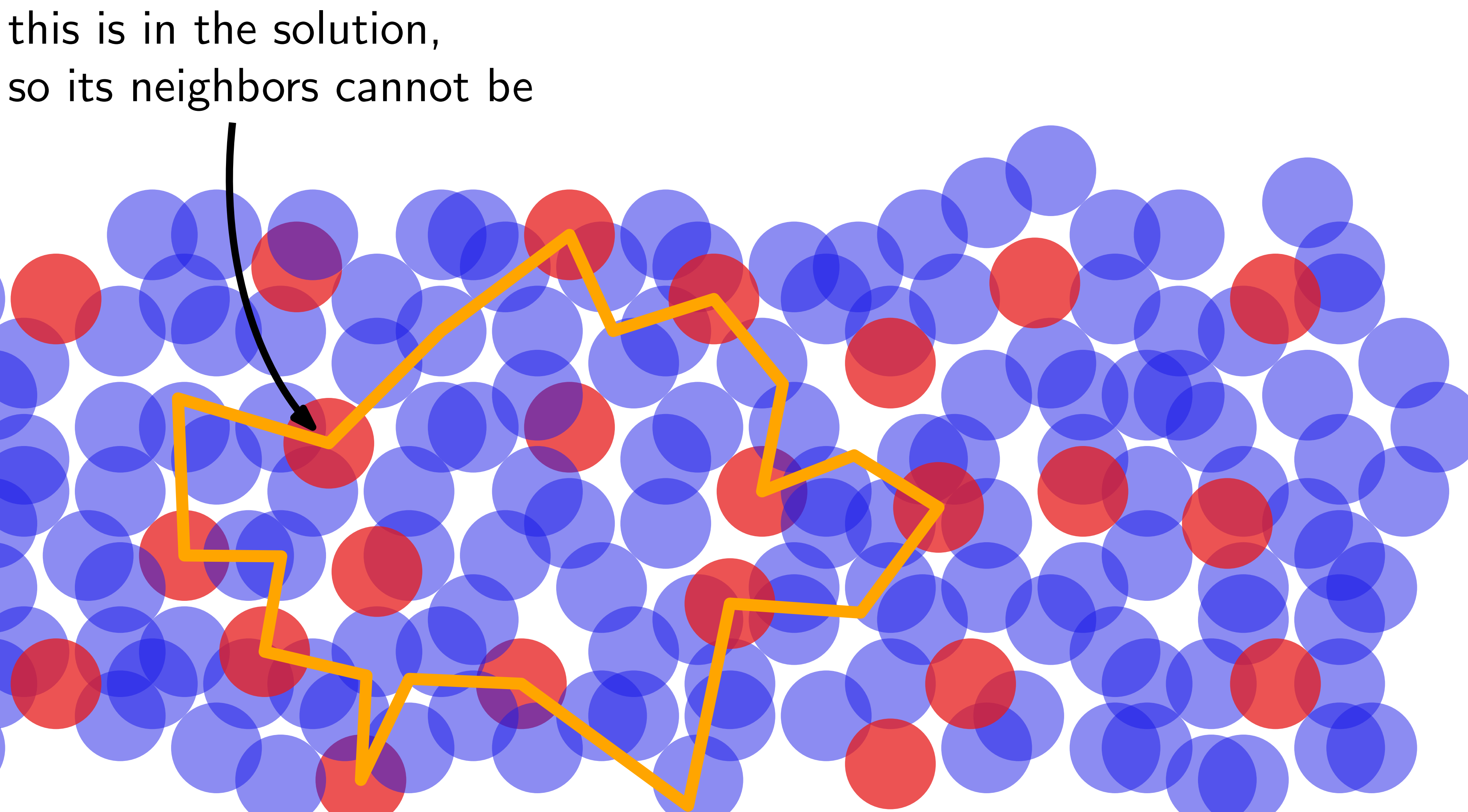

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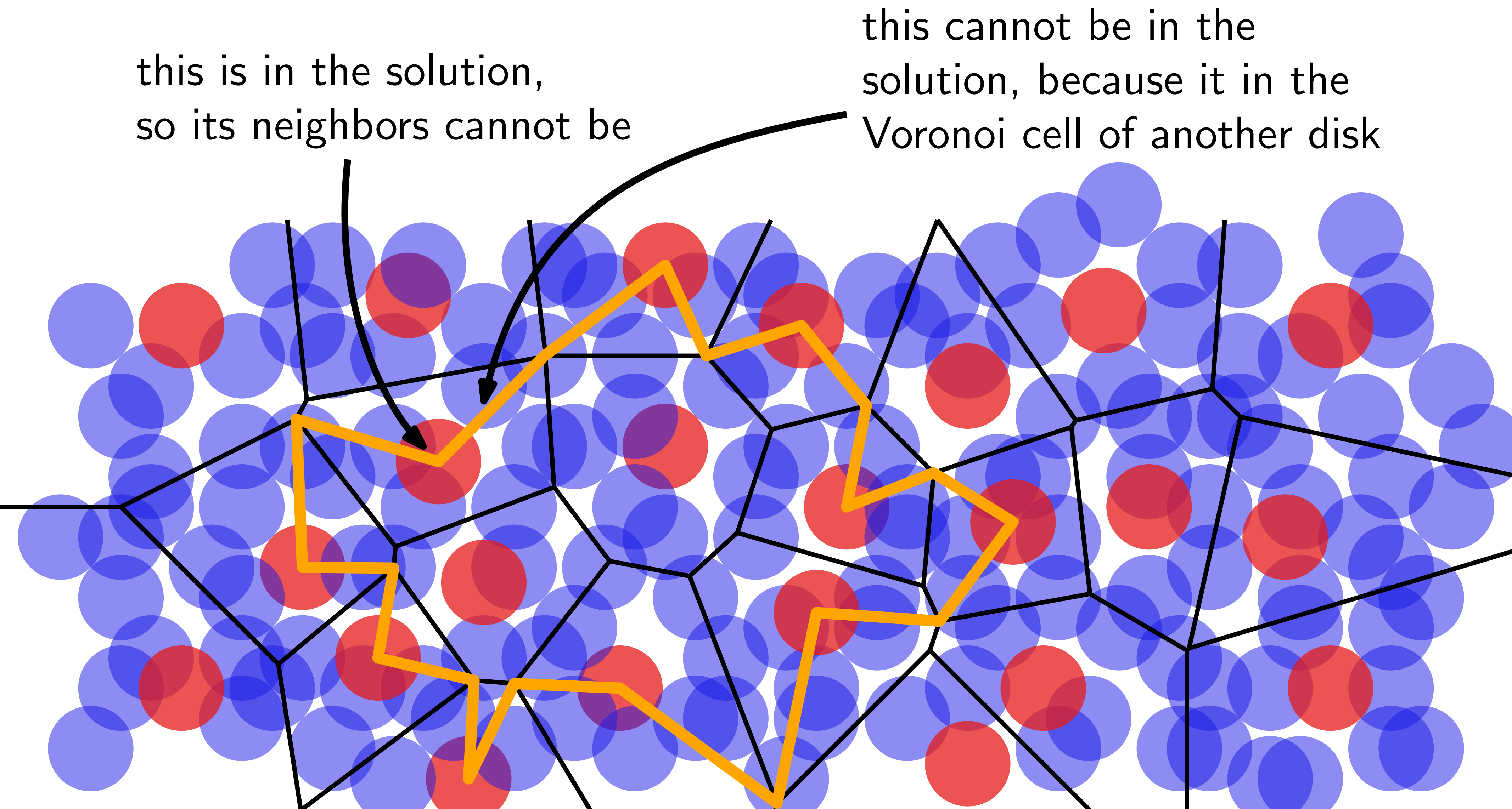

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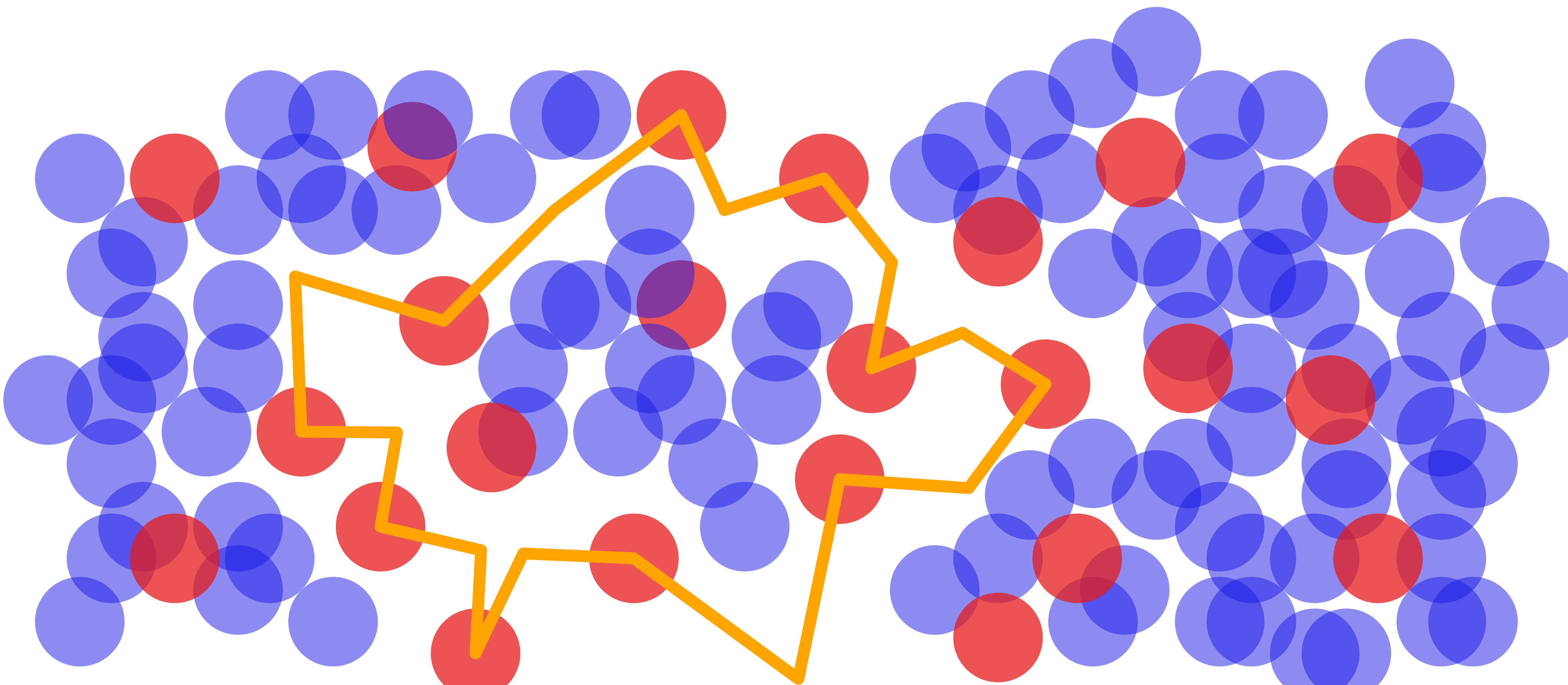
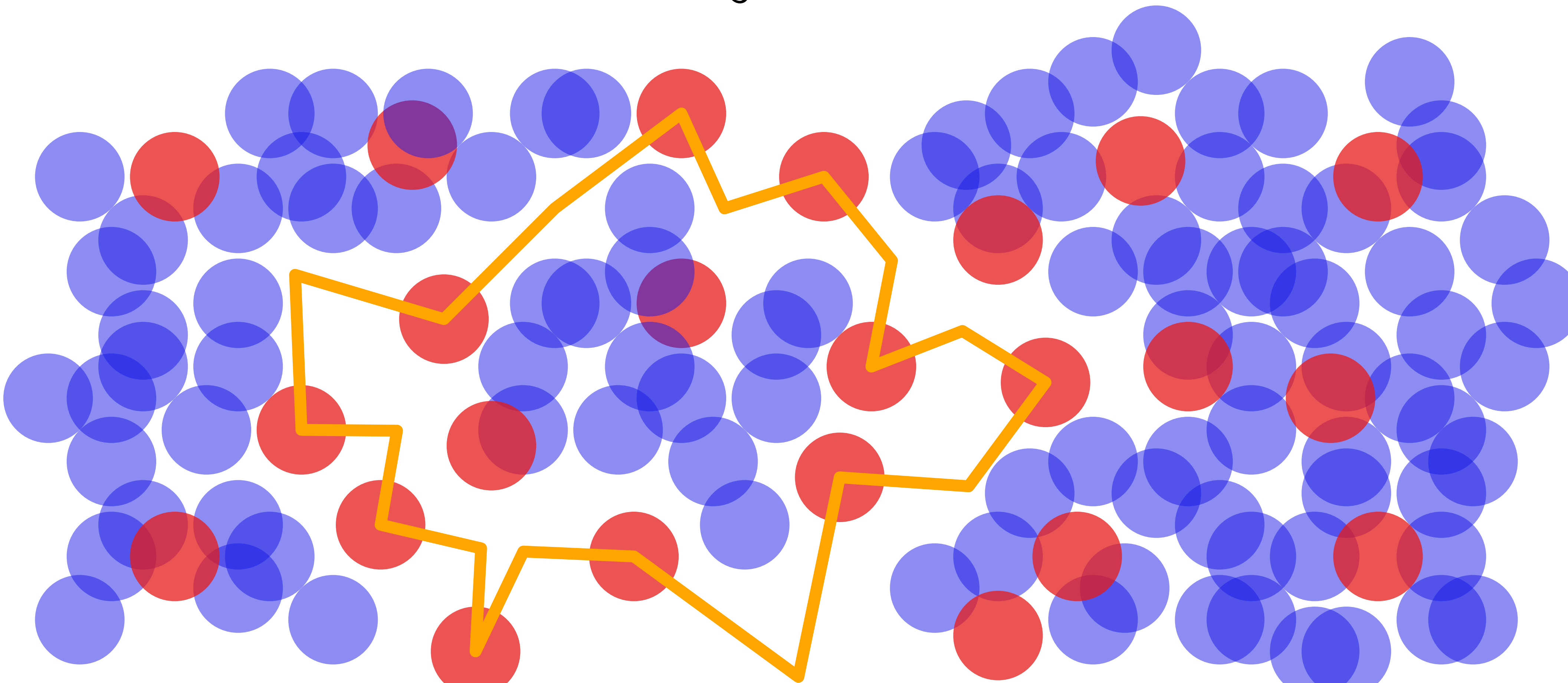


Separators in a solution Voronoi diagram

- every disk touching the outline of the polygon or any of the disks on its vertices can be discarded
- apply recursion to disks inside and outside the polygon, we look for a solutions of size k_1, k_2, where $k_1 + k_2 = k$ and $k_1, k_2 \leq \frac{2}{3} k$



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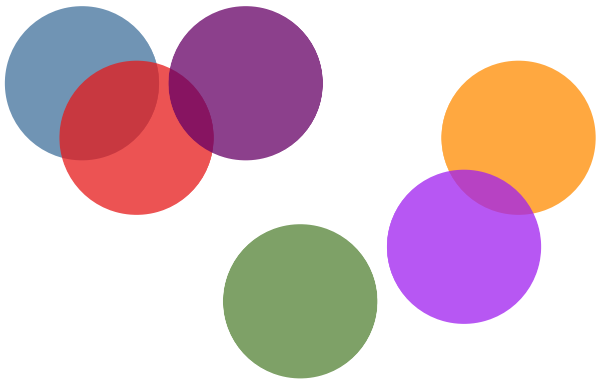
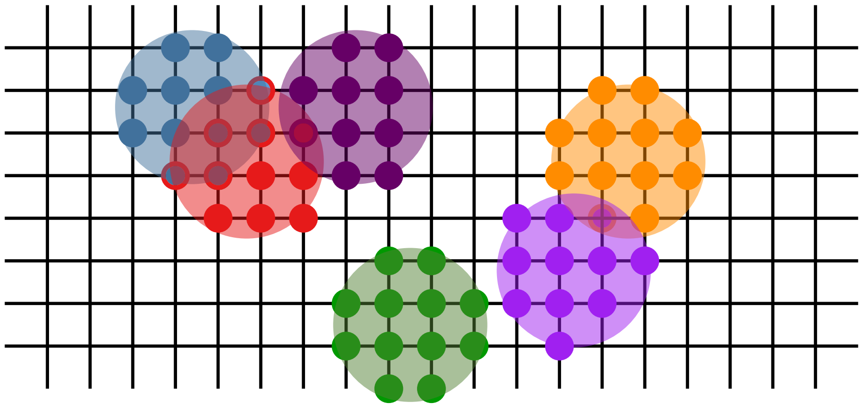
$$T(n, k) \leq n^{\mathcal{O}(\sqrt{k})} \cdot k^2 \cdot 2T(n, \tfrac{2}{3}k) = n^{\mathcal{O}(\sqrt{k})}$$

From disks to other geometric objects

- disks can be seen as connected subgraphs of a fine grid





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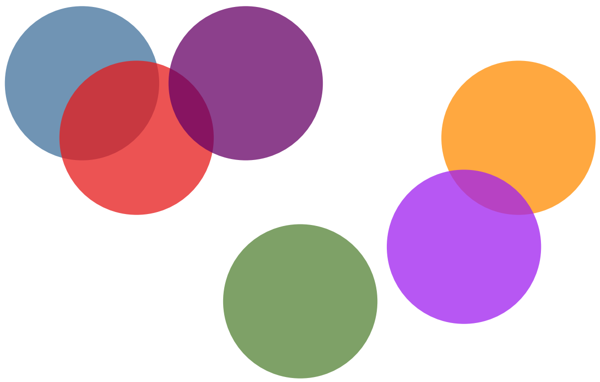
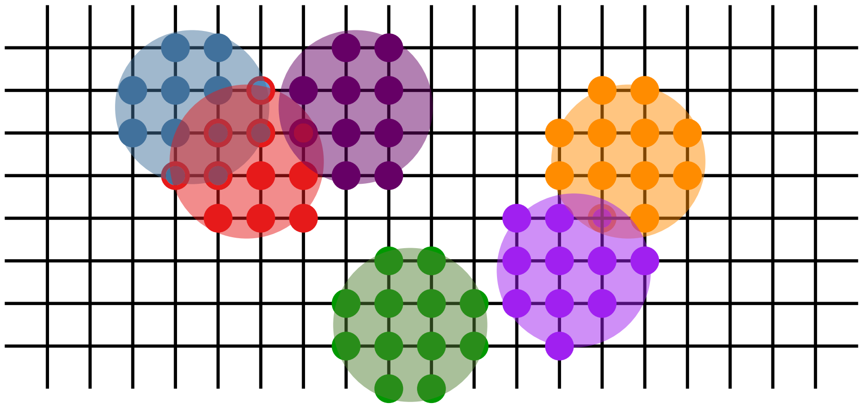
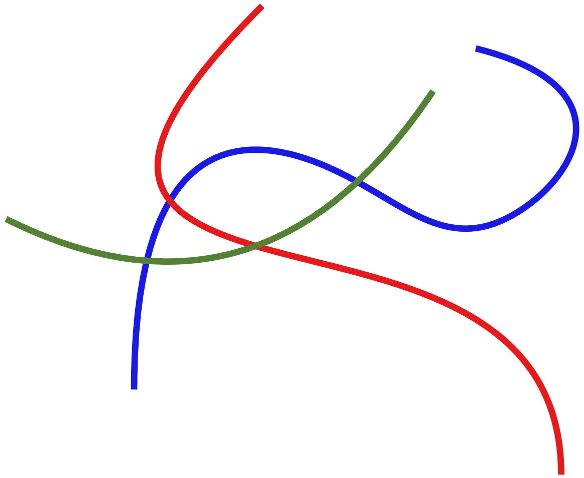
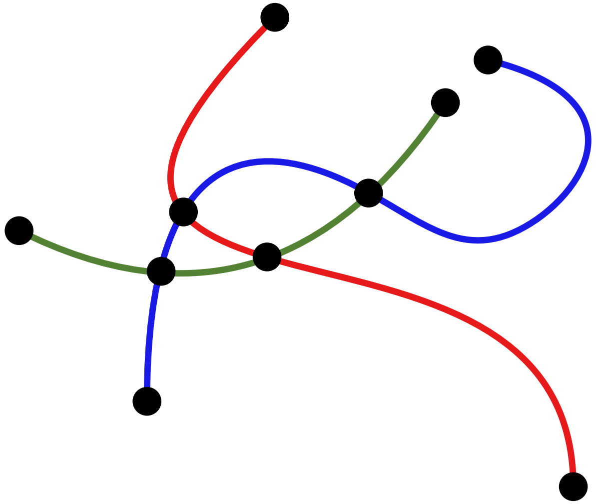




- string graphs = intersection graphs of connected subgraphs of planar graphs





General statement

- the whole approach can be re-interpreted in terms of packing disjoint subgraphs of planar graphs

Theorem [Marx, Pilipczuk '15].

Given a planar graph G with r vertices and n connected subgraphs of G, in time $n^{\mathcal{O}(\sqrt{k})} \cdot \mathrm{poly}(r)$ we can decide if there is a collection of k disjoint subgraphs.

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- no assumptions on area
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- for disks and segments $r = \mathrm{poly}(n)$
- Open question: For disk graphs, is there a robust algorithm for Independent Set with complexity $2^{o(k)}$ or $2^{\widetilde{\mathcal{O}}(\sqrt{n})}$?

Lower bounds
for parameterized algorithms

Parameterized lower bounds

- we know that k-Independent Set can be solved in time $n^{\mathcal{O}(\sqrt{k})}$ in disk graphs
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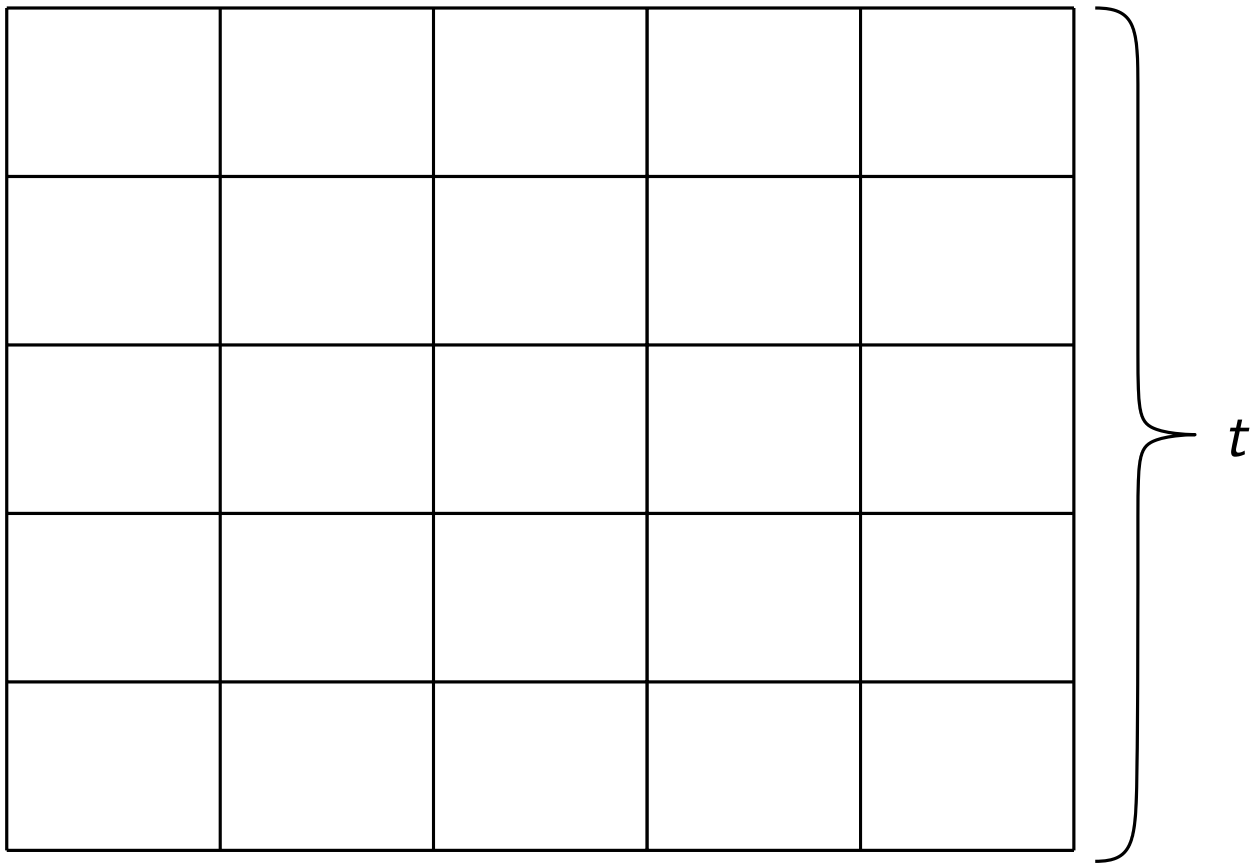
- we will need the following

Theorem.
Assuming the ETH, k-Clique cannot be solved in time $n^{o(k)}$.

- proof by a textbook reduction from 3-Sat

Grid Tiling

- we are given a square $t \times t$ grid



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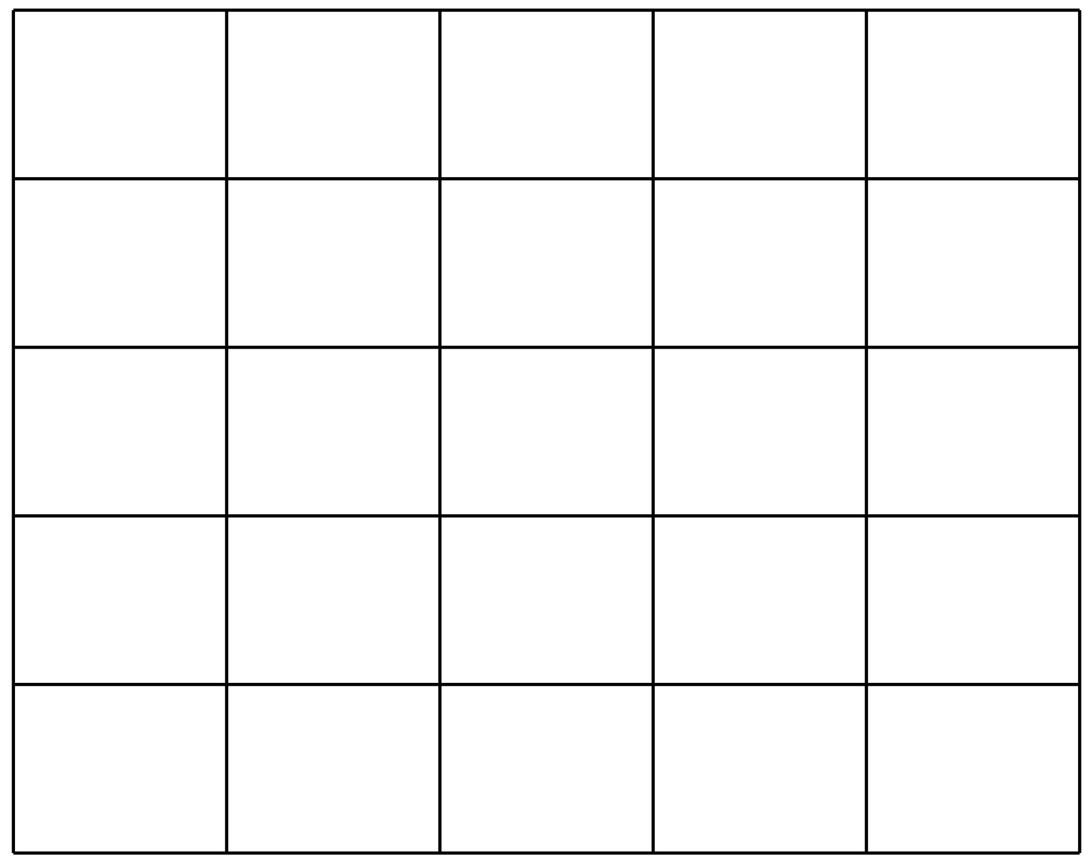

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- Sets for the cell (i, j):
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  - $(x, y) \in S_{i,j}$ if $xy \in E$

(i, i) $i \in [n]$	(i, j) $ij \in [n]$	(i, j) $ij \in [n]$	(i, j) $ij \in [n]$	(i, j) $ij \in [n]$
(i, j) $ij \in [n]$	(i, i) $i \in [n]$	(i, j) $ij \in [n]$	(i, j) $ij \in [n]$	(i, j) $ij \in [n]$
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Hardness of Grid Tiling

- $t \times t$ grid, each cell with some pairs from $[n] \times [n]$

Theorem. Grid Tiling cannot be solved in time $n^{o(t)}$, unless the ETH fails.

- reduction from k-Clique with vertices $1, 2, \ldots, n$, $t = k$
- Sets for the cell (i, j):
  - $(x, y) \in S_{i,i}$ if $x = y$
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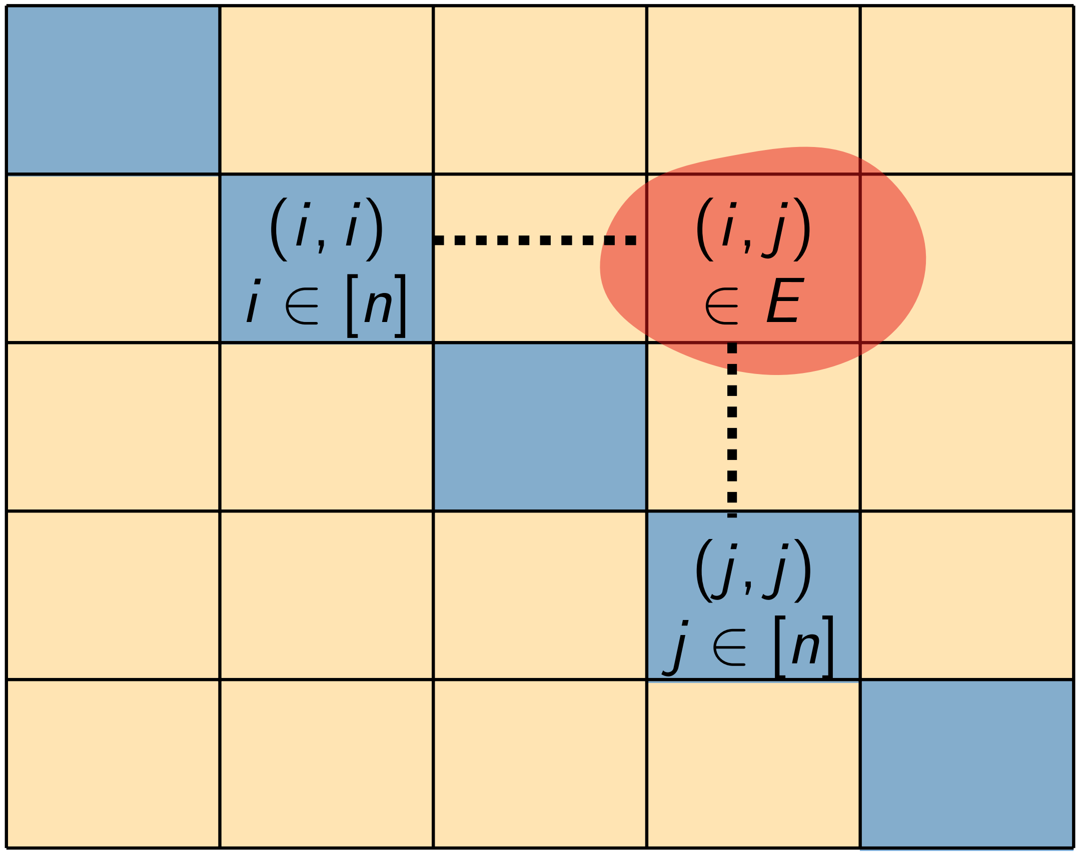

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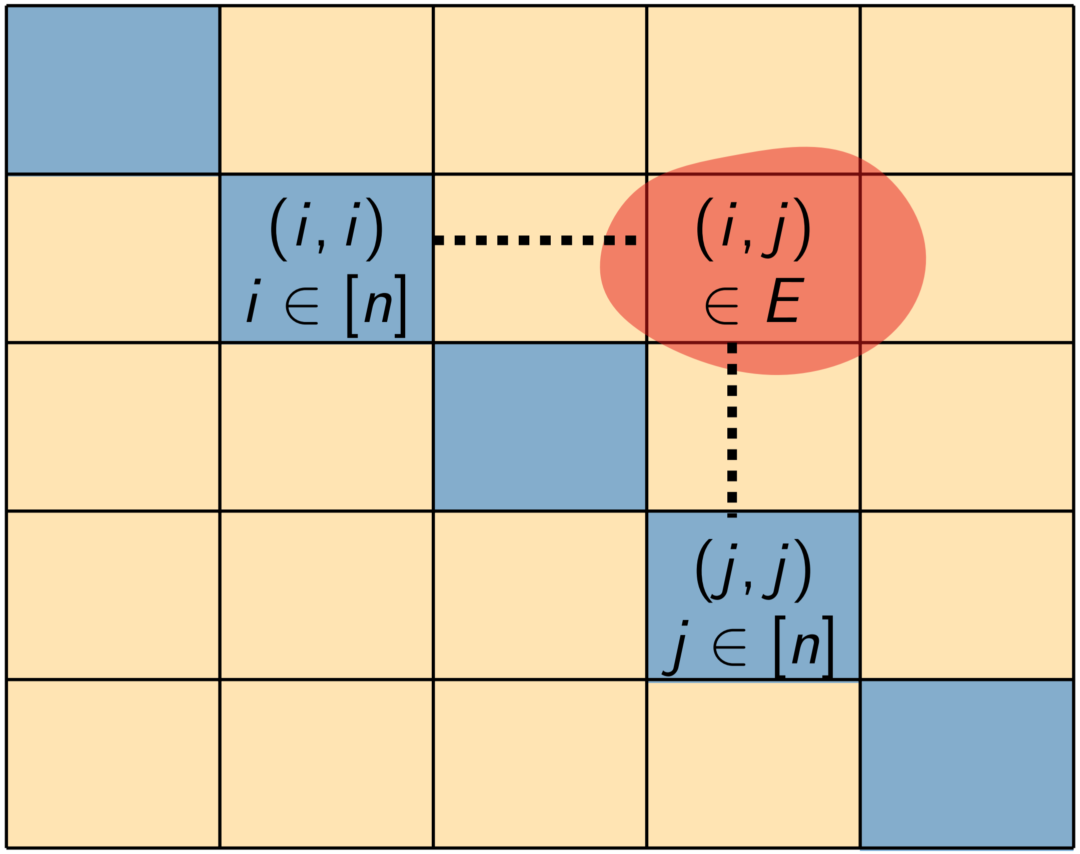
- solving Grid Tiling in time $n^{o(t)} \to$ solving k-Clique in time $n^{o(k)}$



Grid Tiling

- we are given a square $t \times t$ grid
- in each cell (i, j) we have $S_{i,j} \subseteq [n] \times [n]$
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- each set $S_{i,j}$ can be seen as points of $n \times n$ grid

(1,1)(1,2)(1,3)
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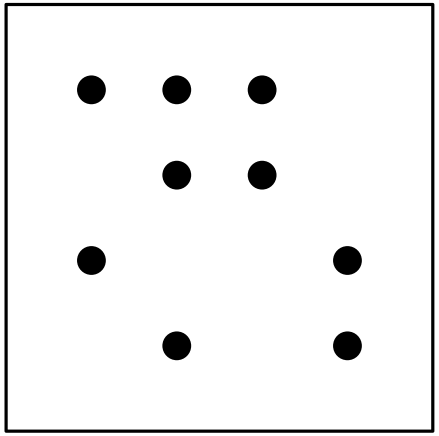

Hardness of Independent Set in UDGs

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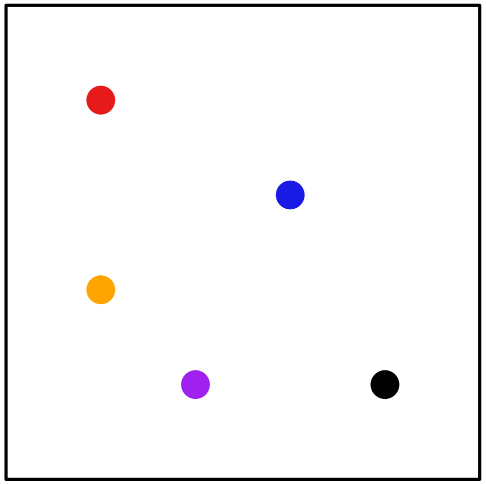

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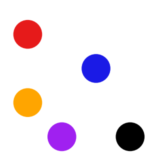

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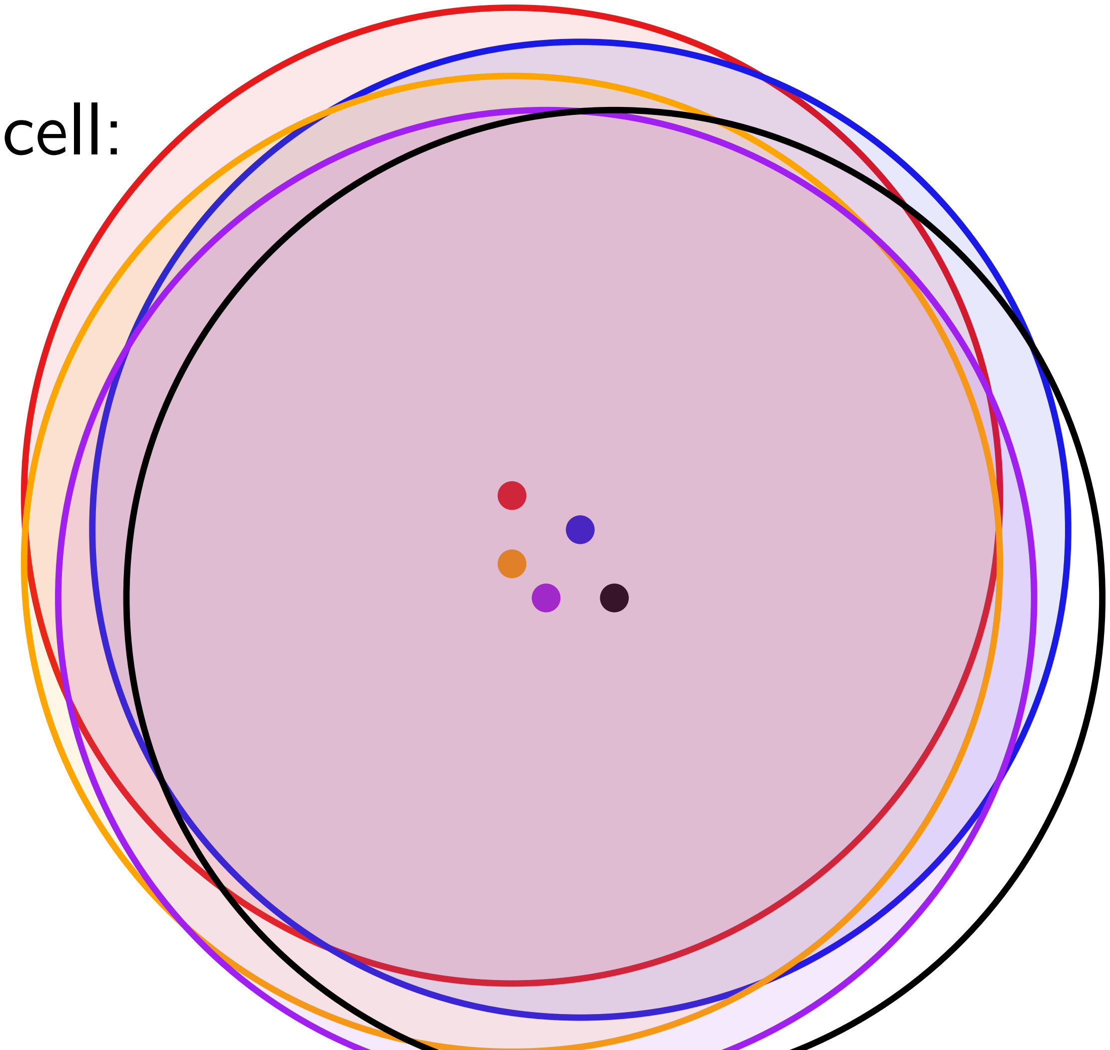
introduce unit disks centered at these points



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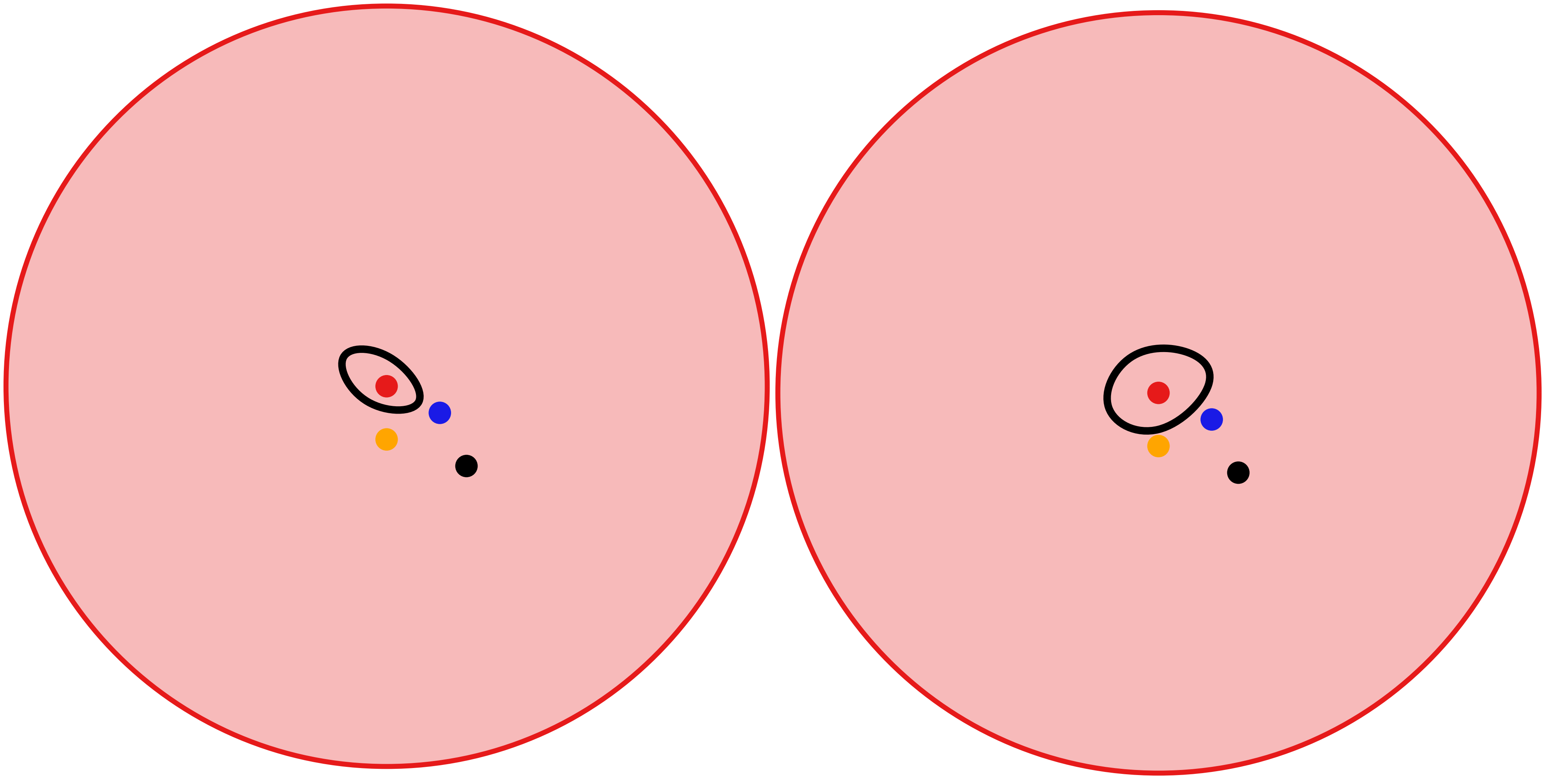

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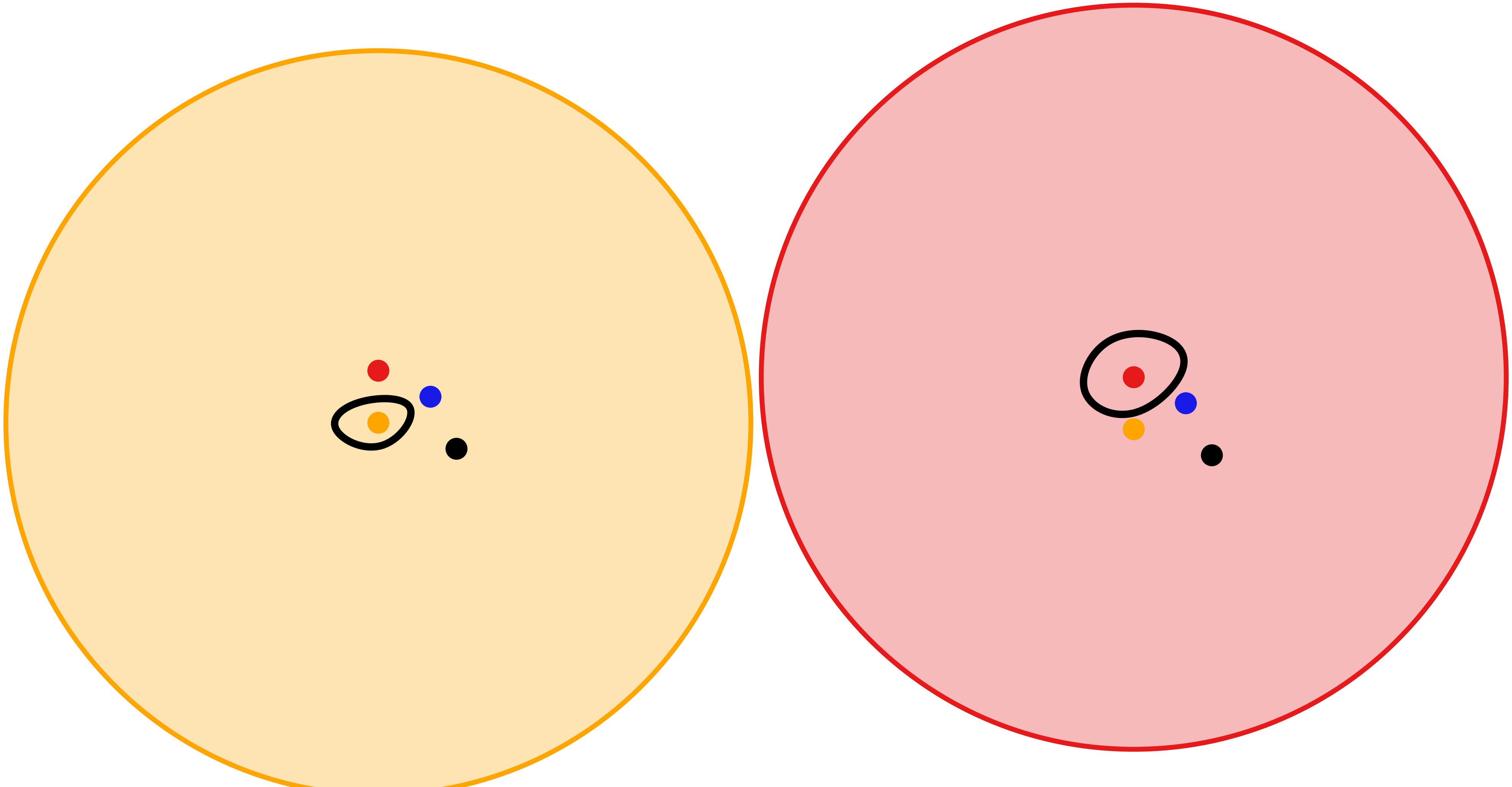

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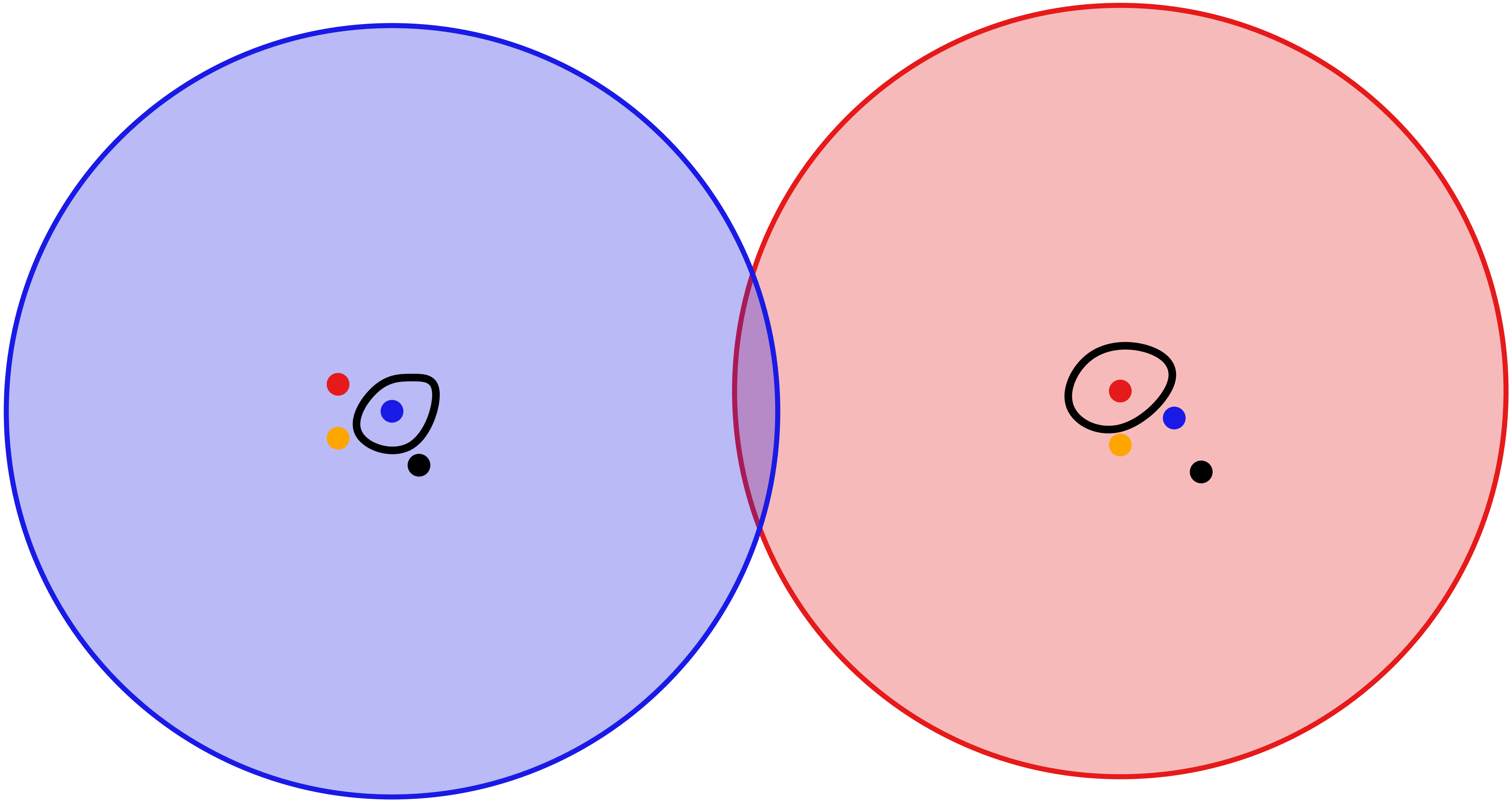

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- so the solution of size $k = t^2$ exists if and only if there is a solution for Grid Tiling
- number of disks $N \leq t^2 \cdot n^2$
- solving Independent Set in time $N^{o(\sqrt{k})}$
  $\rightarrow$ solving Grid Tiling in time $n^{o(t)}$ $\rightarrow$ the ETH fails $\square$

Other faces of Grid Tiling

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- there are also versions for any dimension d:

for INDEPENDENT SET: $2^{\mathcal{O}(k^{1-1/d})}$ [Marx, Sidiropoulos '15]
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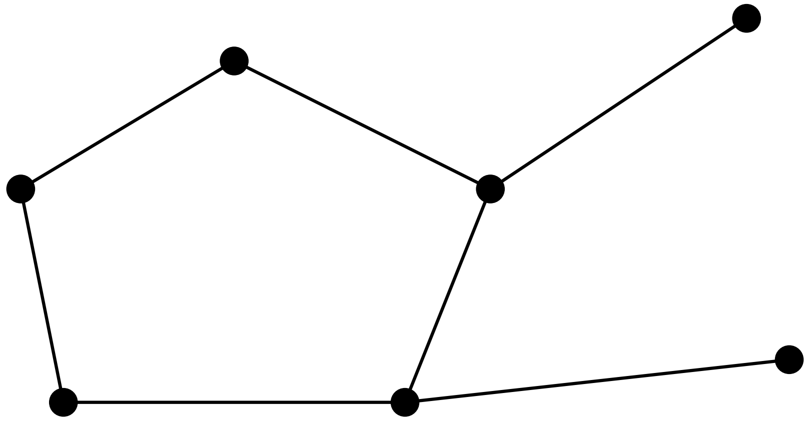
... but it's a different story

Bidimensionality in geometric graphs

Minors

- minor = a graph obtained by deleting vertices/edges and contracting edges



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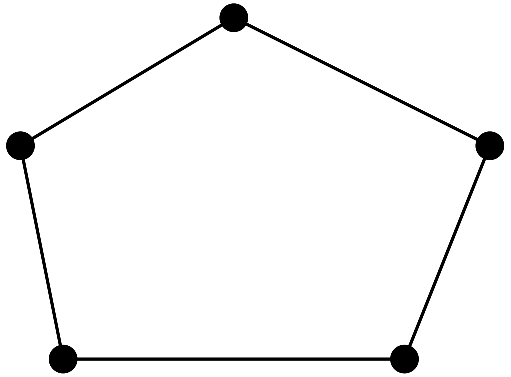

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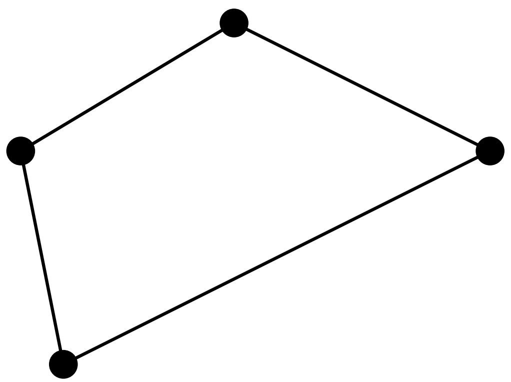

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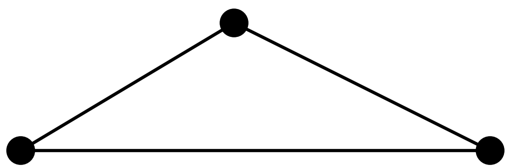

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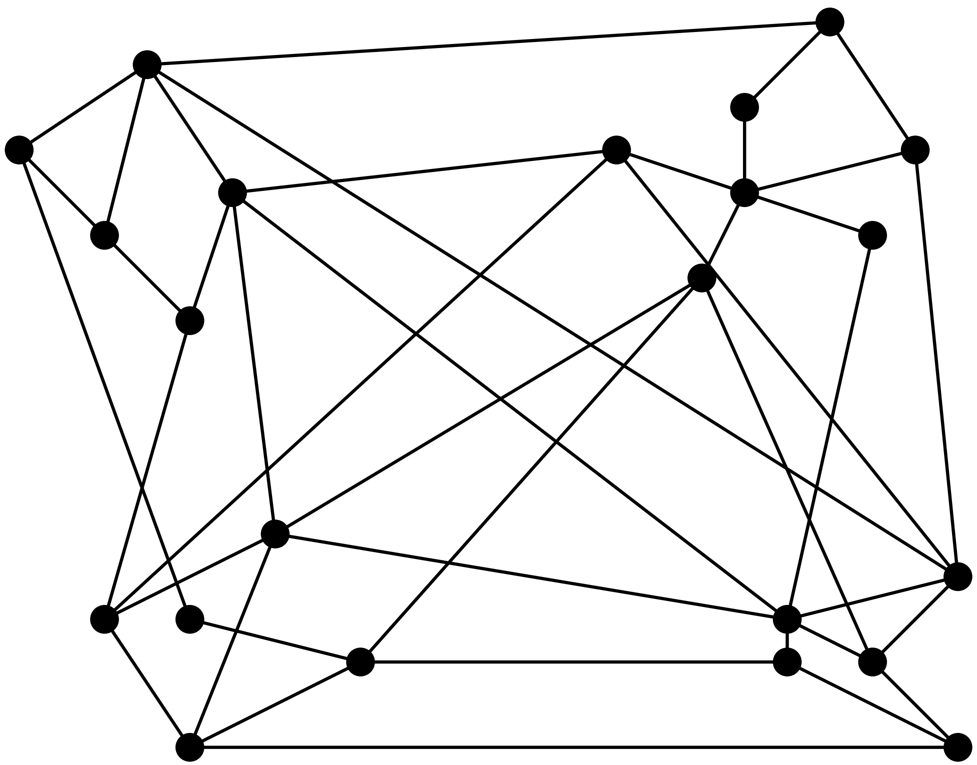

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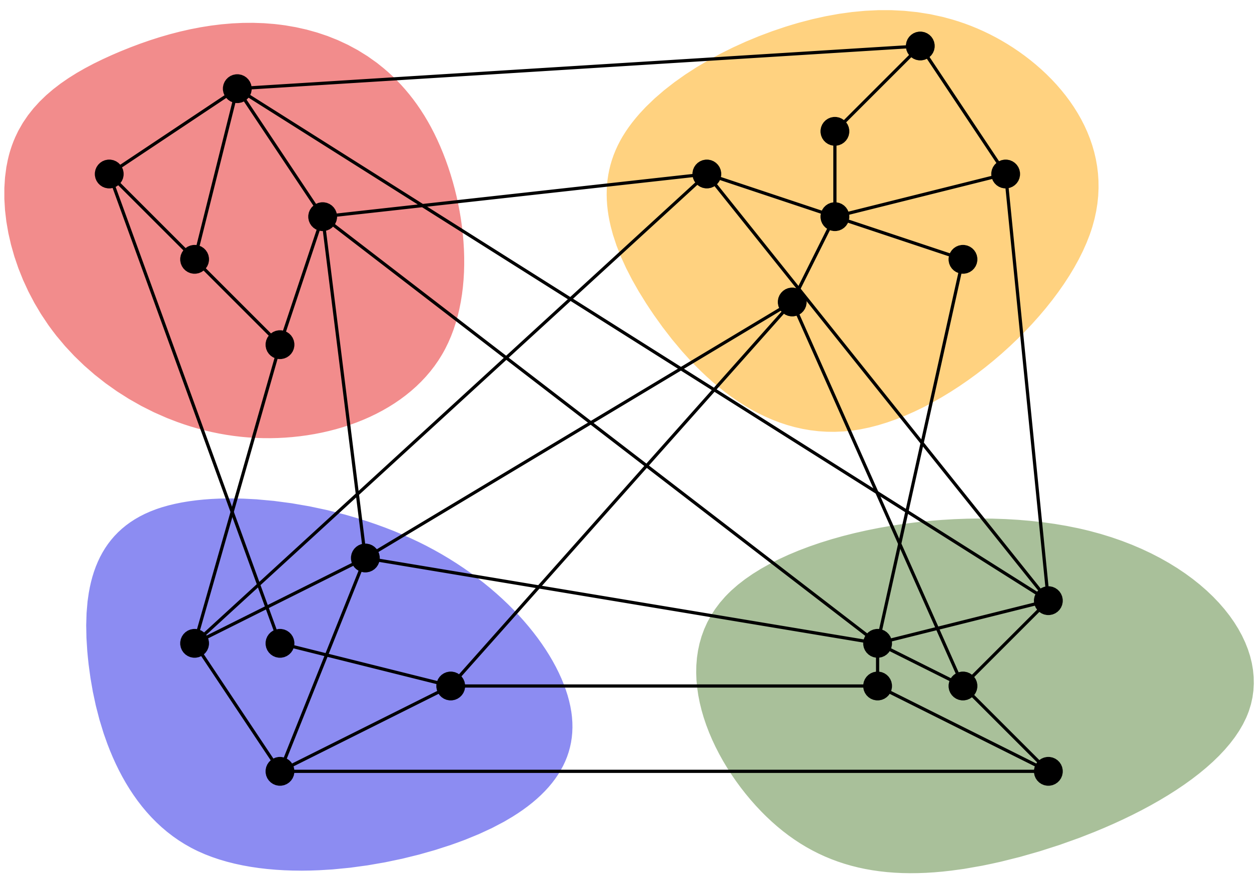

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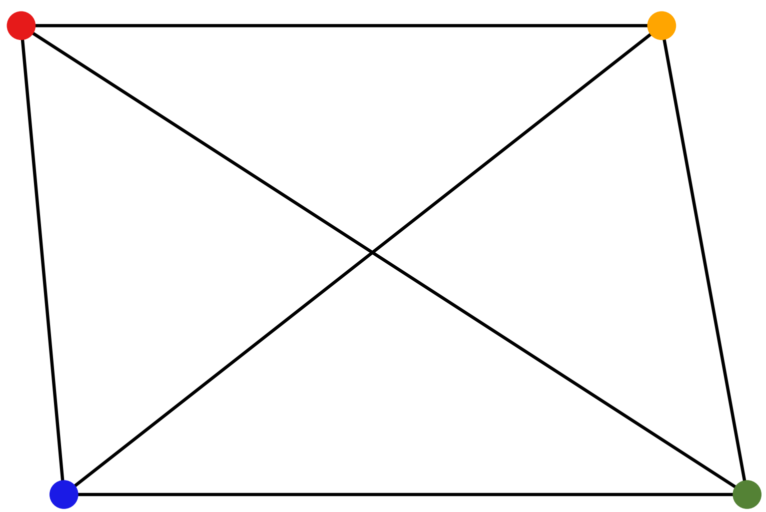
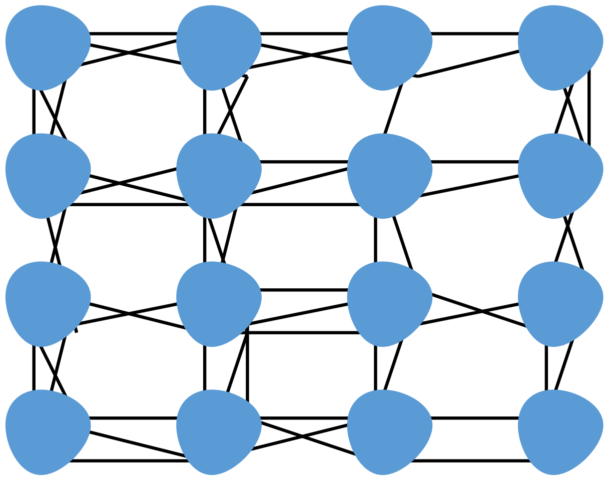


Grid minor theorem

- the presence of $t \times t$ grid minor forces treewidth $\geq t$



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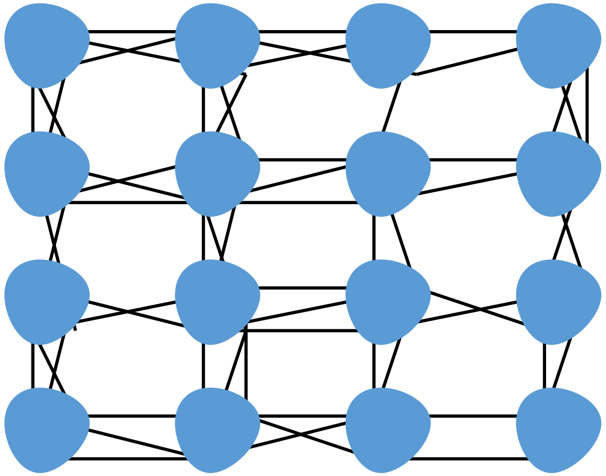

Grid minor theorem [Robertson, Seymour '86].
Every graph with treewidth $\geq f(t)$ contains a $t \times t$ grid minor.

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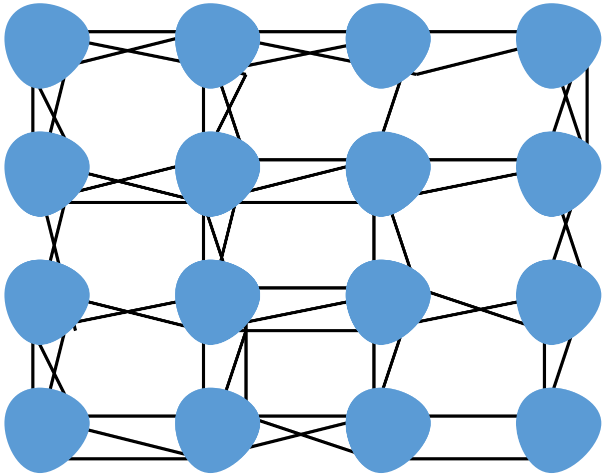

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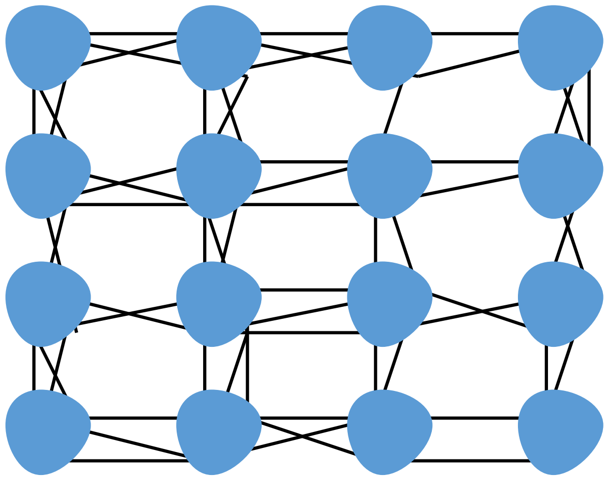

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Planar grid minor theorem [Robertson, Seymour, Thomas '94, Gu, Tamaki '12].
Every planar graph with treewidth $\geq 9/2 \cdot t$ contains a $t \times t$ grid minor. There is a poly-time algorithm for finding a grid or a tree decomposition.

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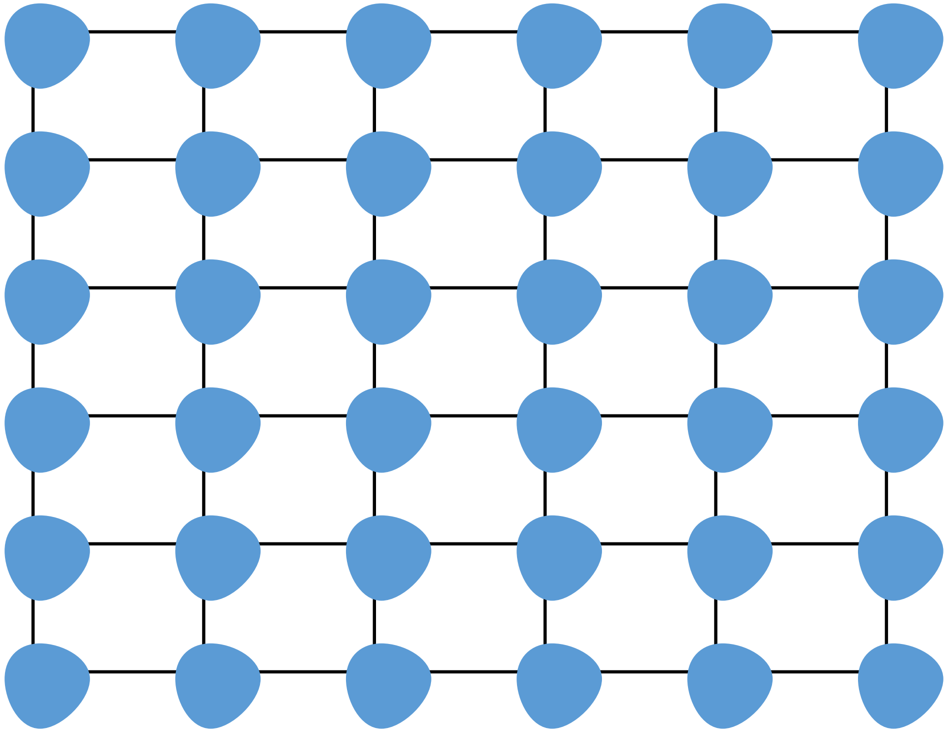

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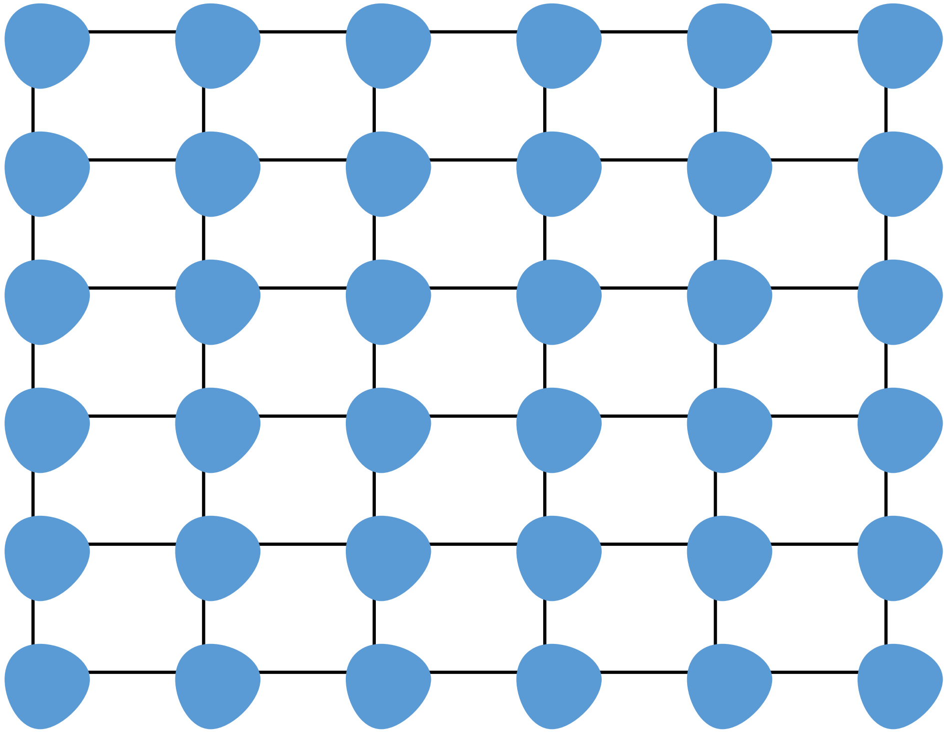

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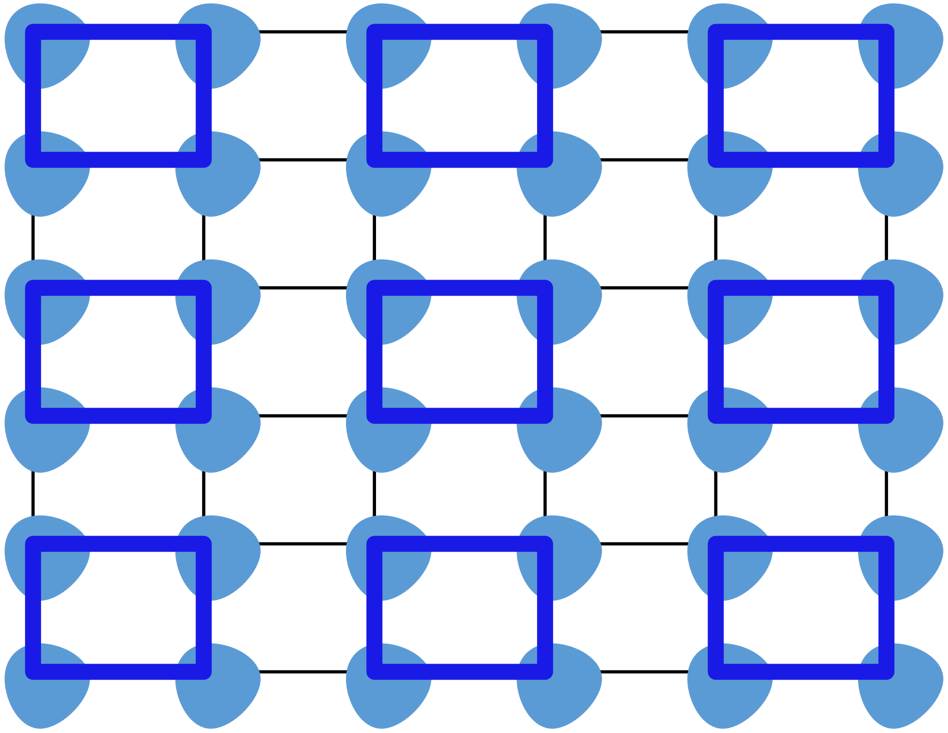

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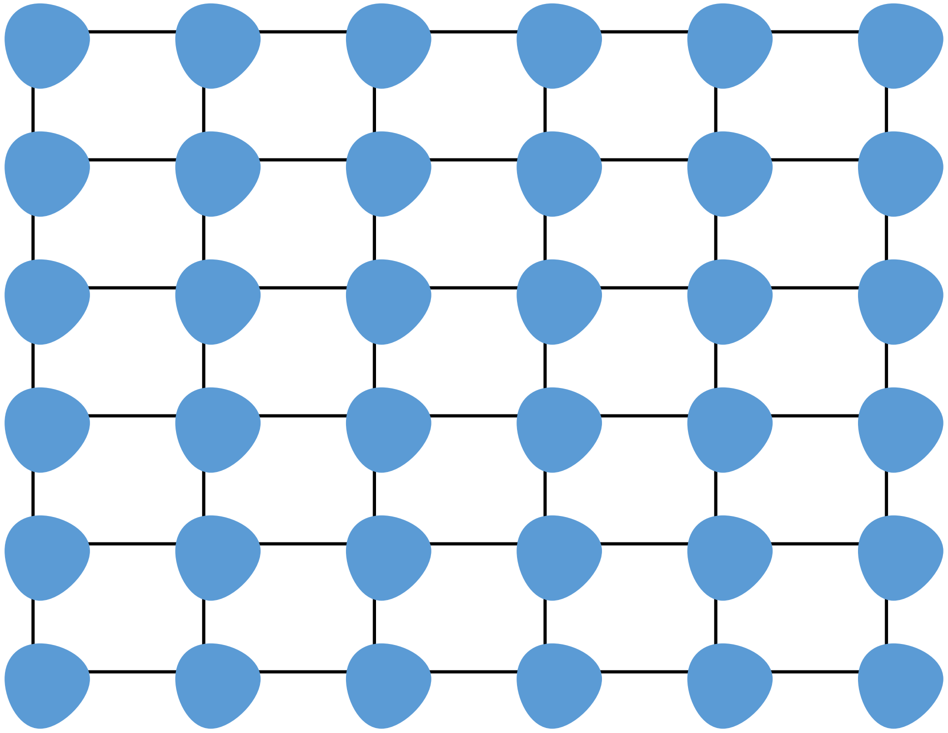

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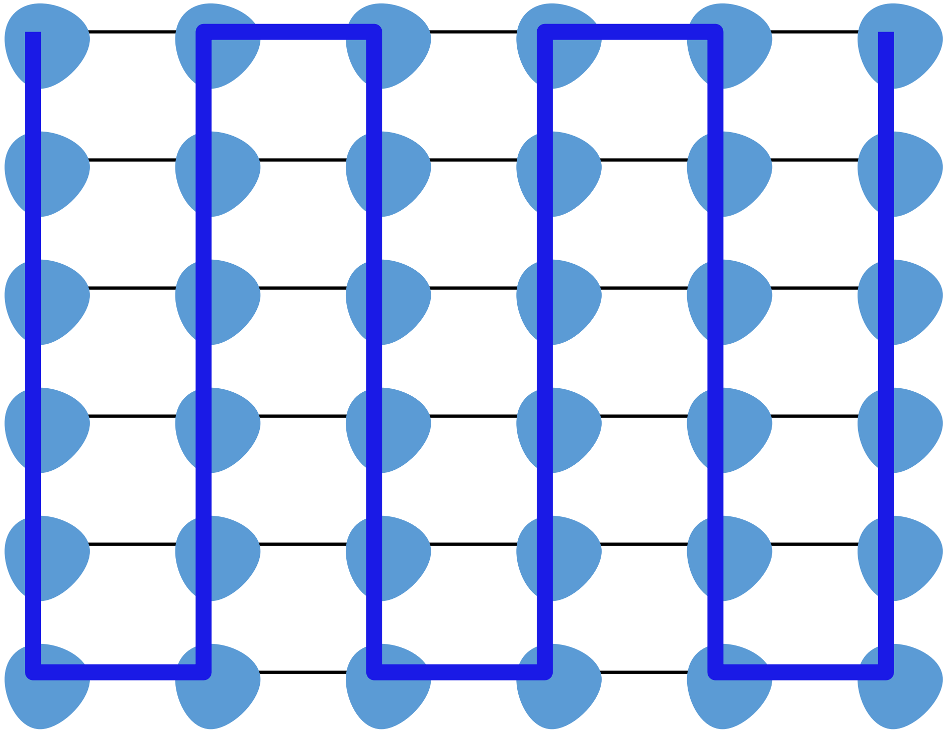

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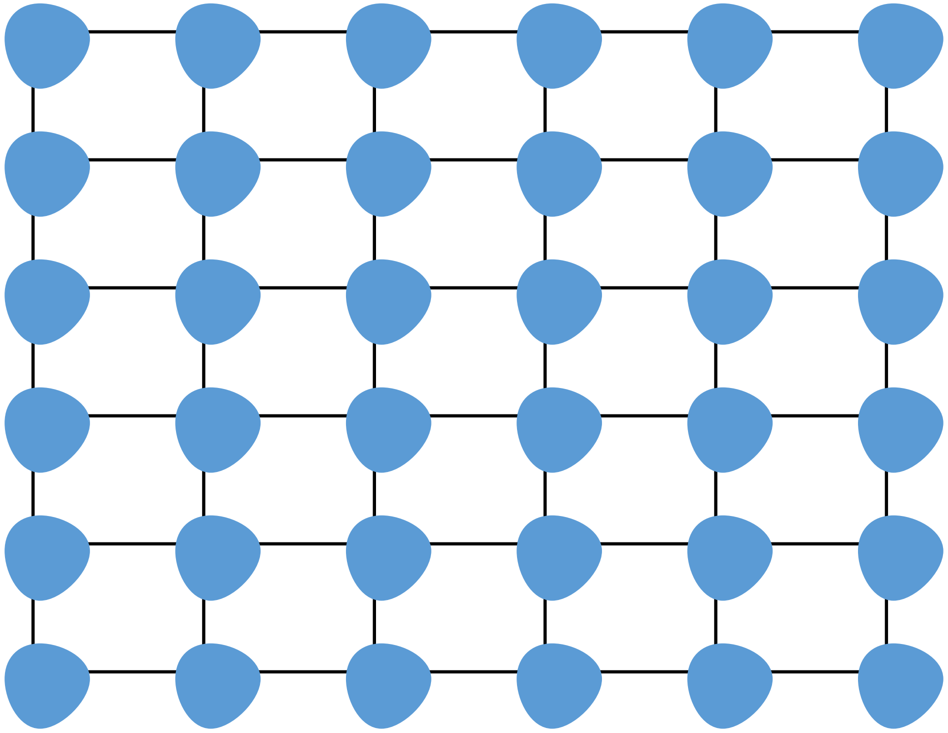

- $2^{\widetilde{\mathcal{O}}(\sqrt{k})} \cdot \text{poly}(n)$-algorithms for many parameterized problems

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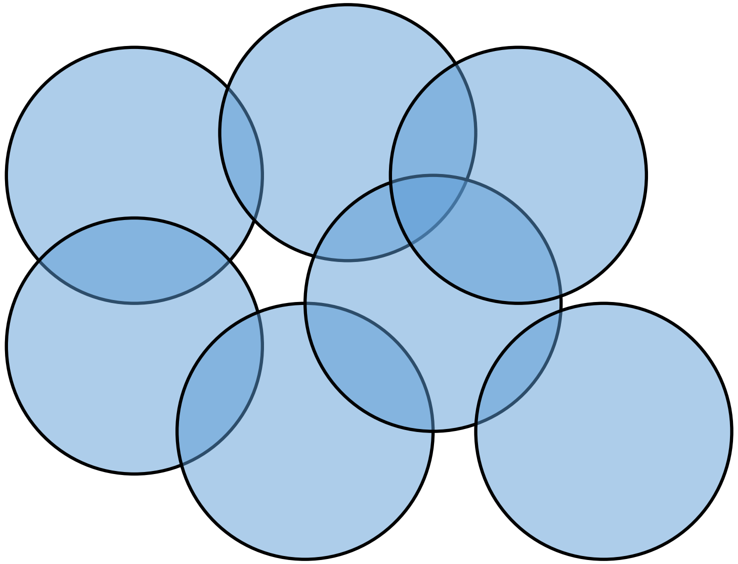

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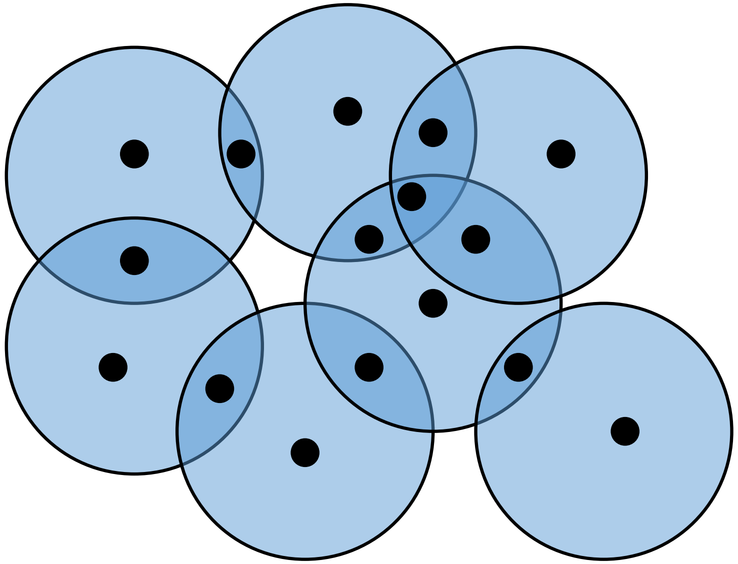

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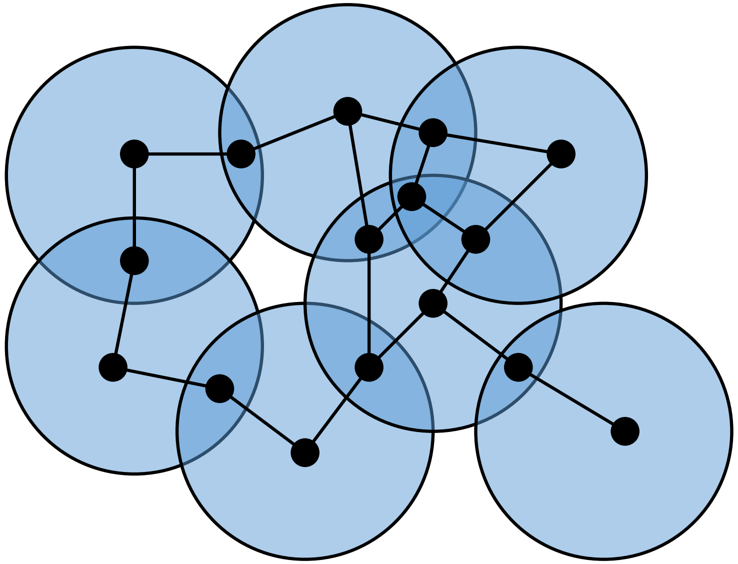

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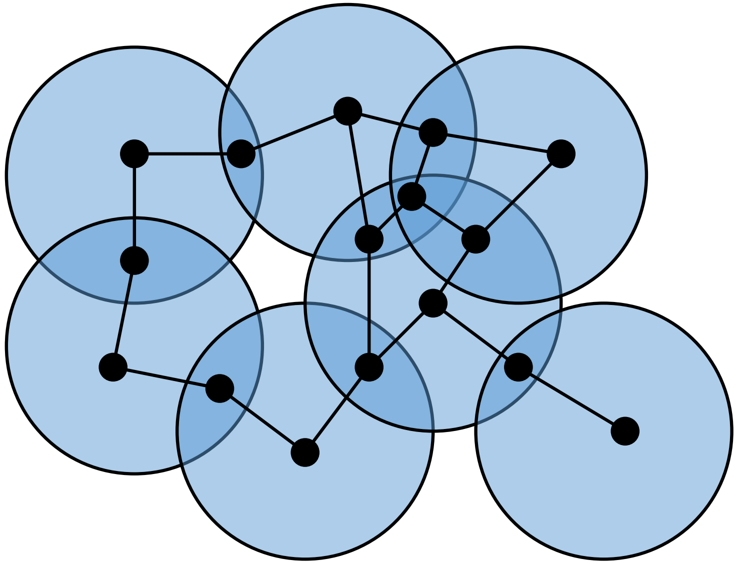

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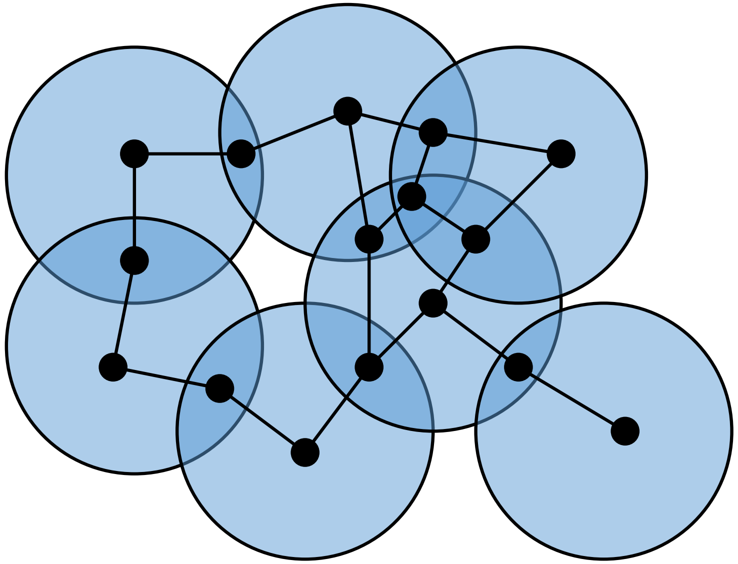

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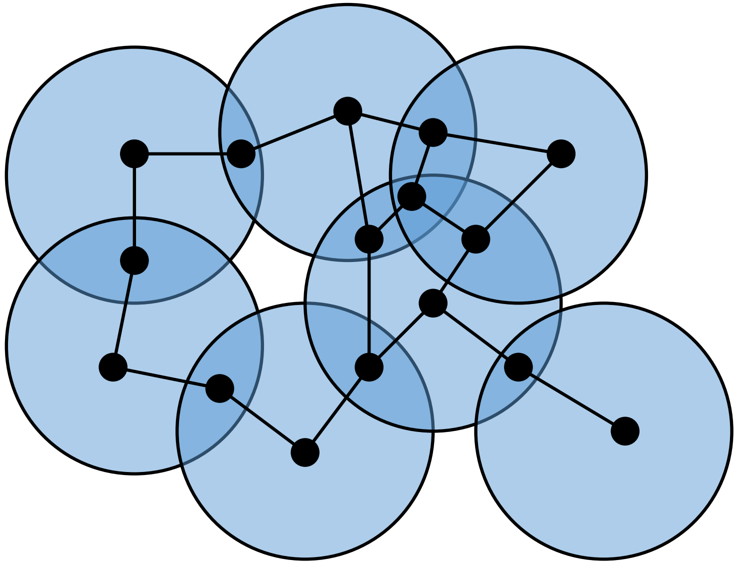

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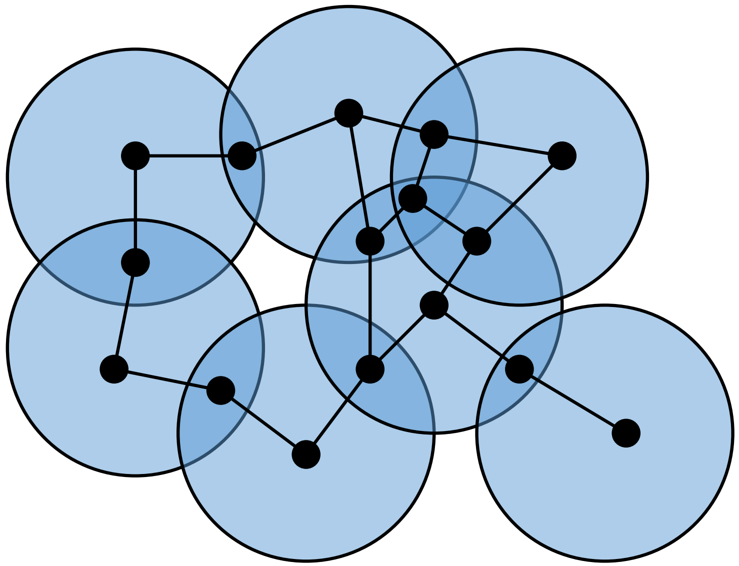

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- using this, we construct a $t' \times t'$ grid minor in G, where $t' = \mathcal{O}(t) = \mathcal{O}(\mathrm{tw}(G))$

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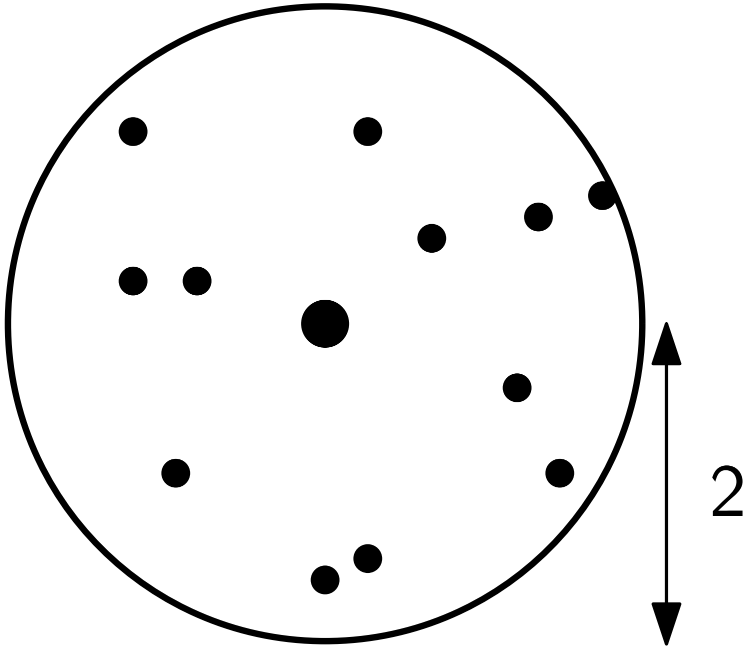
- take a vertex of degree Δ
- centers or all neighbors are in the radius-2 disk



Grid minor theorem for unit disk graphs

Lemma[Fomin, Lokshtanov, Saurabh '11].

Every unit disk graph G with bounded maximum degree and treewidth $\Omega(t)$ has a $t \times t$ grid minor.

- if G has no clique of size p, then $\Delta \leq 6p$
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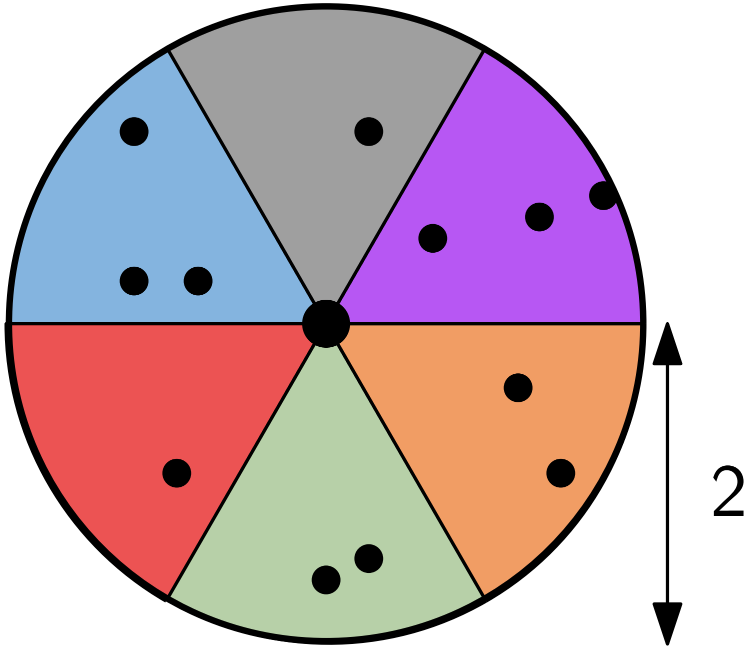

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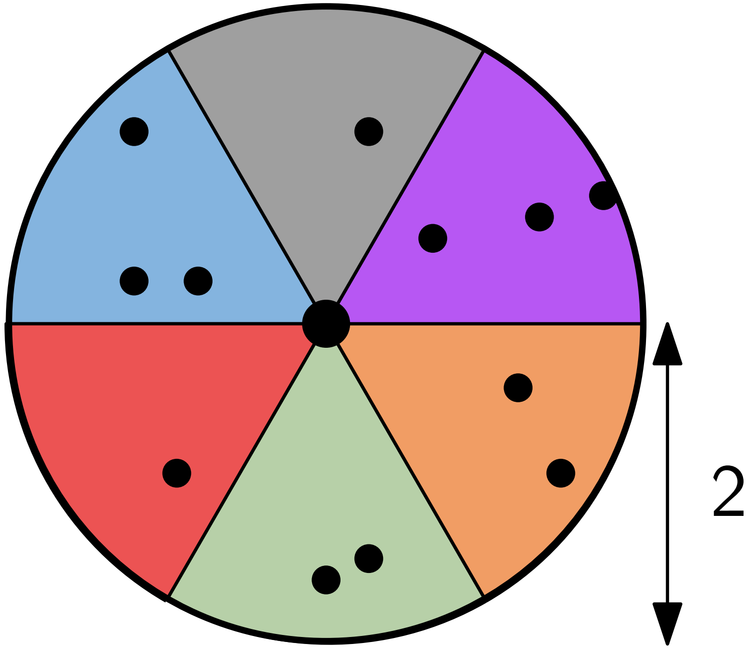
- add some technical magic



Theorem [FLS '11].
Every unit disk graph with no p-clique and treewidth $\Omega(p \cdot t)$ has a $t \times t$ grid minor.

Yet another win-win algorithm

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$$T(n,k) \leq k^{2\varepsilon} \cdot T(n, k - k^{\varepsilon}) \leq \exp\{k^{1-\epsilon} \log k\} \cdot \text{poly}(n)$$

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Overall running time is $2^{\mathcal{O}(k^{0.75} \cdot \log k)} \cdot \mathrm{poly}(n)$.

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