Explicit description of all separating hyperplanes of two convex polyhedral sets with RHS-parameters

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Abstract. Separation theorem is a basic theorem of convex analysis and convex programming. Its applicability to nonlinear programming methods depends on whether we can describe a separating hyperplane of two or more convex sets. But often in practice data are not fixed and we have to deal with parameters. A calculation of all separating hyperplanes of two convex polyhedral sets, which depend on right-hand-side parameters, is dealt with in this article.

Introduction

The explicit description of all separating hyperplanes of two fixed convex polyhedral sets was derived by *Grygarová* [1997, 1998]. In this article we will use the following two kinds of separability (for other definitions see for example *Klee* [1969]).

Definition 1. *Sets* $X, Y \subset \mathbb{R}^n$ *are called* weakly separable *if there exists a hyperplane* $\mathcal{R}$ *such that* $X \subset \overline{\mathcal{R}^-}, Y \subset \overline{\mathcal{R}^+}$ *holds*[1].
Sets $X, Y \subset \mathbb{R}^n$ *are called* strongly separable *if* X, Y *are weakly separable and* $dim X = dim Y = n$ *holds.*

In this article we deal with two families of convex polyhedral sets ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$):

$$M_1(\boldsymbol{\lambda}) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \boldsymbol{\lambda}\}, \tag{1}$$

$$M_2(\boldsymbol{\nu}) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \le \boldsymbol{\nu}\}, \tag{2}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\nu} \in \mathbb{R}^l$ are vectors of parameters. We want to know for which parameters $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\nu} \in \mathbb{R}^l$ the convex polyhedral sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ are nonempty, weakly or strongly separable etc.

Solution set

Definition 2. A solution set *is the set of all* $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ *for which* $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ *are strongly separable.*

Now we introduce

$$\mathcal{P}_1 \equiv \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid dim M_1(\boldsymbol{\lambda}) = n\}, \tag{3}$$

$$\mathcal{P}_2 \equiv \{\boldsymbol{\nu} \in \mathbb{R}^l \mid dim M_2(\boldsymbol{\nu}) = n\}, \tag{4}$$

$$\mathcal{P} \equiv \mathcal{P}_1 \times \mathcal{P}_2, \tag{5}$$

$$\mathcal{U} \equiv \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid int M_1(\boldsymbol{\lambda}) \cap int M_2(\boldsymbol{\nu}) \ne \emptyset\}. \tag{6}$$

Theorem 1. *The set* $\mathcal{P} \setminus \mathcal{U}$ *forms the solution set.*

Proof. Follows from basic separation theorem, see e. g. *Kemp and Kimura* [1978]. □

Theorem 2. *The set* $\mathcal{P}_1$ *has description*

$$\mathcal{P}_1 = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T \boldsymbol{\lambda} > 0 \quad \forall i \in I\}, \tag{7}$$

where $\mathbf{h}_i \ge \mathbf{0}$*,* $i \in I$*, are vectors in directions of all edges of the convex polyhedral cone*

$$N = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{A}^T\mathbf{y} = \mathbf{0},\ \mathbf{y} \ge \mathbf{0}\}.$$

Analogically for $\mathcal{P}_2$*,* $\mathcal{P}$*.*

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[1] If $\mathcal{R} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} = s\}$, then $\overline{\mathcal{R}^+} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} \ge s\}$, $\overline{\mathcal{R}^-} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} \le s\}$.

Proof. $\mathcal{P}_1$ is a set of all $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} < \boldsymbol{\lambda}\} \neq \emptyset$, it means $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} - \boldsymbol{\varepsilon}\} \neq \emptyset$ for an infinitesimal vector $\boldsymbol{\varepsilon} > \mathbf{0}$. It is true iff the problem

$$\max\{\mathbf{0}^T\mathbf{x};\ \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} - \boldsymbol{\varepsilon}\}$$

has an optimal solution. It follows from the theory of duality in linear programming that this is equivalent to the condition, that the problem

$$\min\{(\boldsymbol{\lambda} - \boldsymbol{\varepsilon})^T\mathbf{y};\ \mathbf{y} \in N\}, \tag{8}$$

has an optimal solution. The set N represents a convex polyhedral set with a vertex $\mathbf{0}$ and it is included in the first orthant. Vectors $\mathbf{h}_i \geq \mathbf{0}$, $i \in I$, are vectors in directions of all edges of N. It follows from the theory of polar cones, that the problem (8) has an optimal solution iff

$$\mathbf{h}_i^T.(\boldsymbol{\lambda} - \boldsymbol{\varepsilon}) \geq 0 \quad \forall i \in I \text{ and an infinitesimal } \boldsymbol{\varepsilon} > \mathbf{0}$$

holds. Hence the description of $\mathcal{P}_1$ is following

$$\mathcal{P}_1 = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T\boldsymbol{\lambda} \geq \mathbf{h}_i^T\boldsymbol{\varepsilon} \ \ \forall i \in I \text{ and an infinitesimal } \boldsymbol{\varepsilon} > \mathbf{0}\} = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T\boldsymbol{\lambda} > 0 \ \ \forall i \in I\}.$$

□

Theorem 3. *It holds*

$$\overline{\mathcal{P}_1} = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid M_1(\boldsymbol{\lambda}) \neq \emptyset\}.$$

Analogically for $\overline{\mathcal{P}_2}$, $\overline{\mathcal{P}}$.

Proof. The proof is quite similar to the proof of Theorem 2. The description of $\overline{\mathcal{P}_1}$ is

$$\overline{\mathcal{P}_1} = \{\boldsymbol{\lambda} \in \mathbb{R}^m \mid \mathbf{h}_i^T\boldsymbol{\lambda} \geq 0 \quad \forall i \in I\}, \tag{9}$$

where vectors $\mathbf{h}_i$ have the same meaning as in the proof of Theorem 2. □

Theorem 4. *The set* $\mathcal{U}$ *has description*

$$\mathcal{U} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid \mathbf{h}'^T_i.(\boldsymbol{\lambda}, \boldsymbol{\nu}) > 0 \quad \forall i \in I'\}, \tag{10}$$

where $\mathbf{h}'_i \geq \mathbf{0}$, $i \in I'$, *are vectors in directions of all edges of the convex polyhedral cone*

$$N' = \{(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{m+l} \mid \mathbf{A}^T\mathbf{y} + \mathbf{C}^T\mathbf{z} = \mathbf{0},\ \mathbf{y} \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}.$$

Proof. $\mathcal{U}$ is a set of all $\boldsymbol{\lambda}$, $\boldsymbol{\nu}$ such that $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} < \boldsymbol{\lambda}, \mathbf{C}\mathbf{x} < \boldsymbol{\nu}\} \neq \emptyset$. It means $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} - \boldsymbol{\varepsilon}^1, \mathbf{C}\mathbf{x} \leq \boldsymbol{\nu} - \boldsymbol{\varepsilon}^2\} \neq \emptyset$ for infinitesimal vectors $\boldsymbol{\varepsilon}^1 > \mathbf{0}$, $\boldsymbol{\varepsilon}^2 > \mathbf{0}$. This is true iff the problem

$$\max\{\mathbf{0}^T\mathbf{x};\ \mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} - \boldsymbol{\varepsilon}^1,\ \mathbf{C}\mathbf{x} \leq \boldsymbol{\nu} - \boldsymbol{\varepsilon}^2\}$$

has an optimal solution. ¡it follows from the theory of duality in linear programming that this is equivalent to the condition, that the problem

$$\min\{(\boldsymbol{\lambda} - \boldsymbol{\varepsilon}^1)^T\mathbf{y} + (\boldsymbol{\nu} - \boldsymbol{\varepsilon}^2)^T\mathbf{z};\ (\mathbf{y}, \mathbf{z}) \in N'\}, \tag{11}$$

has an optimal solution. The set N' represents a convex polyhedral set with a vertex $\mathbf{0}$ and is included in the first orthant. Vectors $\mathbf{h}'_i \geq \mathbf{0}$, $i \in I'$, are vectors in directions of all edges of N'. It follows from the theory of polar cones, that

$$\begin{aligned}\mathcal{U} &= \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid \mathbf{h}'^T_i.(\boldsymbol{\lambda} - \boldsymbol{\varepsilon}^1, \boldsymbol{\nu} - \boldsymbol{\varepsilon}^2) \geq 0 \quad \forall i \in I' \text{ and infinitesimal } \boldsymbol{\varepsilon}^1 > \mathbf{0},\ \boldsymbol{\varepsilon}^2 > \mathbf{0}\} = \\ &= \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid \mathbf{h}'^T_i.(\boldsymbol{\lambda}, \boldsymbol{\nu}) > 0 \quad \forall i \in I'\}.\end{aligned}$$

□

Assertion 1. *Let* $M_1(\boldsymbol{\lambda}^0)$ *be nonempty and bounded for a certain* $\boldsymbol{\lambda}^0 \in \mathbb{R}^m$. *Then* $M_1(\boldsymbol{\lambda}^0)$ *is bounded for all* $\boldsymbol{\lambda} \in \mathbb{R}^m$.
Analogically for $M_2(\boldsymbol{\nu})$.

Proof. See *Schrijver* [1998], $M_1(\boldsymbol{\lambda})$ has the same characteristic cone for all $\boldsymbol{\lambda} \in \mathbb{R}^m$. □

Assertion 2. *It holds $\mathcal{U} \subset \mathcal{P}$. Moreover, either the set $\mathcal{P} \setminus \mathcal{U}$ is empty or $dim(\mathcal{P} \setminus \mathcal{U}) = n$.*

Proof. The statement is evident. □

Theorem 5. *It holds*

$$\overline{\mathcal{P} \setminus \mathcal{U}} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l} \mid M_1(\boldsymbol{\lambda}),\ M_2(\boldsymbol{\nu}) \text{ are nonempty and weakly separable}\}.$$

Proof.
Inclusion $\subset$: Let be $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \overline{\mathcal{P} \setminus \mathcal{U}}$. Then there exists a sequence $(\boldsymbol{\lambda}_n, \boldsymbol{\nu}_n) \in \mathcal{P} \setminus \mathcal{U}$, $(\boldsymbol{\lambda}_n, \boldsymbol{\nu}_n) \to (\boldsymbol{\lambda}, \boldsymbol{\nu})$ and separating hyperplanes $\mathcal{R}_n = \{\mathbf{x} \mid \mathbf{r}_n^T\mathbf{x} = s_n\}$, $\|(\mathbf{r}_n, s_n)\| = 1$, for which $M_1(\boldsymbol{\lambda}_n) \subset \overline{\mathcal{R}_n^-}$, $M_2(\boldsymbol{\nu}_n) \subset \overline{\mathcal{R}_n^+}$ hold. Next we w. l. o. g. assume that $(\mathbf{r}_n, s_n) \to (\mathbf{r}, s)$ (otherwise we can choose a convergent subsequence). From the continuity of linear constraints it follows that $M_1(\boldsymbol{\lambda}) \subset \overline{\mathcal{R}^-}$, $M_2(\boldsymbol{\nu}) \subset \overline{\mathcal{R}^+}$.
Inclusion $\supset$: Let us have $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ and a hyperplane $\mathcal{R} = \{\mathbf{x} \mid \mathbf{r}^T\mathbf{x} = s\}$ such that $\emptyset \neq M_1(\boldsymbol{\lambda}) \subset \overline{\mathcal{R}^-}$, $\emptyset \neq M_2(\boldsymbol{\nu}) \subset \overline{\mathcal{R}^+}$. Let us consider a convex polyhedral set $M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon}) = \{\mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} + \boldsymbol{\varepsilon}\}$, where $\boldsymbol{\varepsilon} > \mathbf{0}$ is infinitesimal. Then $dim M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon}) = n$. A problem $\max\{(\mathbf{r}^T\mathbf{x} - s);\ \mathbf{x} \in M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon})\}$ has an optimal solution (characteristic cones for $M_1(\boldsymbol{\lambda})$ and $M_1(\boldsymbol{\lambda} + \boldsymbol{\varepsilon})$ are identical – see *Schrijver* [1998]). Denote $h(\boldsymbol{\varepsilon})$ the optimal value of this problem. This function $h(\boldsymbol{\varepsilon})$ is continuous. Therefore a set $\{\mathbf{A}(\mathbf{x} + h(\boldsymbol{\varepsilon})^+\mathbf{r}) \leq \boldsymbol{\lambda} + \boldsymbol{\varepsilon}\} = \{\mathbf{A}\mathbf{x} \leq \boldsymbol{\lambda} + \boldsymbol{\varepsilon} - \mathbf{A}h(\boldsymbol{\varepsilon})^+\mathbf{r}\}$ has dimension n, is situated in the halfspace $\overline{\mathcal{R}^-}$ and for $\boldsymbol{\varepsilon} \to \mathbf{0}$ converges to $M_1(\boldsymbol{\lambda})$.
Analogically for $M_2(\boldsymbol{\nu})$. □

Definition 3. *Let us define*

$$\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \equiv \left\{(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathbb{R}_+^{m+l+2} \mid \begin{pmatrix} \mathbf{A}^T & \mathbf{0} & \mathbf{C}^T & \mathbf{0} \\ \boldsymbol{\lambda}^T & 1 & \boldsymbol{\nu}^T & 1 \\ \mathbf{1}^T & 1 & \mathbf{1}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ u_{m+1} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Theorem 6. *(Explicit description of all separating hyperplanes) Let be $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$.*

(i) For an arbitrary $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\mathbf{b}, \mathbf{d})$, for which $\mathbf{u}^T\mathbf{A} \neq \mathbf{0}$, a set

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = u_{m+1}\}$$

represents a separating hyperplane of $M_1(\mathbf{b})$, $M_2(\mathbf{d})$.

(ii) An arbitrary separating hyperplane $\mathcal{R}$ of two convex polyhedral sets $M_1(\mathbf{b})$, $M_2(\mathbf{d})$ has a description

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = u_{m+1}\},$$

where $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\mathbf{b}, \mathbf{d})$, $\mathbf{u}^T\mathbf{A} \neq \mathbf{0}$.

Proof. Obviously from Theorem 2 in *Grygarová* [1998]. □

Assertion 3. *It holds*

$$\mathcal{P} \setminus \mathcal{U} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathcal{P} \mid \mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \neq \emptyset\}.$$

Proof. Obviously from Theorem 5 in *Grygarová* [1998]. □

Stability sets

In this section we deal with so called stability sets (for definition of other linear models see e. g. *Nožička* [1974]). It is natural to define stability sets as sets of all $\boldsymbol{\lambda}$, $\boldsymbol{\nu}$ under which all feasible bases of the convex polytope $\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$ (from Definition 3) remain feasible. But for the sake of simplicity we will use the set $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ from (13) instead of $\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$. Next we will w. l. o. g. assume that

$$rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} = n + 1. \tag{12}$$

Otherwise there are two possibilities:

(i) If $rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} = rank \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \end{pmatrix}$, then the set N' from (11) has its description $N' = \{\mathbf{0}\}$ and therefore $\mathcal{U} = \mathbb{R}^{m+l}$ and the solution set $\mathcal{P} \setminus \mathcal{U} = \emptyset$.

(ii) If $rank\begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{1}^T & \mathbf{1}^T \end{pmatrix} > rank\begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \end{pmatrix}$, then $rank\begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \end{pmatrix} < n$ and in the description of $\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$ there are linear dependent equations, which we can remove.

Definition 4. A stability set *is a closure of a set of all* $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ *under which all feasible bases of the convex polyhedral set* $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ *remain feasible, where*

$$\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu}) \equiv \left\{ (\mathbf{u}, \mathbf{v}, v_{l+1}) \in \mathbb{R}_+^{m+l+1} \mid \mathbf{Z}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ v_{l+1} \end{pmatrix} = \mathbf{z} \right\}, \tag{13}$$

and

$$\mathbf{Z}(\boldsymbol{\lambda}, \boldsymbol{\nu}) \equiv \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T & \mathbf{0} \\ \boldsymbol{\lambda}^T & \boldsymbol{\nu}^T & 1 \\ \mathbf{1}^T & \mathbf{1}^T & 0 \end{pmatrix}, \quad \mathbf{z} \equiv \begin{pmatrix} \mathbf{0} \\ 0 \\ 1 \end{pmatrix}.$$

Assertion 4. *For the description of all separating hyperplanes of two convex polyhedral sets* $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$ *we can use a set* $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ *instead of* $\mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

Proof.

(i) Let $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$. Since $u_{m+1} + v_{l+1} < 1$, we can for $\alpha = 1 - u_{m+1} - v_{l+1} > 0$ define

$$\mathbf{u}^* \equiv \frac{\mathbf{u}}{\alpha}, \quad \mathbf{v}^* \equiv \frac{\mathbf{v}}{\alpha}, \quad v_{l+1}^* \equiv \frac{u_{m+1} + v_{l+1}}{\alpha}.$$

Therefore $(\mathbf{u}^*, \mathbf{v}^*, v_{l+1}^*) \in \mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

(ii) Let $(\mathbf{u}^*, \mathbf{v}^*, v_{l+1}^*) \in \mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$. Then for $\alpha = \sum_{i=1}^m u_i^* + \sum_{j=1}^l v_j^* + v_{l+1}^* > 0$ define

$$\mathbf{u} \equiv \frac{\mathbf{u}^*}{\alpha}, \quad \mathbf{v} \equiv \frac{\mathbf{v}^*}{\alpha}, \quad u_{m+1} \in \langle 0, \frac{v_{l+1}^*}{\alpha} \rangle \text{ arbitrarily}, \quad v_{l+1} \equiv \frac{v_{l+1}^*}{\alpha} - u_{m+1}.$$

Therefore $(\mathbf{u}, u_{m+1}, \mathbf{v}, v_{l+1}) \in \mathcal{Q}(\boldsymbol{\lambda}, \boldsymbol{\nu})$.

□

Now we derive a description of a stability set. Let $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$. Denote B an arbitrary feasible basis (it means a basis for which corresponding vertex is feasible) of the convex polytope $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ and define $\mathbf{D} \equiv \mathbf{Z}_B(\mathbf{b}, \mathbf{d})$ (i. e. the restriction of the matrix $\mathbf{Z}(\mathbf{b}, \mathbf{d})$ to the basic columns). The basis B will be preserved for all $(\boldsymbol{\lambda}, \ \boldsymbol{\nu}) \in \mathbb{R}^{m+l}$ such, that

$$(\mathbf{Z}_B(\boldsymbol{\lambda}, \boldsymbol{\nu}))^{-1}\mathbf{z} \geq \mathbf{0}. \tag{14}$$

Let us introduce vectors $\mathbf{p} \in \mathbb{R}^{n+2}$, $\widetilde{\mathbf{q}}, \mathbf{q} \in \mathbb{R}^{m+l+1}$ in this way:

$$\mathbf{p} \equiv \mathbf{e}_{n+1} = \begin{pmatrix} \mathbf{0} \\ 1 \\ 0 \end{pmatrix}, \quad \widetilde{\mathbf{q}} \equiv \begin{pmatrix} \boldsymbol{\lambda} - \mathbf{b} \\ \boldsymbol{\nu} - \mathbf{d} \\ 0 \end{pmatrix}, \quad \mathbf{q} \equiv \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\nu} \\ 0 \end{pmatrix}.$$

Let us assume

$$1 + \widetilde{\mathbf{q}}_B^T \mathbf{D}^{-1}\mathbf{p} > 0, \tag{15}$$

where $\widetilde{\mathbf{q}}_B$ is the restriction of the vector $\widetilde{\mathbf{q}}$ to the basic elements. For the choice $\boldsymbol{\lambda} = \mathbf{b}$, $\boldsymbol{\nu} = \mathbf{d}$ the condition (15) holds (see also Remark 1). Then from the well known Sherman–Morrison formula follows:

$$(\mathbf{Z}_B(\boldsymbol{\lambda}, \boldsymbol{\nu}))^{-1} = (\mathbf{Z}_B(\mathbf{b}, \mathbf{d}) + \mathbf{p}\widetilde{\mathbf{q}}_B^T)^{-1} = \mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\mathbf{p}\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}}{1 + \widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}\mathbf{p}}.$$

Let us rearrange expression (14) under our assumption (15):

$$\left(\mathbf{D}^{-1} - \frac{\mathbf{D}^{-1}\mathbf{e}_{n+1}\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}}{1 + \widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}\mathbf{e}_{n+1}}\right)\mathbf{e}_{n+2} \geq \mathbf{0},$$

$$\mathbf{D}^{-1}_{\cdot,n+2} + \mathbf{D}^{-1}_{\cdot,n+2}(\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\widetilde{\mathbf{q}}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}. \tag{16}$$

For the vector $\widetilde{\mathbf{q}}_B^T$ holds:

$$\widetilde{\mathbf{q}}_B^T = \begin{cases} \mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot}, & \text{if } (m+l+1) \notin B, \\ \mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot} + \mathbf{e}_k^T, & \text{if } (m+l+1) = B_k \text{ for a certain } k \in \{1, \ldots, n+2\}. \end{cases}$$

(i) Let $(m+l+1) \notin B$. Then the expression (16) is equivalent to

$$\mathbf{D}^{-1}_{\cdot,n+2} + \mathbf{D}^{-1}_{\cdot,n+2}\left((\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot})\mathbf{D}^{-1}_{\cdot,n+1}\right) - \mathbf{D}^{-1}_{\cdot,n+1}\left((\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot})\mathbf{D}^{-1}_{\cdot,n+2}\right) \geq \mathbf{0},$$

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}. \tag{17}$$

(ii) Let $(m+l+1) = B_k$ for a certain $k \in \{1,\ldots,n+2\}$. Then $\mathbf{D}_{\cdot,k} = \mathbf{e}_{n+1}$, and therefore $\mathbf{D}^{-1}_{\cdot,n+1} = \mathbf{e}_k$. The absolute term of the expression (16) is equal to

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{e}_k^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{e}_k^T\mathbf{D}^{-1}_{\cdot,n+2}) = \mathbf{D}^{-1}_{\cdot,n+2} - \mathbf{e}_k\mathbf{D}^{-1}_{k,n+2} = \begin{cases} 0 & \text{for row } k, \\ > 0 & \text{for row } \neq k. \end{cases}$$

Nonabsolute terms of the expression (16) have the following description (we know, that $\mathbf{q}_{B_k} = 0$):

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) = -\mathbf{e}_k(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) = \begin{cases} -\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2} & \text{for row } k, \\ 0 & \text{for row } \neq k. \end{cases}$$

Remark 1. *Let us investigate the expression* (15)*.*

(i) If $(m+l+1) \notin B$*, then the expression* (15) *can be simplified to*

$$\begin{aligned} 1 + (\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot})\mathbf{D}^{-1}_{\cdot,n+1} &> 0, \\ \mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1} &> 0. \end{aligned}$$

The sum of inequalities from (17) *results to (we know, that* $\mathbf{1}^T = \mathbf{D}_{n+2,\cdot}$*)*

$$\begin{aligned} (\mathbf{1}^T\mathbf{D}^{-1}_{\cdot,n+2})(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - (\mathbf{1}^T\mathbf{D}^{-1}_{\cdot,n+1})(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) &\geq 0, \\ \mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1} &\geq 0. \end{aligned}$$

Since the stability set is always closed (from its definition), the condition (15) *is redundant.*

(ii) If $(m+l+1) = B_k$ *for a certain* $k \in \{1,\ldots,n+2\}$*, then the expression* (15) *can be simplified to (we know, that* $\mathbf{D}^{-1}_{\cdot,n+1} = \mathbf{e}_k$*,* $\mathbf{q}_{B_k} = 0$*)*

$$\begin{aligned} 1 + (\mathbf{q}_B^T - \mathbf{D}_{n+1,\cdot} + \mathbf{e}_k^T)\mathbf{D}^{-1}_{\cdot,n+1} &> 0, \\ 1 &> 0 \end{aligned}$$

and this inequality holds trivially. The condition (15) *is redundant.*

The following remark summarizes the description of stability sets.

Remark 2. *For each feasible basis* B *of the convex polytope* $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$ *we obtain a system of inequalities* (14)*, which forms a linear system of inequalities with respect to parameters* $\boldsymbol{\lambda}$*,* $\boldsymbol{\nu}$*. The description of a stability set represents a union of all these systems of inequalities for all feasible bases of* $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$*.*

(i) If $(m+l+1) \notin B$ *then the system of inequalities for the basis* B *has its description*

$$\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2}) \geq \mathbf{0}.$$

(ii) If $(m+l+1) \in B$ *then we obtain only one inequality for the basis* B *with its description*

$$-\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2} \geq 0.$$

All vertices of $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$ *in a given stability set follow:*

(i) If $(m+l+1) \notin B$*, then the vertex of* $\mathcal{Q}^*(\boldsymbol{\lambda},\boldsymbol{\nu})$ *corresponding to the basis* B *has the following description:*

$$(\mathbf{u},\mathbf{v},v_{l+1})_B = \frac{\mathbf{D}^{-1}_{\cdot,n+2}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}) - \mathbf{D}^{-1}_{\cdot,n+1}(\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+2})}{\mathbf{q}_B^T\mathbf{D}^{-1}_{\cdot,n+1}},\ (\mathbf{u},\mathbf{v},v_{l+1})_N = \mathbf{0}.$$

(ii) If $(m + l + 1) = B_k$*, then the vertex of* $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ *corresponding to the basis* B *has the following description:*

$$(\mathbf{u}, \mathbf{v}, v_{l+1})_B = \mathbf{D}^{-1}_{\cdot,n+2} - \mathbf{e}_k \mathbf{D}^{-1}_{k,n+2} - \mathbf{e}_k(\mathbf{q}^T_B \mathbf{D}^{-1}_{\cdot,n+2}),\ (\mathbf{u}, \mathbf{v}, v_{l+1})_N = \mathbf{0}.$$

Assertion 5. *There exists a finite number of stability sets.*

Proof. Each stability set corresponds to a particular system of feasible bases. There is a finite number of all systems of feasible bases, therefore there is a finite number of stability sets. □

Lemma 1. *Given* $(\mathbf{b}, \mathbf{d}) \in \mathcal{P}$ *let* S *be a corresponding stability set and* $\mathfrak{S}$ *a system of all feasible bases of* $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$*. Then for all* $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in intS$ *the set* $\mathfrak{S}$ *is the set of all feasible bases of the convex polytope* $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$*.*

Proof. For all $(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in S$ there are all bases from $\mathfrak{S}$ feasible for a convex polytope $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$. Now we show that $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ has no other feasible basis. Each vertex of $\mathcal{Q}^*(\boldsymbol{\lambda}, \boldsymbol{\nu})$ is formed by an intersection of a certain hyperplane $\{\mathbf{x} \mid (\boldsymbol{\lambda}, \boldsymbol{\nu}, 1).\mathbf{x} = 0\}$ and an edge or a vertex of $\mathcal{Q}^*$. Let us assume for contradiction that there exists $(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0) \in intS$ such, that $\mathcal{Q}^*(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0)$ has a new feasible basis, and the corresponding vertex is an intersection of the hyperplane $\{\mathbf{x} \mid (\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0, 1).\mathbf{x} = 0\}$ and an edge $\mathcal{Q}^*$. Let us consider the function $\mathbf{f} : t \mapsto (\mathbf{b}, \mathbf{d}) + t.(\boldsymbol{\lambda} - \mathbf{b}, \boldsymbol{\nu} - \mathbf{d})$ and the minimal t such that $\mathcal{Q}^*(\mathbf{f}(t))$ contains a new feasible basis. Let us denote $\mathbf{w}$ the vertex, corresponding to this new feasible basis. The vertex $\mathbf{w}$ originated from the intersection of the hyperplane $\mathcal{R} = \{\mathbf{x} \mid (\mathbf{f}(t), 1).\mathbf{x} = 0\}$ and a vertex of $\mathcal{Q}^*$ (in other case t is not minimal). There are edges directed from the vertex $\mathbf{w}$ to the halfspace $\mathcal{R}^+$ as well as to the halfspace $\mathcal{R}^-$ (since $t \in (0, 1)$). Increasing of t leads to the loss of some feasible bases from $\mathfrak{S}$, that corresponding vertices are situated in $\mathcal{R}^+$ or in $\mathcal{R}^-$. □

Assertion 6. *The intersection of two stability sets is a set of the dimension less than* n*.*

Proof. Follows from Lemma 1. □

Assertion 7. *Let* S *be a stability set corresponding to a certain* $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$*. Then* $intS \subset \mathcal{P}$ *holds.*

Proof. For contradiction assume that $\exists(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0) \in intS \setminus \mathcal{P}$, w. l. o. g. $\boldsymbol{\lambda}^0 \notin \mathcal{P}(\mathbf{b})$. Since $\boldsymbol{\lambda}^0 \notin \mathcal{P}(\mathbf{b})$, the dual problem (8) has no optimal solution for the objective function $(\boldsymbol{\lambda}^0 - \boldsymbol{\varepsilon})^T\mathbf{y}$, where $\boldsymbol{\varepsilon} > \mathbf{0}$ is infinitesimal. Therefore $\exists \mathbf{y}^0 \in N, \mathbf{y}^0 \neq \mathbf{0}$, that $(\boldsymbol{\lambda}^0 - \boldsymbol{\varepsilon})^T\mathbf{y}^0 < 0$, i. e. $\boldsymbol{\lambda}^{0T}\mathbf{y}^0 \leq 0$. Let us define

$$(\mathbf{u}^0, \mathbf{v}^0, u^0_{l+1}) \equiv \frac{(\mathbf{y}^0, \mathbf{0}, -\boldsymbol{\lambda}^{0T}\mathbf{y}^0)}{\mathbf{1}^T\mathbf{y}^0} \in \mathcal{Q}^*(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0).$$

An arbitrary element $(\mathbf{u}, \mathbf{v}, u_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0)$ can be described as a convex combination of all vertices of $\mathcal{Q}^*(\boldsymbol{\lambda}^0, \boldsymbol{\nu}^0)$ and thus there exists a feasible basis $B^0 \subset \{1, \ldots, m\} \cup \{m + l + 1\}$. It follows from Lemma 1 that B^0 is feasible for $\mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ as well. For the basis B^0 and the corresponding vertex $(\mathbf{u}^1, \mathbf{0}, v^1_{l+1}) \in \mathcal{Q}^*(\mathbf{b}, \mathbf{d})$ we obtain:

$$\mathbf{A}^T\mathbf{u}^1 = \mathbf{0},\ \mathbf{b}^T\mathbf{u}^1 \leq 0,$$

hence for all $\mathbf{x} \in M_1(\mathbf{b}, \mathbf{d})$ we have $0 = \mathbf{u}^{1T}\mathbf{A}\mathbf{x} \leq \mathbf{u}^{1T}\mathbf{b} \leq 0$ and $\mathbf{u}^{1T}\mathbf{A}\mathbf{x} = \mathbf{u}^{1T}\mathbf{b}$, $dimM_1(\mathbf{b}, \mathbf{d}) < n$, which contradicts the assumption that $(\mathbf{b}, \mathbf{d}) \in \mathcal{P} \setminus \mathcal{U}$. □

Examples

Example 1 Let matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are given and we want to compute all stability sets. First, we will compute sets $\mathcal{P}$, $\mathcal{U}$. They have the following description:

$$\mathcal{P}_1 = \{\boldsymbol{\lambda} \in \mathbb{R}^3 \mid \lambda_1 + \lambda_2 + \lambda_3 > 0\},\ \mathcal{P}_2 = \mathbb{R}^2,$$
$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \lambda_2 + \lambda_3 > 0\},$$
$$\mathcal{U} = \{(\boldsymbol{\lambda}, \boldsymbol{\nu}) \in \mathbb{R}^5 \mid \lambda_1 + \nu_1 > 0,\ \lambda_2 + \nu_2 > 0,\ \lambda_1 + \lambda_2 + \lambda_3 > 0\}.$$

1. Choose $(\mathbf{b}^1, \mathbf{d}^1) \in \mathcal{P} \setminus \mathcal{U}$, e. g. $\mathbf{b}^1 = (0, 0, 2)^T$, $\mathbf{d}^1 = (1, -1)^T$. All feasible bases of $\mathcal{Q}^*(\mathbf{b}^1, \mathbf{d}^1)$ with the corresponding systems of inequalities are the following:

$$\begin{aligned}
&\text{Basis } (1,2,4,5) : -\lambda_2 - \nu_2 \geq 0,\ \lambda_1 + \nu_1 \geq 0.\\
&\text{Basis } (1,2,5,6) : -\lambda_2 - \nu_2 \geq 0.\\
&\text{Basis } (2,3,5,6) : -\lambda_2 - \nu_2 \geq 0.\\
&\text{Basis } (2,4,5,6) : -\lambda_2 - \nu_2 \geq 0.\\
&\text{Basis } (1,2,3,5) : -\lambda_2 - \nu_2 \geq 0,\ \lambda_1 + \lambda_2 + \lambda_3 \geq 0.
\end{aligned}$$

The first stability set (denote it as S_1) is described as follows: $\lambda_1 + \nu_1 \geq 0,\ -\lambda_2 - \nu_2 \geq 0,\ \lambda_1 + \lambda_2 + \lambda_3 \geq 0$.

2. Choose $(\mathbf{b}^2, \mathbf{d}^2) \in \mathcal{P} \setminus \mathcal{U} \setminus S_1$, e. g. $\mathbf{b}^2 = (0, 0, 2)^T$, $\mathbf{d}^2 = (-1, -1)^T$. The second stability set (denote it as S_2) has the following description: $-\lambda_1 - \nu_1 \geq 0,\ -\lambda_2 - \nu_2 \geq 0,\ \lambda_1 + \lambda_2 + \lambda_3 \geq 0$.

3. Analogically the last stability set. Choose $(\mathbf{b}^3, \mathbf{d}^3) \in \mathcal{P} \setminus \mathcal{U} \setminus S_1 \setminus S_2$ e. g. $\mathbf{b}^3 = (0, 0, 2)^T$, $\mathbf{d}^3 = (-1, 1)^T$. The third stability set has the following description: $-\lambda_1 - \nu_1 \geq 0,\ \lambda_2 + \nu_2 \geq 0,\ \lambda_1 + \lambda_2 + \lambda_3 \geq 0$.

We have obtained three stability sets (except degenerated stability sets on borders) the solution set $\mathcal{P} \setminus \mathcal{U}$ consists of. The geometric interpretation of the three stability sets is illustrated by Figure 1 – 3; each stability set determines a special position of the sets $M_1(\boldsymbol{\lambda})$, $M_2(\boldsymbol{\nu})$.

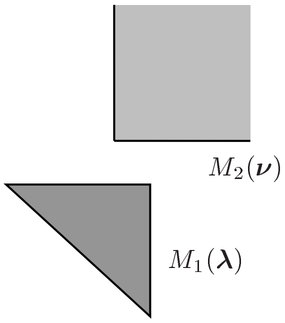

Figure 1. Stability set no. 1.

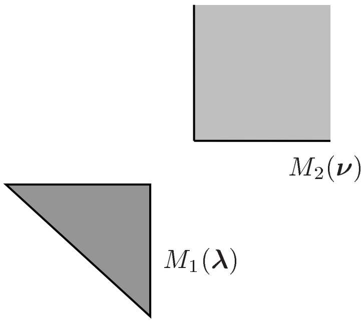

Figure 2. Stability set no. 2.

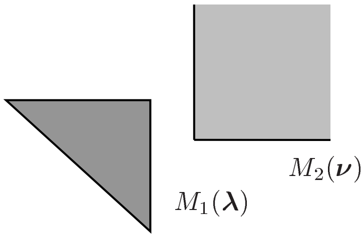

Figure 3. Stability set no. 3.

The following two tables contain further results obtained by a programm written for MATLAB. With the increase of m, l, n the number of stability sets and the computing time increases very rapidly.

Table 1. Examples for $\mathbb{R}^2$.

matrix A	matrix C	number of stability sets	computing time
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	1 s
$\begin{pmatrix} 1 & 0 \\ -3 & 5 \\ -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -4 & -1 \\ 3 & -4 \\ 2 & 3 \end{pmatrix}$	20	2 s
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$	8	3 s
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$	7	1 s
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}$	29	6 s

Table 2. Examples for $\mathbb{R}^3$, $\mathbb{R}^4$ – pseudorandom data.

matrix A	matrix C	number of stability sets	computing time
$\begin{pmatrix} -2 & 5 & 3 \\ -1 & -3 & -3 \\ 4 & -2 & 3 \\ 1 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & 3 & 1 \\ 0 & 3 & 5 \\ 0 & 2 & -3 \end{pmatrix}$	103	1 m 25 s
$\begin{pmatrix} -5 & -1 & 3 & 0 \\ 3 & 3 & -5 & 2 \\ -1 & 0 & 2 & -1 \\ 4 & -3 & -1 & -2 \\ -1 & 4 & -1 & -4 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 & -1 & -4 \\ 0 & 3 & 4 & -1 \\ -5 & 4 & 4 & 3 \\ 3 & 2 & -1 & -5 \\ -1 & 1 & 2 & 4 \end{pmatrix}$	446	21 m 38 s
$\begin{pmatrix} 5 & 3 & 1 & -1 \\ -3 & 0 & 3 & 4 \\ -5 & 4 & -2 & 0 \\ -4 & 0 & 3 & 2 \\ 3 & 3 & 3 & -4 \end{pmatrix}$	$\begin{pmatrix} 5 & 1 & -2 & 5 \\ -1 & -2 & 5 & -2 \\ 4 & 2 & -3 & -4 \\ 0 & -4 & 3 & 3 \\ -2 & 4 & -3 & -3 \end{pmatrix}$	1666	91 m 42 s

Special cases

To stability sets, defined in the previous section, we can directly apply all special kinds of postoptimality analyses – parametric analysis, sensitivity analysis or tolerance analysis. But in this section we will discuss parametrization for special cases of families of convex polyhedral sets (1), (2).

One parametr

We deal with two families of convex polyhedral sets (with λ as a parameter)

$$M_1(\lambda) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \begin{pmatrix} \lambda \\ \mathbf{b} \end{pmatrix}\},$$
$$M_2 = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \mathbf{d}\}.$$

Using (7) and (10) we obtain the description of the solution set $\mathcal{P} \setminus \mathcal{U}$ as an interval in $\mathbb{R}$. Therefore the solution set for this case is always convex.

Parameter with fixed direction

We deal with two families of convex polyhedral sets (with λ as a parameter)

$$M_1(\lambda) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b} + \lambda\mathbf{b}'\},$$
$$M_2(\lambda) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{x} \leq \mathbf{d} + \lambda\mathbf{d}'\}.$$

Using (7) and (10) we obtain the description of the solution set $\mathcal{P} \setminus \mathcal{U}$ as a difference of two intervals in $\mathbb{R}$. Therefore the solution set for this case is not generally convex.

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