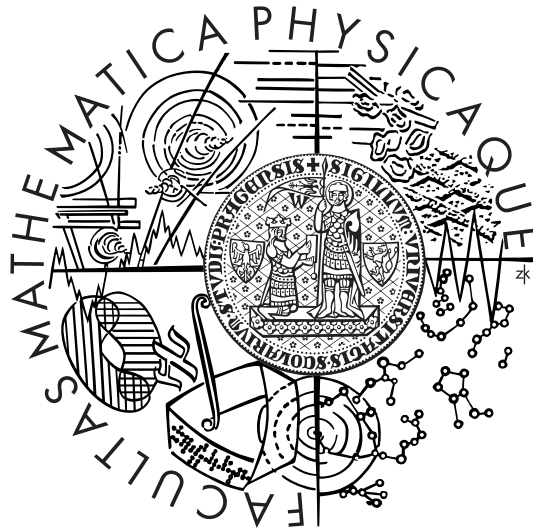


Charles University in Prague  
Faculty of Mathematics and Physics

## BACHELOR'S THESIS



Jan Bok

## Cooperative interval games

Department of Applied Mathematics

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In Prague, July 29, 2014

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**Název práce:** Kooperativní intervalové hry

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**Klíčová slova:** kooperativní teorie her, intervalová analýza, jádro, Shapleyho hodnota

**Abstrakt:**

V této práci studujeme kooperativní intervalové hry, zobecněný model kooperativních her ve kterém hodnota každé koalice koresponduje s uzavřeným intervalem, reprezentujícím všechny možné výsledky jejich kooperace.

Nejprve dáváme stručné úvody do klasické kooperativní teorie her a intervalové analýzy a následně uvádíme čtenáře do kooperativních intervalových her, a to se speciálním důrazem na selekce, což jsou všechny možné výsledky hry ve kterých už není žádná další neurčitost.

Představujeme nové třídy her podle vlastností jejich selekcí a dokazujeme jejich charakterizace a vztahy s již existujícími třídami. Ukazujeme nové výsledky týkající se imputací a jader. Zavádíme definici silné imputace a silného jádra a zkoumáme problém rovnosti dvou různých typů jader – hlavního stabilního řešení kooperativních intervalových her. Nakonec ukazujeme nová pozorování ohledně Shapleyho hodnoty intervalových her.

**Title:** Cooperative interval games

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**Keywords:** cooperative game theory, interval analysis, core, Shapley value

**Abstract:**

In this thesis, we study cooperative interval games, a generalized model of cooperative games in which worth of every coalition corresponds with a closed interval representing all possible outcomes of their cooperation.

We give a brief introduction into classical cooperative games, interval analysis and finally introduction to cooperative interval games with focus on selections, that is on all possible outcomes of interval game with no additional uncertainty.

We introduce new selection-based classes of interval games and prove their characterizations and relation to existing classes. We show a new results regarding core and imputations. We introduce a definition of strong imputation and core and examine a problem of equality of two different versions of core – the main stability solution of cooperative interval games. Finally, we make some new remarks on Shapley value of interval games.





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# 1. Introduction

While game theory is well developed field of mathematics, hundreds of new papers are published every year and many applications exist, its usage in real world problems is quite limited due to the fact that in reality, we very often do not know exact data to analyze players' behavior. To be able to deal with different types of uncertainty is therefore of high importance.

This thesis is about cooperative games with interval uncertainty. In cooperative interval games, players are allowed to cooperate and every group of players (*coalition*) knows worst and best possible outcome of its cooperation. This situation can be naturally modeled with intervals encapsulating all possibilities that can occur. Bounds of interval then correspond to optimistic and pessimistic expectations. To study this interval uncertainty model, we will use results of both classical cooperative game theory and interval analysis.

Structure of this thesis is as follows: In the second chapter we present basic definitions and facts on classical cooperative games, their most important solution concepts – core and Shapley value – and the main classes of cooperative games together with their properties.

Third chapter is a brief introduction into interval analysis with emphasis on interval arithmetic, functions and linear algebra.

Second and third chapter can be considered as the preliminaries sections to the study of cooperative interval games.

Finally, fourth chapter is devoted to cooperative interval games – to motivation, history, applications, definitions, basic facts, open problems and to new results as well. Our results are presented in Section 4.3, 4.4 and 4.5. See the beginning of Chapter 4 for more details.

## On mathematical notation

- We will use  $\leq$  relation on real vectors. For every  $x, y \in \mathbb{R}^N$  we write  $x \leq y$  if  $x_i \leq y_i$  holds for every  $i \in N$ .
- We do not use symbol  $\subset$  in the thesis. Instead,  $\subseteq$  and  $\subsetneq$  are used for subset and proper subset, respectively, to avoid ambiguity.



## 2. Cooperative game theory

In this chapter our aim is to introduce reader to the theory of cooperative games (which is sometimes called classical cooperative game theory to distinguish between another models). Knowledge of this chapter is necessary to understanding theory of cooperative interval games since many of its concepts are extended results of classical games.

The structure of this chapter is as follows: In the first section we will briefly say what game theory is, what do we mean by cooperation and how can we further divide cooperative games. Next section deals with basic definitions. Third and fourth section are focused on solutions - namely on core and Shapley value. Last section presents important classes of cooperative games and discusses some of their properties.

### 2.1 History and informal introduction

What is game theory? Perhaps the most elegant answer is Myerson's [31]: "*Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.*" History of this area of mathematics goes back to eighteenth century but it was extensively developed mainly in the second half of the twentieth century thanks to work of John von Neumann and Oskar Morgenstern who established game theory in a more uniformed way and gave the basis for future research. Most important and influential was their book *Theory of Games and Economic Behavior* [39] published in 1944.

Word "game" can be somewhat misleading, since, as we now know, game theory is mainly analysis of conflicts and interactions and that includes a lot more than games only. Term game theory comes from the times when the field was in its beginnings and when studying strategies for various games was its main application. In fact, game theory has many applications in many other fields like biology, politics, economy, psychology etc. Various fields of mathematics like set theory and combinatorics benefited from game theory research as well. So far, ten game theorists were awarded with Nobel Prize in Economics.

Game theory can be further divided by game types. Most common division is to non-cooperative and cooperative games. While non-cooperative game theory does not allow communication and bargaining between players (for example combinatorial games - chess, go, nim etc), cooperative game theory allows it and players are therefore able to form into groups (coalitions) and coordinate their actions in order to achieve higher profit. Applications of cooperative approach include for example insurance problems [24], games on graphs and matroids [8], games arising from combinatorial optimization problems [14] or fair division problems [30].

Cooperative game theory is further divided into two categories by transferability of utilities. For simplification, we can think of utility as of reward given to each

player. Transferability is, informally said, an ability to redistribute players' gains inside of coalition. This thesis deals with transferable utility (TU) games only, so from now on, we will use the term cooperative game or game but we will mean a transferable utility cooperative game. For further details on non-transferable utility games (NTU), we recommend the reader to check the comprehensive book by Peleg and Südholtzer [32]. This book covers TU games as well.

## 2.2 Basic definitions

We will start with the formal definition of a cooperative game.

**Definition 2.1.** (Cooperative game) *Cooperative game is an ordered pair  $(N, v)$ , where  $N = \{1, 2, \dots, n\}$  is a set of players and  $v : 2^N \rightarrow \mathbb{R}$  is a characteristic function of cooperative game. We further assume that  $v(\emptyset) = 0$ .*

The set of all cooperative games with player set  $N$  is denoted by  $G^N$ .

Subsets of  $N$  are called *coalitions* and  $N$  itself is called a *grand coalition*.

Observe that the characteristic function tells us what payoff is coalition able to get if all of its members cooperate together as a unit and that all the information about the game is encoded in its characteristic function.

The central goal of cooperative game theory is to analyze situations in which grand coalition forms, that is a situation in which all players cooperate together and receive  $v(N)$ . The problem is to split reward in such way that no coalition breaks of, i.e. so the grand coalition is stable situation.

To further analyze players' gains, we will introduce *payoff vector* which can be interpreted as a proposed distribution of reward between players.

**Definition 2.2.** (Payoff vector) *Payoff vector for a cooperative game  $(N, v)$  is a vector  $x \in \mathbb{R}^N$  with  $x_i$  denoting reward given to  $i$ th player.*

It is quite natural to introduce some restrictions on payoff vectors. The most important are efficiency and individual rationality. While efficiency ensures that whole grand coalition surplus is distributed, individual rationality states that the reward given to  $i$ th player is as high as a reward which would  $i$ th player achieve on his own.

**Definition 2.3.** (Efficiency) *A payoff vector is efficient if  $\sum_{i \in N} x_i = v(N)$ .*

**Definition 2.4.** (Individual rationality) *A payoff vector is individually rational if for every player  $i$  we have  $x_i \geq v(\{i\})$ .*

In some literature, efficiency is also called *group rationality*.

To distinguish between general payoff vectors and those which satisfy efficiency or both efficiency and individual rationality, we introduce *preimputation* and *imputation*.

**Definition 2.5.** (Preimputation) *A payoff vector satisfying efficiency is called preimputation.*

**Definition 2.6.** (Imputation) *A payoff vector satisfying efficiency and individual rationality is called imputation.*

*The set of all imputations of a given cooperative game  $(N, v)$  is denoted by  $I(v)$*

**Note 2.7.** *Formally, it would be better to write  $I((N, v))$  instead of  $I(v)$  but we will use the latter since it does not cause any confusion. Throughout this thesis, we will often write only  $v$  in place of  $(N, v)$  because we can easily identify game with its characteristic function without loss of generality.*

Group and individual rationality are reasonable restrictions but as we will see in the next section, even imputations do not give us stable payoff distributions.

## 2.3 Core

One of the most important solution concepts in cooperative game theory is *core*, introduced by Gillies in 1959 [17]. It gives us those imputations which are guaranteed to be stable in such a way that grand coalition forms and no subcoalition has an incentive to split off. First let us have a look at an example of game and an unstable imputation.

**Example 2.8.** (Unstable imputation) *Let us have a game  $G = (\{1, 2, 3\}, v)$  where  $v(\{1\}) = v(\{2\}) = v(\{3\}) = 1$ ,  $v(\{1, 2\}) = v(\{2, 3\}) = v(\{1, 3\}) = 3$  and  $v(\{1, 2, 3\}) = 4$ . Then in imputation  $(1, 1, 2)$  the coalition  $\{1, 2\}$  has an incentive to split off since it can achieve 3 on its own, instead of 2 which is imposed by this imputation.*

In the light of this example, it is quite natural to define core in a following way.

**Definition 2.9.** (Core) *The core of game  $(N, v)$  is a subset of imputations of  $(N, v)$  where each element satisfies following condition:*

$$\forall S \in 2^N \setminus \{\emptyset\} : \sum_{i \in S} x_i \geq v(S).$$

*The core of a given cooperative game  $(N, v)$  is denoted by  $C(v)$ .*

Clearly, every subcoalition that would leave grand coalition would not be able to get greater reward than the reward imposed by the payoff vector from core.

Natural question is to find sufficient and necessary conditions under which the core is a nonempty set. This characterization is known as *Bondareva-Shapley theorem*. It was stated and proved by Shapley [36] and Bondareva [9] independently on each other in 1960s. The following formulation of theorem and related definitions are from [32]. We refer reader to [32, p. 29] for a proof and a sharp form of the theorem as well.

**Definition 2.10.** (Characteristic vector of coalition) Let  $S \subseteq N$ . The characteristic vector  $\chi_S$  of  $S$  is the member of  $\mathbb{R}^N$  which is given by

$$\chi_S^i = \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \in N \setminus S. \end{cases}$$

**Definition 2.11.** (Balanced collection and balancing weights)

A collection  $\mathcal{B} \subseteq 2^N$ ,  $\emptyset \notin \mathcal{B}$ , is called *balanced* (over  $N$ ) if positive numbers  $\delta_S$ ,  $S \in \mathcal{B}$ , exist such that

$$\sum_{S \in \mathcal{B}} \delta_S \chi_S = \chi_N.$$

The collection  $(\delta_S)_{S \in \mathcal{B}}$  is called a *system of balancing weights*.

**Theorem 2.12.** (Bondareva-Shapley) A necessary and sufficient condition that the core of a game  $(N, v)$  is not empty is that for each balanced collection  $\mathcal{B}$  and each system  $(\delta_S)_{S \in \mathcal{B}}$  of balancing weights

$$v(N) \geq \sum_{S \in \mathcal{B}} \delta_S v(S).$$

Theorem 2.12 motivates us to define *balanced game* as a game with a nonempty core.

**Example 2.13.** (Game with empty core) Let us have a game  $G = (N, v)$  where  $N = \{1, 2, 3\}$  and  $v$  is the following characteristic function:

$x$	$v(x)$
$\emptyset$	0
$\{1\}$	1
$\{2\}$	2
$\{3\}$	3
$\{1, 2\}$	4
$\{1, 3\}$	4
$\{2, 3\}$	4
$\{1, 2, 3\}$	5

Collection  $A = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$  is balanced because there exists system of balancing weights ( $1/2$  for each coalition in  $A$ ). However,  $\frac{1}{2}v(\{1, 2\}) + \frac{1}{2}v(\{1, 3\}) + \frac{1}{2}v(\{2, 3\}) = 6 > v(N) = 5$  so from Bondareva-Shapley theorem we get that the core of this game is empty.

## 2.4 Shapley value

Shapley value, introduced and named by Lloyd Shapley [35] (Nobel Memorial Prize in Economic Science winner in 2012), is another important solution. Shapley value is a payoff vector which has two very desired properties. First one is that it always exists and second is that it gives to players rewards depending on their importance in coalition forming. Thus it can be considered as an “universal fair solution”.

Shapley theorem states that such desired function exists and that it is unique.



**Note 2.14.** We will use notation  $f_i(v)$  instead of  $f((N, v))_i$ .

**Theorem 2.15.** (Shapley) *There exists a unique function  $f : G^N \rightarrow \mathbb{R}^N$  satisfying following properties for every  $(N, v) \in G^N$ .*

(Efficiency) *It holds that  $\sum_{i \in N} f_i(v) = v(N)$ .*

(Dummy player) *It holds  $f_i(v) = 0$  for every  $i \in N$  such that for every  $S \setminus \{i\} \subseteq N$  equality  $v(S \cup \{i\}) = v(S)$  holds.*

(Symmetry) *If for every  $S \subseteq N$  with the property that  $\{i, j\} \not\subseteq S$  holds  $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ , then  $f_i(v) = f_j(v)$ .*

(Additivity) *For every two games  $(N, a)$  and  $(N, b)$  and for every  $i \in N$ , equality  $f_i(a + b) = f_i(a) + f_i(b)$  holds.*

It can be shown that the following function  $\phi$  satisfies these axioms.

**Definition 2.16.** (Shapley value function) *Shapley value function  $\phi : G^N \rightarrow \mathbb{R}^N$  is defined as*

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! (n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S)).$$

Definition of  $\phi$  can be interpreted as this: We give to each player his average marginal contribution to each coalition. We consider all orders in which coalition can be formed equally likely.

As we said, unlike core, Shapley value always exists, which is easily seen. As a corollary to this, we see that Shapley value does not necessarily has to lie inside of core (take for example a game with an empty core). But even a game with an nonempty core does not have to contain its Shapley value in the core.

**Example 2.17.** (Game with nonempty core and Shapley value not contained in core [16]) *Let us have a game  $G = (N, v)$  where  $N = \{1, 2, 3\}$  and  $v$  is the following characteristic function:*

$x$	$v(x)$
$\emptyset$	0
$\{1\}$	0
$\{2\}$	0
$\{3\}$	0
$\{1, 2\}$	100
$\{1, 3\}$	150
$\{2, 3\}$	0
$\{1, 2, 3\}$	150

*Shapley payoff vector  $(\phi_1(v), \phi_2(v), \phi_3(v))$  is equal to  $(91\frac{2}{3}, 16\frac{2}{3}, 41\frac{2}{3})$  as one can verify by inserting numbers into formula. Core of this game can be expressed like this:*

$$C(v) = \left\{ (x_1, x_2, x_3) \mid 100 \leq x_1 \leq 150, x_2 = 0, x_3 = 150 - x_1 \right\}.$$

*As we can easily check, core is nonempty and Shapley value does not lie in the core since  $\phi_2(v) \neq 0$ .*

## 2.5 Special classes of cooperative games

In this section, we will briefly speak about classes of cooperative games with special characteristic functions. We will also show some relations between them.

**Definition 2.18.** (Monotonic game) *A game  $(N, v)$  is monotonic if for every  $T \subseteq S \subseteq N$  holds*

$$v(T) \leq v(S).$$

Informally, in monotonic games, bigger coalitions are stronger.

**Definition 2.19.** (Superadditive game) *A game  $(N, v)$  is superadditive if for every  $S, T \subseteq N$ ,  $S \cap T = \emptyset$  holds*

$$v(T) + v(S) \leq v(S \cup T).$$

In superadditive game, coalition has no incentive to divide itself, since together, they will always achieve at least as much as separated.

Superadditive game is not necessarily monotonic. Conversely, monotonic game is not necessarily superadditive. However, these classes have a nonempty intersection. Check Caulier's paper [13] for more details on relation of these two classes.

We will define *subadditive* and *additive* game in a similar manner.

**Definition 2.20.** (Subadditive game) *A game  $(N, v)$  is subadditive if for every  $S, T \subseteq N$ ,  $S \cap T = \emptyset$  holds*

$$v(T) + v(S) \geq v(S \cup T).$$

**Definition 2.21.** (Additive game) *A game  $(N, v)$  is additive if for every  $S, T \subseteq N$ ,  $S \cap T = \emptyset$  holds*

$$v(T) + v(S) = v(S \cup T).$$

Observe that additive games lie in the intersection of subadditive and superadditive games.

Another important type of game is *convex game*.

**Definition 2.22.** (Convex game) *A game  $(N, v)$  is convex if its characteristic function is supermodular. The characteristic function is supermodular if for every  $S \subseteq T \subseteq N$  holds*

$$v(T) + v(S) \leq v(S \cup T) + v(S \cap T).$$

Clearly, supermodularity implies superadditivity.

To understand convex games more, let us look on one of its characterizations.

**Theorem 2.23.** (Characterization of convex games) *A game  $(N, v)$  is convex if and only if for every  $i \in N$ ,  $S \subseteq T \subseteq N \setminus \{i\}$ :*

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

*Proof.* Proof can be found e.g. in Section V.1 of [15]. □

Thus we can say that in convex game, player's contribution to coalition's reward grows with the size of coalition. Or, as Shapley says [37]: "*The incentives for joining a coalition increase as the coalition grows.*"

Convex games have many nice properties. we show two of them which we think are the most important.

**Theorem 2.24.** *If a game  $(N, v)$  is convex, then its core is nonempty.*

**Theorem 2.25.** *If a game  $(N, v)$  is convex, then a vector  $(\phi_1(v), \dots, \phi_n(v))$  is an element of its core.*

Both of these results are due to Shapley and proofs can be found in [37].



## 3. Interval analysis

This chapter is intended as an introduction to interval analysis. Tools of this chapter will be heavily used in studying cooperative interval games. In Section 1 we introduce interval analysis and show the motivation to study it. Next we introduce interval arithmetic. In the third section, we will present basic terminology regarding interval matrices and interval linear systems. Section 4 shows some basic definitions and facts regarding the use of intervals with functions. The last section shows how can we compare intervals.

### 3.1 Introduction and motivation

Interval analysis is a field of mathematics which uses intervals for computing rigorous bounds on problems dealing with inexact data. It was introduced by R. E. Moore in the late 1950s during his doctoral studies on Stanford. One of the first sources on how to work with intervals is Moore's dissertation [27].

Original motivation to build the theory of interval analysis was to avoid rounding errors and inaccuracy following from the fact that irrational and some rational numbers do not have a finite binary expansion. In general, these problems cannot be solved by recomputing with higher precision arithmetic and therefore some other tool is needed. To illustrate this motivation, take a look on a following example given by Rump [34].

**Example 3.1.** *Let's have a variable  $c$  defined in a following way:*

$$c = 332.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + a/(2b)$$

*with  $a = 77617.0$  and  $b = 33096.0$ . Computing powers of  $a$  and  $b$  successively by multiplications on an IBM 370 using different precisions will yield a following results:*

$$\begin{aligned} \text{(single precision)} \quad c &= \underline{1.17260361} \dots \\ \text{(double precision)} \quad c &= \underline{1.17260394005317847} \dots \\ \text{(extended precision)} \quad c &= \underline{1.17260394005317863185} \dots \end{aligned}$$

*From these results, one would say that the exact result has to be close to the number 1.172603 but it is  $-0.827396 \dots$*

However, this is not the only use of interval analysis. Interval analysis can also be used to solve problems regarding uncertainty. Suppose we do not know precise input data but we are able to encapsulate them with intervals that surely contain input. Then, under some additional conditions and assumptions, we can count with these intervals instead and get rigorous bounds on solution. This is in contrast with approach of choosing some concrete approximate value in the beginning and ending up with results with possibly unknown level of inaccuracy.

Interval analysis found applications in global optimization, fluid mechanics, artificial intelligence, economics, robotics, CSP (constraint satisfaction problems) and was even used for computer-aided proof of the Kepler conjecture [19]. We refer reader to Section 4 of [22] for more information on applications.

Of course, interval analysis is not a holy grail and does have its own pitfalls. We need to work with interval arithmetic carefully. Otherwise, obtained solutions can be not tight enough. One of the common issues is also an interval dependency problem which will be described in more detail in Section 3.4.

## 3.2 Interval arithmetic

Let us first formally define intervals.

**Definition 3.2.** (Interval) *The interval  $X$  is a set*

$$X := [\underline{X}, \overline{X}] = \{x \in \mathbb{R} : \underline{X} \leq x \leq \overline{X}\}.$$

With  $\underline{X}$  being the lower bound and  $\overline{X}$  being the upper bound of the interval.

So from now on when we say an interval we mean a closed interval. The set of all real intervals is denoted by  $\mathbb{IR}$ .

It is useful to introduce the following definitions as well.

**Definition 3.3.** (Width) *The width  $w(X)$  of an interval  $X$  is defined as*

$$w(X) := \overline{X} - \underline{X}.$$

**Definition 3.4.** (Midpoint) *The midpoint  $m(x)$  of an interval  $X$  is defined as*

$$m(X) := \frac{1}{2}(\overline{X} + \underline{X}).$$

It is time to show how to count with intervals. This is how are defined basic operations.

**Definition 3.5.** *For every  $X, Y, Z \in \mathbb{IR}$  and  $0 \notin Z$  holds*

$$X + Y := \{x + y : x \in X, y \in Y\} = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}],$$

$$X - Y := \{x - y : x \in X, y \in Y\} = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}],$$

$$X \cdot Y := \{x \cdot y : x \in X, y \in Y\} = [\min S, \max S], S = \{\underline{X}\overline{Y}, \overline{X}\underline{Y}, \underline{X}\underline{Y}, \overline{X}\overline{Y}\},$$

$$X / Z := \{x / z : x \in X, z \in Z\} = [\min S, \max S], S = \{\underline{X}/\overline{Z}, \overline{X}/\underline{Z}, \underline{X}/\underline{Z}, \overline{X}/\overline{Z}\}.$$

Notice that any real number can be associated to a degenerate interval (lower and upper bound being equal).

It is important to note that while interval addition and multiplication is associative and commutative, distributivity does not hold in general. Take for example intervals  $A = [1, 2]$ ,  $B = [1, 1]$  and  $C = [-1, -1]$ . Then  $A(B + C) = [0, 0]$  but  $AB + AC = [-1, 1]$ . However, we can introduce somewhat weaker property called *subdistributivity*.

**Proposition 3.6.** (Subdistributivity) *For any intervals  $X, Y$  and  $Z$  holds*

$$X(Y + Z) \subseteq XY + XZ.$$

*Proof.* Trivial with use of definitions of arithmetic operations. □

**Note 3.7.** *Reader who is already familiar with interval analysis may wonder why we do not use its standardized notation [23]. It is because papers on cooperative interval games are not using it and we felt that it is better to be consistent with existing literature on cooperative interval games.*

### 3.3 Interval linear algebra

In this section, we will briefly discuss generalizing of interval notion to vectors and matrices.

The following definitions of *interval vector* and *interval matrix* are used in [29].

**Definition 3.8.** (Interval vector) *By interval  $n$ -dimensional vector we mean an ordered  $n$ -tuple of intervals:*

$$(X_1, \dots, X_n).$$

*We denote set of all interval  $n$ -dimensional vectors by  $\mathbb{IR}^n$ .*

**Definition 3.9.** (Interval matrix) *By interval matrix we mean matrix whose elements are intervals. We denote set of all interval  $n \times m$  matrices by  $\mathbb{IR}^{n \times m}$ .*

**Definition 3.10.** (Realization/selection) *Realization (or selection) of an interval matrix  $A \in \mathbb{IR}^{n \times m}$  is a matrix  $A' \in \mathbb{R}^{n \times m}$  such that*

$$\underline{A}_{ij} \leq A'_{ij} \leq \bar{A}_{ij} \text{ for every } i, 1 \leq i \leq n, \text{ and } j, 1 \leq j \leq m.$$

We can use the term of realization and selection for interval vectors in a similar manner.

Let us now formally define *interval linear systems* and their properties and two different kind of solutions.

**Definition 3.11.** (Interval linear system) *Let  $A \in \mathbb{IR}^{k \times m}$ ,  $C \in \mathbb{IR}^{l \times m}$ ,  $b \in \mathbb{IR}^k$  and  $d \in \mathbb{IR}^l$ . Interval linear system is a system*

$$Ax = b, Cx \leq d.$$

**Definition 3.12.** (Weak solution and weakly solvable interval system) *A weak solution of interval linear system (with notation as in Definition 3.11) is a vector  $x \in \mathbb{R}^m$  satisfying system*

$$A'x = b', C'x \leq d'$$

*for some  $A' \in A$ ,  $b' \in b$ ,  $C' \in C$  and  $d' \in d$ . Interval linear system is weakly solvable if it does have some weak solution.*

**Definition 3.13.** (Strongly solvable interval system) *An interval linear system is strongly solvable if each of its realizations is solvable.*

**Definition 3.14.** (Strong solution of interval system) *A vector  $x' \in \mathbb{R}^m$  is a strong solution of interval linear system (with notation as in Definition 3.11) if it is solution for all realizations of  $A, b, C$  and  $d$ .*

Note that while an existence of a strong solution implies strong solvability, converse does not hold in general.

## 3.4 Intervals and functions

Imagine that we have some function  $f$  which is mapping from the set of real numbers. We would like to know what is the image of this function for some given interval. We will call this image *united extension*.

**Definition 3.15.** (United extension) *United extension of function  $f : \mathbb{R} \rightarrow \mathbb{R}$  over interval  $X$  is a set*

$$f(X) = \{f(x) \mid x \in X\}.$$

For further discussion, we need to define an *interval extension*.

**Definition 3.16.** (Interval extension [29]) *We say that  $F : \mathbb{IR} \rightarrow \mathbb{R}$  is an interval extension of  $f : \mathbb{R} \rightarrow \mathbb{R}$ , if  $F(X) \supseteq \{f(x) \mid x \in X\}$  and for degenerate interval arguments,  $F$  agrees with  $f$  (that is  $F([x, x]) = f(x)$ ).*

The united extension and the interval extension can be easily extended to multi-variable functions.

As we said in this chapter's introduction, one of the problems with the use of intervals is interval dependency. Take for example functions  $f : \mathbb{IR} \rightarrow \mathbb{IR}$  and  $g : \mathbb{IR} \rightarrow \mathbb{IR}$ :

$$\begin{aligned} f(X) &= [0, 0], \\ g(X) &= X - X. \end{aligned}$$

Clearly,  $f([0, 1]) = [0, 0]$ , but  $g([0, 1]) = [0, 1] - [0, 1] = [-1, 1]$ .

Overestimation in  $g$  is due to fact that in  $[0, 1] - [0, 1]$  we take numbers from first and from second interval independently, e.g. 0 and 1. When dealing with interval computations, we have to be very careful about this phenomenon.

The function  $g$  illustrates an important fact that not all interval extensions of a real function yield an united extension. A simple replacement of real variables by intervals might not be a good idea in general.

However, a classical result in interval analysis (see e.g. [28]) states, that an interval extension  $F$  of function  $f$  in which every variable appears only once and only basic arithmetic operations ( $+$ ,  $-$ ,  $\cdot$ ,  $/$ ) are used always yields an united extension of  $f$ .



**Note 3.17.** In [28], statement and proof of this result and related definitions are more complicated and more formal. We made a few simplifications in this text, so the reader is not overwhelmed by formality and by too many definitions.

### 3.5 Comparing intervals

It will come in handy to be able to compare intervals. There are basically two ways.

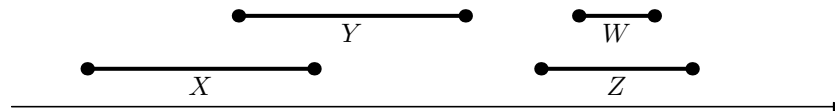
We say that interval  $X$  is greater or equal than interval  $Y$  if  $\underline{X} \geq \underline{Y}$ . We denote it by  $X \succeq Y$  or  $Y \preceq X$ . Intervals are equal with respect to this relation if  $\underline{X} = \underline{Y}$  and  $\overline{X} = \overline{Y}$ . Interval  $X$  is greater than  $Y$  ( $X \succ Y$ ) if  $X \succeq Y$  and  $X$  is not equal to  $Y$ .

Second way is to use relation *weakly better*. We say that interval  $X$  is weakly better than  $Y$  ( $X \succeq Y$  or  $Y \preceq X$ ) if  $\underline{Y} \leq \underline{X}$  and  $\overline{Y} \leq \overline{X}$ . Two intervals are equal with respect to this relation if  $X \succeq Y$  and  $Y \succeq X$ . Interval  $X$  is better than  $Y$  ( $X \succ Y$ ) if  $X \succeq Y$  and  $X$  is not equal to  $Y$ .

We easily see that in both cases equality occurs if and only if  $\underline{X} = \underline{Y}$  and  $\overline{X} = \overline{Y}$ , i.e. when both intervals represent the same set. We denote it by  $X = Y$ .

Finally, it is important to note that both of these relations are partial orderings of  $\mathbb{IR}$  but not total orderings. As one can see in the following example, it is possible to have pair of intervals which are not comparable by these relations.

**Example 3.18.** Consider intervals  $X, Y, Z$  and  $W$  with  $X = [1, 4]$ ,  $Y = [3, 6]$ ,  $Z = [7, 9]$  and  $W = [7.5, 8.5]$ . Figure 3.1 shows their mutual position. Note that all of these intervals are from  $\mathbb{IR}$  but for better readability are not drawn on the same line.



**Figure 3.1:** Interval  $Y$  is weakly better than  $X$ . Interval  $Z$  is greater than both  $X$  and  $Y$ . Interval  $W$  is not comparable with  $Z$  by any of two relations.



# 4. Cooperative interval games

This chapter has the following structure. In Section 1 we talk about motivation and history of cooperative interval games. In Section 2 we present necessary basic definitions and facts. Subsequent sections show our new results.

Section 4.3 is devoted to selection-based special classes of interval cooperative games which are direct analogy of some special classes in classical cooperative games. Fourth section aims on new results regarding imputations and core. We examine a problem stated in [1] which asks for characterization of games for which set of vectors generated by interval core equals to the selection core. Then a definition of strong imputation and core is introduced as a new concept in cooperative interval games. Section 4.5 sheds a new light on interval Shapley value. Finally, the last section shows some of the open problems.

## 4.1 Introduction

In Chapter 2 we gave a quick overview of classical cooperative game theory. We will now introduce another model, this time incorporating interval uncertainty. As we saw, cooperative game is defined as a pair  $(N, v)$  with characteristic function  $v$  encoding all information about game. However, in many real world situations, e.g. in operations research (OR), biology or political sciences, we do not know precise worths of coalitions. On the other hand, we are very often able to enclose possible coalition value by its lower and upper bound representing worst and best case scenario. Therefore, it is natural to extend characteristic function to the set of real closed intervals and get the definition of cooperative interval game.

Cooperation under interval uncertainty was first considered by Branzei, Dimitrov and Tijs in 2003 to study bankruptcy situations [11] and later further extensively studied by Gök in her PhD thesis [1] and other papers written together with Branzei et al. (see the references section of [10] for more).

It is worth noting that there exist other non-classical models of cooperative games incorporating some kind of uncertainty such as games with random payoffs [38], games with fuzzy uncertainty [26] or even combination of fuzzy and interval games – cooperative fuzzy interval games [25].

## 4.2 Basic facts and definitions

**Definition 4.1.** (Cooperative interval game) *Cooperative game is an ordered pair  $(N, w)$ , where  $N = \{1, 2, \dots, n\}$  is a set of players and  $w : 2^N \rightarrow \mathbb{IR}$  is a characteristic function of cooperative game. We further assume that  $w(\emptyset) = [0, 0]$ .*

*Set of all interval cooperative games on player set  $N$  is denoted by  $IG^N$ .*

**Note 4.2.** *We often write  $w(i)$  instead of  $w(\{i\})$ .*

**Remark 4.3.** *Each cooperative interval game in which characteristic function maps to degenerate intervals only can be associated with some classical cooperative game. Converse holds as well.*

It will be useful to name the following important games associated with each element of  $IG^N$ .

**Definition 4.4.** (Border games) *For every  $(N, w) \in IG^N$ , border games  $(N, \underline{w})$  (lower border game) and  $(N, \bar{w})$  (upper border game) are given by  $\underline{w}(S) = \underline{w}(S)$  and  $\bar{w}(S) = \overline{w(S)}$  for every  $S \in 2^N$ .*

**Definition 4.5.** *Length game of  $(N, w) \in IG^N$  is a game  $(N, |w|) \in G^N$*

$$|w|(S) = \bar{w}(S) - \underline{w}(S), \quad \forall S \in 2^N.$$

Basic term of our approach will be a selection which is a direct analogy of a definition of the selection for interval matrices and vectors 3.10.

**Definition 4.6.** (Selection) *Game  $(N, v) \in G^N$  is a selection of  $(N, w) \in IG^N$  if for every  $S \in 2^N$  holds  $v(S) \in w(S)$ . Set of all selections of  $(N, w)$  is denoted by  $\text{Sel}(w)$ .*

With the knowledge of this definition, we can define imputations and core.

**Definition 4.7.** (Interval selection imputation) *Set of interval selection imputations (or just selection imputations) of  $(N, w) \in IG^N$  is defined as*

$$\mathcal{SI}(w) = \bigcup \{I(v) \mid v \in \text{Sel}(w)\}.$$

**Definition 4.8.** (Interval selection core) *Interval selection core (or just selection core) of  $(N, w) \in IG^N$  is defined as*

$$\mathcal{SC}(w) = \bigcup \{C(v) \mid v \in \text{Sel}(w)\}.$$

Gök et al. choose an approach using a weakly better operator (see 3.5). This was inspired by [33]. Their definition of imputation and core is as follows.

**Definition 4.9.** (Interval imputation) *Set of interval imputations of  $(N, w) \in IG^N$  is defined as*

$$\mathcal{I}(w) := \left\{ (I_1, I_2, \dots, I_n) \in \mathbb{IR}^N \mid \sum_{i \in N} I_i = w(N), I_i \succeq w(i), \forall i \in N \right\}.$$

**Definition 4.10.** (Interval core) *An interval core of  $(N, w) \in IG^N$  is defined as*

$$\mathcal{C}(w) := \left\{ (I_1, I_2, \dots, I_n) \in \mathcal{I}(w) \mid \sum_{i \in S} I_i \succeq w(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

**Note 4.11.** *Proposition 2.1 and Theorem 3.1 of [2] show an analogy of Bondareva-Shapley theorem for  $\mathcal{SC}$  and  $\mathcal{C}$ , respectively. As we will not need these results, we do not show them here.*

Important difference between these four definitions is that selection concepts yield a payoff vectors from  $\mathbb{R}^N$ , while  $\mathcal{I}$  and  $\mathcal{C}$  yield vectors from  $\mathbb{IR}^N$ . That means that the selection-based approach gives us payoffs with no additional uncertainty. However, this approach was never systematically studied and very little is known. This thesis is trying to fix this and we will concentrate almost purely on selections.

**Note 4.12.** (Notation) *Throughout the papers on cooperative interval games, notation, especially of core and imputations, is not unified. It is therefore possible to encounter different notation from ours.*

*Also, in these papers, selection core is called core of interval game. We consider that confusing and that is why do we use term selection core instead. Term selection imputation is used because of its connection with selection core.*

## 4.3 Special classes of interval games

### 4.3.1 Existing classes of interval games

This subsection aims on presenting existing classes of interval games which have been studied earlier (see e.g. [3]). This is necessary later when we introduce selection-based classes and show their relations with the existing ones.

**Definition 4.13.** (Size monotonicity) *A game  $(N, w) \in IG^N$  is size monotonic if for every  $T \subseteq S \subseteq N$  holds*

$$|w|(T) \leq |w|(S).$$

*That is when its length game is monotonic.*

*Class of size monotonic games on player set  $N$  is denoted by  $SMIG^N$ .*

As we can see, size monotonic games capture situations in which an interval uncertainty grows with the size of coalition.

**Definition 4.14.** (Superadditive interval game) *A game  $(N, w) \in IG^N$  is super-additive interval if for every  $S, T \subseteq N$ ,  $S \cap T = \emptyset$  holds*

$$w(T) + w(S) \preceq w(S \cup T).$$

*We denote by  $SIG^N$  class of superadditive interval games on player set  $N$ .*

We should be careful with an analogy of convex game, since now supermodularity is not the same as convexity.

**Definition 4.15.** (Supermodular interval game) *An interval game  $(N, v)$  is supermodular interval if for every  $S \subseteq T \subseteq N$  holds*

$$v(T) + v(S) \preceq v(S \cup T) + v(S \cap T).$$

We get immediately that interval game is supermodular interval if and only if its border games are convex.

**Definition 4.16.** (Convex interval game) *An interval game  $(N, v)$  is convex interval if its border games and length game are convex.*

We write  $\text{CIG}^N$  for a set of convex interval games on player set  $N$ .

Convex interval game is supermodular as well but converse does not hold in general.

See [3] for characterizations of convex interval games and discussion on their properties.

### 4.3.2 Selection-based classes of interval games

We will now introduce a new classes of interval games based on the properties of their selections. We think that it is natural way to generalize special classes from classical cooperative game theory. Consequently, we show their characterizations and relation to classes from preceding subsection.

**Definition 4.17.** (Selection monotonic interval game) *An interval game  $(N, v)$  is selection monotonic if all its selections are monotonic games. Class of such games on player set  $N$  is denoted by  $\text{SeMIG}^N$ .*

**Definition 4.18.** (Selection superadditive interval game) *An interval game  $(N, v)$  is selection superadditive if all its selections are superadditive games. Class of such games on player set  $N$  is denoted by  $\text{SeSIG}^N$ .*

**Definition 4.19.** (Selection convex interval game) *An interval game  $(N, v)$  is selection convex if all its selections are convex games. Class of such games on player set  $N$  is denoted by  $\text{SeCIG}^N$ .*

We see that many properties persist. For example, a selection convex game is a selection superadditive as well. Selection monotonic and selection superadditive are not subset of each other but their intersection is nonempty. Furthermore, selection core of selection convex game is nonempty, which is an easy observation.

We will now show characterizations of these three classes and consequently show their relations to existing classes presented in Subsection 4.3.1.

**Proposition 4.20.** *An interval game  $(N, w)$  is selection monotonic if and only if for every  $S, T \in N$ ,  $S \subsetneq T$  holds*

$$\bar{w}(S) \leq \underline{w}(T).$$

*Proof.* For the “only if” part, suppose that  $(N, w)$  is a selection monotonic and  $\bar{w}(S) > \underline{w}(T)$  for some  $S, T \in 2^N$ ,  $S \subsetneq T$ . Then selection  $(N, v)$  with  $v(S) = \bar{w}(S)$  and  $v(T) = \underline{w}(T)$  clearly violates monotonicity and we arrive at a contradiction with assumptions  $\nabla$ .

Now for the “if” part. For any two subsets  $S, T$  of  $N$ , one of the situations  $S \subsetneq T$ ,  $T \subsetneq S$  or  $S = T$  occur. For  $S = T$ , in every selection  $v$ ,  $v(S) \leq v(S)$  holds. As for the other two situations, it is obvious that monotonicity cannot be violated as well since  $v(S) \leq \bar{w}(S) \leq \underline{w}(T) \leq v(T)$ .  $\square$

**Note 4.21.** Notice the importance of using  $S \subsetneq T$  in the formulation of Proposition 4.20. It is because using of  $S \subseteq T$  (thus allowing situation  $S = T$ ) would imply  $\bar{w}(S) \leq \underline{w}(S)$  for every  $S$  in selection monotonic game which is obviously not true in general. In characterizations of selection superadditive and selection convex games, similar situation arises.

**Proposition 4.22.** An interval game  $(N, w)$  is selection superadditive if and only if for every  $S, T \in 2^N$  such that  $S \cap T = \emptyset$ ,  $S \neq \emptyset$ ,  $T \neq \emptyset$  holds

$$\bar{w}(S) + \bar{w}(T) \leq \underline{w}(S \cup T).$$

*Proof.* Similar to proof of Proposition 4.20. □

**Proposition 4.23.** An interval game  $(N, w)$  is selection convex if and only if for every  $S, T \in 2^N$  such that  $S \not\subseteq T$ ,  $T \not\subseteq S$ ,  $S \neq \emptyset$ ,  $T \neq \emptyset$  holds

$$\bar{w}(S) + \bar{w}(T) \leq \underline{w}(S \cup T) + \underline{w}(S \cap T).$$

*Proof.* Similar to proof of Proposition 4.20. □

Now let us look on relation with existing classes of interval games.

For selection monotonic and size monotonic games, their relation is obvious. For nontrivial games (that is games with player set size greater than one), selection monotonic game is not necessarily size monotonic. Converse is the same. Finding of counterexamples is left as an exercise.

**Proposition 4.24.** For every player set  $N$  with  $|N| > 1$ , following assertions hold.

- (i)  $\text{SeSIG}^N \not\subseteq \text{SIG}^N$ .
- (ii)  $\text{SIG}^N \not\subseteq \text{SeSIG}^N$ .
- (iii)  $\text{SeSIG}^N \cap \text{SIG}^N \neq \emptyset$ .

*Proof.* In (i), we can construct the counterexample in the following way.

Let us construct game  $(N, w)$ . For  $w(\emptyset)$ , interval is given. Now for any nonempty coalition, set  $w(S) := [2|S| - 2, 2|S| - 1]$ . For any  $S, T \in 2^N$  with  $S$  and  $T$  being nonempty, the following holds with the fact that  $|S| + |T| = |S \cup T|$  taken into account.

$$\begin{aligned} \bar{w}(S) + \bar{w}(T) &= (2|S| - 1) + (2|T| - 1) \\ &= 2|S| + 2|T| - 2 \\ &\leq 2|S \cup T| - 2 \\ &= \underline{w}(S \cup T) \end{aligned}$$

So  $(N, w)$  is selection superadditive. Its length game, however, is not superadditive, since for any two nonempty coalitions with the empty intersection  $|w|(S) + |w|(T) = 2 \not\leq 1 = |w|(S \cup T)$ .

In (ii), we can construct the following counterexample  $(N, w')$ . Set  $w'(S) = [0, |S|]$  for any nonempty  $S$ . Lower border game is surely superadditive, since  $0 + 0 \leq 0$ . For the upper game,  $\overline{w'}(S) + \overline{w'}(T) = |S| + |T| = |S \cup T| = \overline{w'}(S \cup T)$  for any  $S, T$  with the empty intersection, so the upper game is superadditive. Observe that length game is the same as upper border game. This shows an interval superadditivity.

However,  $(N, w')$  is clearly not selection superadditive because of nonzero upper bounds, zero lower bounds of nonempty coalitions and the characterization of  $\text{SeSIG}^N$  taken into account.

(iii) Nonempty intersection can be argued easily by taking some superadditive game  $(N, c) \in G^N$ . Then we can define corresponding game  $(N, d) \in IG^N$  with

$$d(S) = [c(S), c(S)], \quad \forall S \in 2^N.$$

Game  $(N, d)$  is selection superadditive since its only selection is  $(N, c)$ . And it is superadditive interval game since border games are supermodular and length game is  $|w|(S) = 0$  for every coalition, which trivially implies its superadditivity.  $\square$

**Proposition 4.25.** *For every player set  $N$  with  $|N| > 1$ , following assertions hold.*

- (i)  $\text{SeCIG}^N \not\subseteq \text{CIG}^N$ .
- (ii)  $\text{CIG}^N \not\subseteq \text{SeCIG}^N$ .
- (iii)  $\text{SeCIG}^N \cap \text{CIG}^N \neq \emptyset$ .

*Proof.* For (i), take a game  $(N, w)$  assigning to each nonempty coalitions  $S$  interval  $[2^{|S|} - 2, 2^{|S|} - 1]$ . From the characterization theorem, we get that in inequalities which must hold in order to meet necessary conditions of game to be selection convex,  $|S| < |S \cup T|$  and  $|T| < |S \cup T|$  must hold. This yield the following inequality.

$$\begin{aligned} \overline{w}(S) + \overline{w}(T) &\leq (2^{|S \cup T|} - 1) + (2^{|S \cup T|} - 1) \\ &= 2^{|S \cup T|} - 2 \\ &= \underline{w}(S \cup T) \\ &\leq \underline{w}(S \cup T) + \underline{w}(S \cap T) \end{aligned}$$

This concludes that  $(N, w)$  is selection convex. We see that border games and length game are convex too. To have a game so that it is selection convex and not convex interval game, we can take  $(N, c)$  and set  $c(S) := w(S)$  for  $S \neq N$  and  $v(N) := [\underline{w}(S), \underline{w}(S)]$ . Now the game  $(N, c)$  is still selection convex, but its length game is not convex, so  $(N, v)$  is not convex interval game, which is what we wanted.

In (ii), we can take a game  $(N, w')$  from Proposition 4.24. From the fact that  $|S| + |T| = |S \cup T| + |S \cap T|$ , it is clear that  $\overline{w'}$  is convex. Lower border game is trivially convex and the length game is the same as upper. However, for nonempty  $S, T \in 2^N$  such that  $S \not\subseteq T$ ,  $T \not\subseteq S$ ,  $S \neq \emptyset$ ,  $T \neq \emptyset$ , convex selection games characterization is clearly violated.



As for (iii), we can use the same steps as in (iii) of Proposition 4.24 or we can use a game  $(N, w)$  from (i) of this theorem.  $\square$

## 4.4 New results on imputations and core

### 4.4.1 Core coincidence

In Gök's PhD thesis [1], the following topic is suggested: “A difficult topic might be to analyze under which conditions the set of payoff vectors generated by the interval core of a cooperative interval game coincides with the core of the game in terms of selections of the interval game.”

We decided to examine this topic. We call it a *core coincidence problem*. This subsection shows our results.

**Note 4.26.** *We remind the reader that whenever we talk about relation and maximum/minimum/maximal/minimal vectors, we mean relation  $\leq$  on real vectors unless we say otherwise.*

Main thing to notice is that while the interval core gives us a set of interval vectors, selection core gives us a set of real numbered vectors. To be able to compare them, we need to assign to set of interval vectors a set of real vectors generated by these interval vectors. That is exactly what the following function  $\text{gen}$  does.

**Definition 4.27.** *A function  $\text{gen}$  maps to every set of interval vectors a set of real vectors. It is defined as*

$$\text{gen}(S) = \bigcup_{s \in S} \{x \mid x \text{ is a realization of } s\}.$$

Core coincidence problem can be formulated as this: What are the necessary and sufficient condition to satisfy  $\text{gen}(\mathcal{C}(w)) = \mathcal{SC}(w)$ ?

**Note 4.28.** *Because of incomparability of sets of  $\mathcal{C}$  and  $\mathcal{SC}$ , it is not formally right to speak about the coincidence of these two sets. However, we will use that wording from time to time, since it is clear what it says from the context.*

The main result of this subsection are two following theorems which give an answer to aforementioned question.

**Note 4.29.** *In the following text, by mixed system we mean a system of equalities and inequalities.*

**Theorem 4.30.** (Core coincidence characterization) *For every interval game  $(N, w)$  holds  $\text{gen}(\mathcal{C}(w)) = \mathcal{SC}(w)$  if and only if for every  $x \in \mathcal{SC}(w)$  exist non-*

negative vectors  $l^{(x)}$  and  $u^{(x)}$  such that

$$\sum_{i \in N} x_i - l_i^{(x)} = \underline{w}(N), \quad (4.1)$$

$$\sum_{i \in N} x_i + u_i^{(x)} = \overline{w}(N), \quad (4.2)$$

$$\sum_{i \in S} x_i - l_i^{(x)} \geq \underline{w}(S), \quad \forall S \in 2^N \setminus \{\emptyset\}, \quad (4.3)$$

$$\sum_{i \in S} x_i + u_i^{(x)} \geq \overline{w}(S), \quad \forall S \in 2^N \setminus \{\emptyset\}. \quad (4.4)$$

*Proof.* Idea: We will first prove, that for any game,  $\text{gen}(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$  holds. So for equality, we will only need to take care of  $\text{gen}(\mathcal{C}(w)) \supseteq \mathcal{SC}(w)$ .

For any  $x' \in \text{gen}(\mathcal{C}(w))$ , inequality  $\underline{v}(N) \leq \sum_{i \in N} x'_i \leq \overline{v}(N)$  obviously holds. Furthermore,  $x'$  is in core for any selection of interval game  $(N, s)$  with  $s$  given by

$$s(S) = \begin{cases} \left[ \sum_{i \in N} x'_i, \sum_{i \in N} x'_i \right] & \text{if } S = N \\ \left[ \underline{w}(S), \min(\sum_{i \in S} x'_i, \overline{w}(S)) \right] & \text{if otherwise.} \end{cases}$$

Clearly,  $\text{Sel}(s) \subseteq \text{Sel}(w)$  and  $\text{Sel}(s) \neq \emptyset$ . That concludes  $\text{gen}(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$ .

As for  $\text{gen}(\mathcal{C}(w)) \supseteq \mathcal{SC}(w)$ , we have some  $x \in \mathcal{SC}(w)$ . For this vector, we need to find some interval  $X$  form  $\mathcal{C}(w)$  so that  $x \in \text{gen}(X)$ . This is equivalent to the task of finding two nonnegative vectors  $l^{(x)}$  and  $u^{(x)}$  such that

$$([x'_1 - l_1^{(x)}, x'_1 + u_1^{(x)}], [x'_2 - l_2^{(x)}, x'_2 + u_2^{(x)}], \dots, [x'_n - l_n^{(x)}, x'_n + u_n^{(x)}]) \in \mathcal{C}(w).$$

From the definition of interval core, we can see that these two vectors have to satisfy exactly the mixed system (4.1) – (4.4). That completes the proof.  $\square$

**Example 4.31.** Consider an interval game with  $N = \{1, 2\}$  and  $w(\{1\}) = w(\{2\}) = [1, 3]$  and  $w(N) = [1, 4]$ . Then vector  $(2, 2)$  lies in core of selection with  $v(\{1\}) = v(\{2\}) = 2$  and  $v(N) = 4$ . However, to satisfy equation (4.1), we need to have  $\sum_{i \in N} l_i = 3$  which means that either  $l_1$  or  $l_2$  has to be greater than 1. That means we cannot satisfy (4.3) and we conclude that  $\text{gen}(\mathcal{C}(w)) \neq \mathcal{SC}(w)$ .

The following theorem shows that it suffices to check only minimal and maximal vectors of  $\mathcal{SC}(w)$ .

**Theorem 4.32.** For every interval game  $(N, w)$  and  $x \in \mathbb{R}^N$  holds  $x \in \text{gen}(\mathcal{C}(w))$  if there exist vectors  $q, r \in \mathbb{R}^N$  such that  $q, r \in \text{gen}(\mathcal{C}(w))$  and  $q_i \leq x_i \leq r_i$  for every  $i \in N$ .

*Proof.* Let  $l^{(r)}, u^{(r)}, l^{(q)}, u^{(q)}$  be the corresponding vectors in the sense of Theorem 4.30. We need to find vectors  $l^{(x)}$  and  $u^{(x)}$  satisfying (4.1) – (4.4) of Theorem 4.30.

Let's define vectors  $dq, dr \in \mathbb{R}^N$ :

$$dq_i = x_i - q_i,$$

$$dr_i = r_i - x_i.$$

Finally, we define  $l^{(x)}$  and  $u^{(x)}$  like this:

$$\begin{aligned} l_i^{(x)} &= dq_i + l_i^{(q)}, \\ u_i^{(x)} &= dr_i + u_i^{(r)}. \end{aligned}$$

Now we will check if we can satisfy (4.1) – (4.4).

For example (4.2):

$$\begin{aligned} \sum_{i \in N} x_i - l_i^{(x)} &= \sum_{i \in N} x_i - dq_i - l_i^{(q)} \\ &= \sum_{i \in N} x_i - x_i + p_i - l_i^{(q)} \\ &= \sum_{i \in N} p_i - l_i^{(q)} \\ &= \underline{w}(N). \end{aligned}$$

The last equality holds because of assumptions on  $q$  and  $l^{(q)}$ .

Other three are similar and obviously hold, so we omit their proof.  $\square$

**Theorem 4.33.** *For an interval game  $(N, w)$  with additive border games (see Definition 2.21) a payoff vector  $(\underline{w}(1), \underline{w}(2), \dots, \underline{w}(n)) \in \text{gen}(\mathcal{C}(w))$ .*

*Proof.* First, let us look on an arbitrary additive game  $(A, v_A)$ . From additivity condition and fact that we can write any subset of  $A$  as a union of one-player sets we conclude that  $v_A(A) = \bigcup_{i \in A} v_A(\{i\})$  for every coalition  $A'$ . This implies that vector  $a$  with  $a_i = v_A(\{i\})$  is in the core.

Now described argument can be applied to border games of  $(N, w)$ . Vector  $q \in \mathbb{R}^N$  with  $q_i = \underline{w}(i)$  is an element of core of  $(N, \underline{w})$  and an element of  $\mathcal{SC}(w)$ .

For vector  $q$  we want to satisfy mixed system (4.1)-(4.4) of Theorem 4.30.

Take a vector  $l$  containing zeros only and vector  $u$  with  $u_i = |w|(i)$ . From the additivity, we get that  $\sum_{i \in N} q_i - l_i = \underline{w}(N)$  and  $\sum_{i \in N} q_i + u_i = \bar{w}(N)$ .

Additivity further implies that inequalities (4.3) and (4.4) hold for  $q$ ,  $l$  and  $u$ . Therefore,  $q$  is an element of  $\text{gen}(\mathcal{C}(w))$ .  $\square$

Theorem implicates that for games with additive border games, we need to check the existence of vectors  $l$  and  $u$  from (4.1) – (4.4) of Theorem 4.30 for maximal vectors of  $\mathcal{SC}$  only. We need to check only the maximal vectors, because for any vector  $y \in \mathcal{SC}(w)$  holds  $(\underline{w}(1), \underline{w}(2), \dots, \underline{w}(n)) \leq y$ . In other words,  $(\underline{w}(1), \underline{w}(2), \dots, \underline{w}(n))$  is a minimum vector of  $\mathcal{SC}(w)$ .

## 4.4.2 Strong imputation and core

In this subsection, our aim will be on a new concept of *strong imputation* and *strong core*.

**Definition 4.34.** (Strong imputation) For a game  $(N, w) \in IG^N$  a strong imputation is a vector  $x \in \mathbb{R}^N$  such that  $x$  is an imputation for every selection of  $(N, w)$ .

**Definition 4.35.** (Strong core) For a game  $(N, w) \in IG^N$  a strong core is a union of vectors  $x \in \mathbb{R}^N$  such that  $x$  is an element of core of every selection of  $(N, w)$ .

We show the following four quick facts about the strong core.

**Remark 4.36.** For every interval game with nonempty strong core  $w(N)$  is a degenerate interval.

*Proof.* Easily by the fact that the element  $c$  of strong core must be efficient for every selection and therefore  $\sum_{i \in N} c_i = \underline{w}(N) = \bar{w}(N)$ .  $\square$

**Remark 4.37.** For every element  $c$  of strong core holds  $\sum_{i \in N} c_i = \bar{w}(N)$  and  $\sum_{i \in S} c_i \geq \bar{w}(S)$ ,  $\forall S \in 2^N \setminus \emptyset$ .

**Remark 4.38.** For every element  $c$  of strong core of  $(N, w)$ ,  $c \in \text{gen}(\mathcal{C}(w))$ .

*Proof.* Vector  $c$  has to satisfy mixed system (4.1)-(4.4) of Theorem 4.30 for some  $l, u \in \mathbb{IR}^N$ . We show that  $l_i = u_i = 0$  will do the thing.

Equalities (4.1) and (4.2) are satisfied by taking Remark 4.36 into account. Inequalities (4.3) and (4.4) are satisfied as a consequence of Remark 4.37.  $\square$

**Remark 4.39.** An interval game  $(N, w)$  has an nonempty strong core if and only if  $w(N)$  is a degenerate interval and the upper game  $\bar{w}$  has an nonempty core.

*Proof.* Combination of Remark 4.36 and 4.37.  $\square$

**Note 4.40.** All of the remarks mentioned her can be proved by constructing a corresponding interval matrix  $A$  and  $b$  for an interval game and by applying a characterization results of Hladík's paper [21]. Because it is more straightforward to show these remarks in a way as we did, we do not show how to build these interval matrices. However, reader is invited to do so.

**Note 4.41.** The reason behind the using of name strong core and strong imputation comes form the interval linear algebra. This is more apparent after reading a text in Section 3.3.

**Note 4.42.** One could say why we did not introduce a strong game as game in which each of its selection does have an nonempty core. This is because such games are already defined as strongly balanced games (see e.g. [2]).

## 4.5 Shapley value

The purpose of this section is to only quickly remark a fact about interval extension of Shapley value which was never stated before.

**Definition 4.43.** (Shapley interval extension) *By Shapley interval extension we mean a function  $\phi^{\mathcal{I}} : IG^N \rightarrow \mathbb{IR}^N$  defined as*

$$\phi_i^{\mathcal{I}}(w) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! (n - |S| - 1)!}{n!} (w(S \cup \{i\}) - w(S)).$$

In existing papers on the Shapley value for interval games [4] [7], Shapley interval extension is ignored because it does not satisfy an efficiency in general. Authors further restrict themselves only to size-monotonic games on which the efficiency is guaranteed.

We remark an important fact, that while the Shapley interval extension does not satisfy efficiency in general, each  $\phi_i^{\mathcal{I}}(w)$  contains all Shapley values for any selection. This easily follows from the classical result of interval analysis about which we talked in Section 3.4.

## 4.6 Open problems

Aim of this section is to present some of the open problems we discovered or encountered. We hope that it will stimulate research in the area of cooperative interval games.

### Partially fixed interval Shapley value

Suppose that uncertainty of Shapley value of some players is settled. What effect has this on other players? How do their lower and upper bounds of Shapley interval extension change?

### More applications

There are already some papers on applications of cooperative interval games (see [5] [18] [6] for applications to mountain situations, airport games and forest situations, respectively). For researchers with stronger background in economy or other social science, it could be easier to find some new applications.

### Relation with a new approach

Weibin Han, Hao Sun and Genjiu Xu studied a new approach [20] to interval games based on introducing a total order on  $\mathbb{IR}$ . It could be interesting to find the relation with the existing results.

### Stable sets from selection-based approach

Stable set is, next to the core and Shapley value, another important solution concept and is especially useful in situations when core is empty. Stable set for classical game can be defined as follows (definitions are from [12, p. 18]).

**Definition 4.44.** (Domination of imputation) Let  $(N, v) \in G^N$  and  $x, y \in I(v)$ . We say that  $x$  dominates  $y$ , and denote it by  $x \text{ dom } y$  if there exists  $S \in 2^N \setminus \{\emptyset\}$  such that

$$(i) \quad x_i > y_i \text{ for all } i \in S,$$

$$(ii) \quad \sum_{i \in S} x_i \leq v(S).$$

**Definition 4.45.** (Stable set) For  $(N, v) \in G^N$  a subset  $K$  of  $I(v)$  is called a stable set if the following conditions hold:

$$(i) \quad (\text{Interval stability}) \quad K \cap \text{dom}(K) = \emptyset,$$

$$(ii) \quad (\text{External stability}) \quad I(v) \setminus K \subseteq \text{dom}(K),$$

where by  $\text{dom}(A)$  we denote the set consisting of all imputations that are dominated by some element in  $A$ .

For example, one could examine a union or an intersection of sets of stable sets for each selection.

In [2] Gök et al. studied stable set analogy for interval games, defined with  $\preceq$  relation. In the concluding remarks, authors suggest to study stable sets from the selection point of view as well.

## Games with coincident cores

Interesting, yet as it seems a quite difficult question is what 'nice' classes and type of games do have a coincident cores. Games with degenerate intervals are quite clear example, but they are too simple, since in fact, they contain no uncertainty.

# 5. Conclusion

The main task of this thesis was to examine existing model of cooperative interval games, extend it and to find the characterizations of some of their solutions and properties. Subsequent section summarizes these efforts.

## 5.1 Results of this thesis

We extended interval cooperative games with the definitions of strong core and strong imputation and showed some of their properties.

We introduced new classes of interval games based on their selections, characterized these classes and showed their relation to corresponding existing classes of interval games. We emphasize the implications of Proposition 4.24 and 4.25. They show that studying the selection-based classes gives more sense, since existing classes have two major drawbacks:

- Classes based on weakly better operator may contain games with selections that do not have any link with the properties of their border games and consequently no link with the name of the class. For example, superadditive interval games may contain a selection that is not superadditive.
- Selection-based classes are not contained in corresponding classes based on weakly better operator. Therefore, all the results on existing classes are quite useless for selection-based classes.

We also examined the core coincidence problem and proved that it suffices to find for every selection core element a two special vectors. These vectors have to satisfy a mixed system (4.1) – (4.4) of Theorem 4.30.

We further show that checking minimal and maximal elements of selection core for existence of these vectors is enough to show the coincidence. On the top of that, we proved a statement saying that for interval game with border games ( $\underline{w}$  and  $\bar{w}$ ) being additive, it suffices to check maximal vectors only.

Thesis further summarizes struggles to find interval analogy of Shapley value and makes remark about bounds on possible Shapley values of players in interval game.

## 5.2 Future work

Section 4.6 presents several open problems. We think that theory of cooperative interval games is promising area of mathematics with many open problems and high potential of applicability in real world problems. Author of this thesis plans to attack some of the open problems in the near future.





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