Linear Programming Problems with Absolute Values and Interval Uncertainty

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What is Absolute Value Linear Programming?

Absolute value linear programming (AVLP)

Linear programming with absolute values

$$\max \ c^T x \ \text{ subject to } \ Ax - D|x| \le b$$

Assumption: $D \geq 0$

Negative coefficients can be reformulated as linear constraints

- Example: $2x + |x| \le 3$ rewrite as $2x + y \le 3$, $-y \le x \le y$

Hard and challenging problem: Reduction from integer programming

Consider a 0-1 integer linear program

$$\max \ c^T x \ \text{ subject to } \ Ax \le b, \ x \in \{0,1\}^n.$$

The problem equivalently states

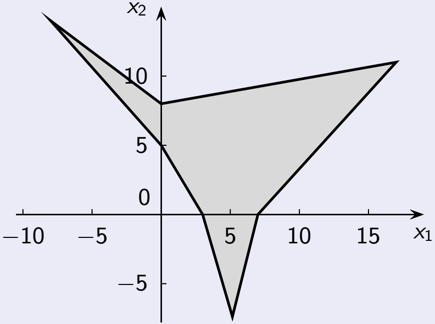
$$\max \ c^T x \ \text{ subject to } \ Ax \le b, \ |2x - e| = e.$$

The AVLP problem $\max\ c^T x$ subject to $Ax - D|x| \leq b$

- nonconvex and nonsmooth optimization problem
- the feasible set can be disconnected: $|x| = e$

Proposition

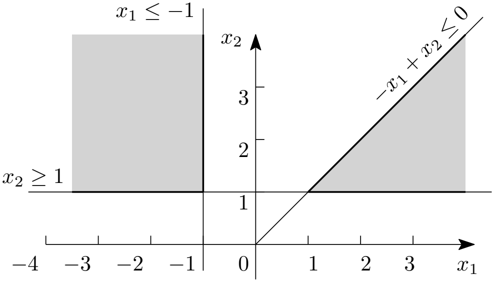
The feasible set is a convex polyhedra set inside each orthant.



Theorem (H., Hartman)

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a polyhedral set and convex in each orthant. Suppose there is no unbounded direction in the boundary of $\mathcal{S}$ that is orthogonal to an axis. Then $\mathcal{S}$ can be described by

$$Ax - D|x| \le b.$$



Theorem (H., Hartman)

The set

$$\mathcal{S} = \bigcup_{i=1}^{m} \{x \in \mathbb{R}^n : A^i x \le b^i\}.$$

can be described as an AVLP system with $\le \log(m)$ additional variables.

The AVLP problem

$$f^* = \max \; c^T x \;\text{ subject to }\; Ax - D|x| \le b.$$

Relation to Interval Linear Programming

- The feasible set is the united solution set of an interval linear system

  $$[A - D, A + D]x \le b.$$

  That is,

  $$\{x : Ax - D|x| \le b\} = \bigcup_{A' \in [A-D, A+D]} \{x : A'x \le b\}$$

- f^* is equal to the best-case optimal value of

  $$\max \; c^T x \;\text{ subject to }\; [A - D, A + D]x \le b.$$

  That is,

  $$f^* = \max_{\tilde{A} \in [A-D, A+D]} \; \max \; c^T x \;\text{ subject to }\; \tilde{A} \le b.$$

The interval AVLP problem

$$\max \ \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x - \boldsymbol{D}|x| \le \boldsymbol{b},$$

where

- $\boldsymbol{c} \in \mathbb{IR}^n$ and $\boldsymbol{b} \in \mathbb{IR}^m$ are interval vectors,
- $\boldsymbol{A}, \boldsymbol{D} \in \mathbb{IR}^{m \times n}$ are interval matrices.

Assumption

- $\underline{D} \ge 0$

Our goal: Range of optimal values

$$\overline{f} = \max \ f(A, b, c, D) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},\ D \in \boldsymbol{D},$$
$$\underline{f} = \min \ f(A, b, c, D) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},\ D \in \boldsymbol{D},$$

where $f(A, b, c, D)$ is the optimal value of the particular AVLP problem.

Proposition (Reduction to one AVLP problem)

We have

$$\overline{f} = \max\ c_c^T x + c_\Delta^T |x| \ \ \textit{subject to}\ \ A_c x - (A_\Delta + \overline{D})|x| \leq \overline{b}$$

Proposition (Reduction to 2^n LP problems)

We have

$$\overline{f} = \max_{s \in \{\pm 1\}^n}\ \max\ (c_c + \operatorname{diag}(s) c_\Delta)^T x$$

$$\textit{subject to}\ \ (A_c - (A_\Delta + \overline{D}) \operatorname{diag}(s))x \leq \overline{b},\ \ \operatorname{diag}(s)x \geq 0.$$

Corollary

$\overline{f}$ is the same as the best case optimal value of the interval LP problem

$$\max\ \boldsymbol{c}^T x \ \ \textit{subject to}\ \ \boldsymbol{A}^* x \leq \boldsymbol{b},$$

where $\boldsymbol{A}^* = \left[A_c - A_\Delta - \overline{D},\ A_c + A_\Delta + \overline{D}\right]$.

Observation

- $\underline{f}$ is attained for $b = \underline{b}$ and $D = \underline{D}$.
- For c and A no reduction can exist.

Proposition (Lower bound)

We have

$$\underline{f} \geq \underline{f}^L,$$

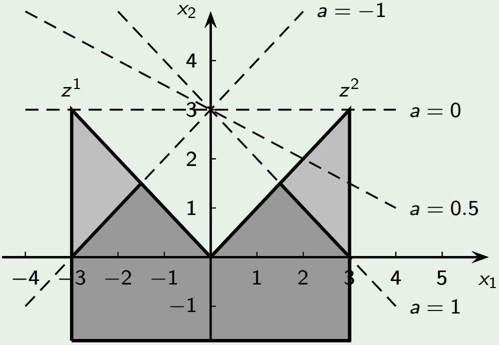
where

$$\underline{f}^L = \max\ c_c^T x - c_\Delta^T |x| \ \ \textit{subject to}\ \ A_c x + (A_\Delta - \underline{D})|x| \leq \underline{b}.$$

Example

$$\max\ x_2\ \text{ subject to }\ -3 \leq x_1 \leq 3,\ x_2 - |x_1| \leq 0,\ \boldsymbol{a}x_1 + x_2 \leq 3$$

- for $a \in [0, 1]$:
  optimum $z^1 = (-3, 3)^T$,
  optimal value 3
- for $a \in [-1, 0]$:
  optimum $z^2 = (3, 3)^T$,
  optimal value 3
- $f(a) = 3\ \forall a \in \boldsymbol{a} = [-1, 1]$
  and $\underline{f} = 3$
- however, $\underline{f}^L = 1.5$



$$\underline{f}^L = \max\ x_2\ \text{ subject to }\ -3 \leq x_1 \leq 3,\ x_2 - |x_1| \leq 0,\ x_2 + |x_1| \leq 3.$$

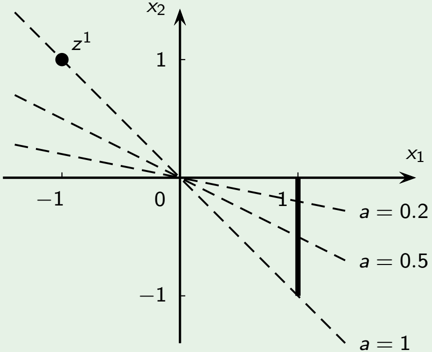
Upper bound iterative method (find a promising realization)

1. Put $b = \underline{b}$ and $D = \underline{D}$.
2. Put $A \coloneqq A_c$ and $c \coloneqq c_c$.
3. Compute
$$\underline{f}^U \coloneqq f(A, \underline{b}, c, \underline{D}).$$
and let s be the sign of the computed optimal solution.
4. Put
$$c \coloneqq c_c - \operatorname{diag}(s) c_\Delta, \quad A \coloneqq A_c + A_\Delta \operatorname{diag}(s).$$
5. Update the upper bound
$$\underline{f}^U \coloneqq \min\left\{\underline{f}^U,\ f(A, \underline{b}, c, \underline{D})\right\}.$$
6. Iterate this process until the upper bound $\underline{f}^U$ is not improved.

Example

$$\max\ x_2 \ \text{ subject to } \ -1 \leq x_1 \leq 1,\ -|x_1| \leq -1,\ 0 \leq x_1 + x_2 \leq 1,$$
$$x_2 \leq 1,\ \boldsymbol{a}x_1 + x_2 \leq 0,\ \boldsymbol{a} = [0,1]$$

- for $a \in [0,1)$: optimum $(1,-a)^T$, optimal value $-a$
- for $a = 1$: optimum $z^1 = (-1,1)^T$, optimal value 1
- Thus $\underline{f} = -1$, but not attained
- Now, $\underline{f}^L = \underline{f} = -1$



Upper bound $\underline{f}^U = -0.5$: $f(a_c) = -0.5$, $s = (1,-1)^T$, update $a := 1$.

Basis Stability

Basis stability of our problem

$$\max\ \boldsymbol{c}^T x : \boldsymbol{A}x - \boldsymbol{D}|x| \le \boldsymbol{b}$$

means basis stability of the interval LP relaxation

$$\max\ \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}^* x \le \boldsymbol{b},$$

where $\boldsymbol{A}^* = \left[A_c - A_\Delta - \overline{D},\, A_c + A_\Delta + \overline{D}\right]$.

That is, there is a basis B that is optimal for each realization.

- In interval LP, basis stability simplifies many problems
- Checking basis stability is co-NP-hard, but sufficient conditions exist

Proposition (Reduction to one LP problem)

Under B-stability, we have

$$\overline{f} = \max\ \overline{b}_B^T y \ \textit{ subject to } \ (A_c - A_\Delta - \overline{D})_B^T y \le \overline{c}, \quad (A_c + A_\Delta + \overline{D})_B^T y \ge \underline{c},\ \ y \ge 0.$$

Proposition (Reduction to one AVLP problem)

Under B-stability, we have

$$\underline{f} = \min \ c_c^T x - c_\Delta^T |x| \ \text{ subject to } \ (A_c)_B x + (A_\Delta - \underline{D})_B |x| \geq \underline{b}_B.$$

Theorem (Reduction to one absolute value system)

Under B-stability, the absolute value system

$$(A_c)_B x + (A_\Delta - \underline{D})_B |x| = \underline{b}_B \qquad (\star)$$

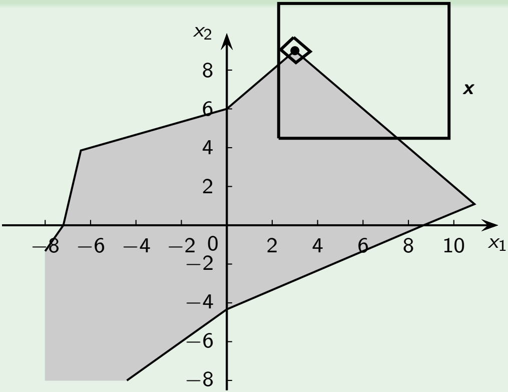
has the unique solution x^* and

$$\underline{f} = \underline{f}^L = c_c^T x^* - c_\Delta^T |x^*|.$$

Remarks

- In general, systems of type $(\star)$ are NP-hard to solve
- In our case of unique solvability, the complexity is unknown.

Example



$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \\ -6 & 2 \\ 4 & -7 \end{pmatrix},\ b = \begin{pmatrix} 12 \\ 18 \\ 36 \\ 26 \end{pmatrix},\ c = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix},\ \text{basis } B = \{1, 2\}.$$

Assume 5% uncertainty in the constraint matrix A:

$$\boldsymbol{A} = [A - 0.05|A|, A + 0.05|A|]$$

Conclusion

$$\begin{bmatrix}\texttt{Group on} \\ \texttt{Interval} \\ \texttt{Methods}\end{bmatrix}$$

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