Matrix Classes
Associated with Absolute Value Equations

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What are Absolute Value Equations (AVE)?

$$Ax + |x| = b \qquad (A \in \mathbb{R}^{n \times n},\ b \in \mathbb{R}^n)$$

Definition

The solution set of AVE is

$$\Sigma = \Sigma(b) = \{x \in \mathbb{R}^n : Ax + |x| = b\}.$$

Proposition

The solution set Σ forms a convex polyhedron in each orthant.

Proof.

In the orthant described $\operatorname{diag}(s)x \geq 0,\ s \in \{\pm 1\}^n$ we have

$$|x| = \operatorname{diag}(s)x.$$

So AVE reads $(A + \operatorname{diag}(s))x = b$. □

The solution set of AVE

- May possess up to 2^n isolated points
  (Example: $|x| = e$, where $e = (1, \ldots, 1)^T$
  Remark: each value between 1 and 2^n is attained)

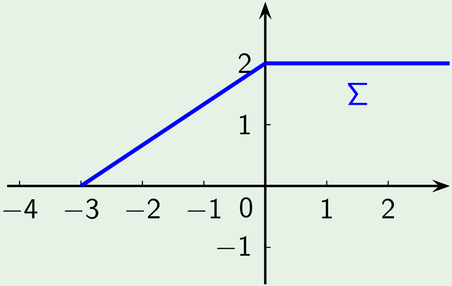
The solution set of AVE – Example 1

Example

Consider the absolute value equations

$$\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} x + |x| = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

Its solution set:



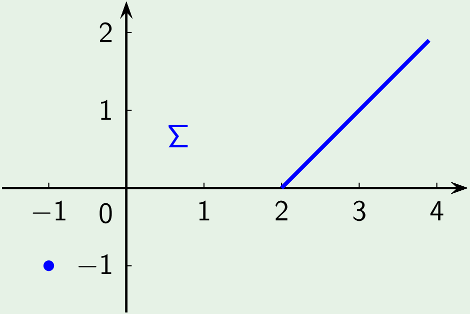
The solution set of AVE – Example 2

Example

Consider the absolute value equations

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} x + |x| = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Its solution set:



Equivalence to LCP (1/3)

The linear complementarity problem (LCP)

$$y = Mz + q, \; y^T z = 0, \; y, z \geq 0.$$

- quadratic programming, bimatrix games, discrete knapsack,...

Reduction AVE → LCP.

- Assume $A + I_n$ is nonsingular (reductions avoiding this exist).
- Write x as $x = x^+ - x^-$, where $x^+, x^- \geq 0$, $(x^+)^T x^- = 0$.
- Then $|x| = x^+ + x^-$
- Now, AVE reads $A(x^+ - x^-) + x^+ + x^- = b$, or after rearranging,

$$x^+ = (A + I_n)^{-1}(A - I_n)x^- + (A + I_n)^{-1}b.$$

Equivalence to LCP (2/3)

The linear complementarity problem (LCP)

$$y = Mz + q, \;\; y^T z = 0, \;\; y, z \geq 0.$$

Reduction LCP $\rightarrow$ AVE (Mangasarian, 2007).

- Assume $M - I_n$ is nonsingular (obtained by scaling M).
- Observation: $ab = 0, \;\; a, b \geq 0 \;\; \Leftrightarrow \;\; a + b = |a - b|$.
- Thus we can write LCP as

$$z + Mz + q = |z - Mz - q|.$$

- Substituting $x \equiv z - Mz - q$, we have $z = (I_n - M)^{-1}(x + q)$ and system reads

$$(M + I_n)(M - I_n)^{-1}x + |x| = 2(I_n - M)^{-1}q.$$

Equivalence to LCP (3/3)

The linear complementarity problem (LCP)

$$y = Mz + q,\ \ y^T z = 0,\ \ y, z \geq 0.$$

More than 50 matrix classes...

matrix type	definition
P-matrix	unique solution for each q
principally nondegenerate	finitely many solutions (incl. 0) for each q
strictly copositive	at least one solution for each q
semimonotone	unique solution for each $q > 0$
column sufficient	the solution set is convex (or empty)
R_0-matrix	the solution set is bounded
Q-matrix	at least one solution for each q

Our goal

- Similar matrix classes for AVE.

Theorem (Mangasarian, 2007)

Checking solvability of AVE is NP-complete.

Proof.

Reduction from Set-Partitioning:

$$\text{Given } a \in \mathbb{Z}^n, \text{ exists } x \in \{\pm 1\}^n : a^T x = 0?$$

Write it as

$$|x| = e, \ \ a^T x = 0.$$

Equivalently in the canonical form

$$\begin{aligned} |x| &= e, \\ a^T x + |x_{n+1}| &= 0, \\ -a^T x + |x_{n+2}| &= 0. \end{aligned}$$

□

Conditions for unsolvability (1/2)

From Reduction AVE $\to$ LCP:

$$(A + I_n)x^+ + (I_n - A)x^- = b,\ (x^+)^T x^- = 0,\ x^+, x^- \geq 0.$$

$$(A + I_n)x^+ + (I_n - A)x^- = b,\ \cancel{(x^+)^T x^- = 0,}\ x^+, x^- \geq 0.$$

Now, apply the Farkas lemma.

Theorem (Mangasarian & Meyer, 2006)

If

$$-y \leq A^T y \leq y, \quad b^T y < 0$$

is solvable, then AVE is unsolvable.

Theorem

The AVE is unsolvable if

$$\rho(|A|) < 1 \;\; \textit{and} \;\; (I_n - |A|)^{-1} b \textit{ is not nonnegative.}$$

Lemma

If $\rho(|A|) < 1$, then each solution x of AVE satisfies

$$|x| \leq (I_n - |A|)^{-1} b.$$

Proof.

For each solution

$$|x| = -Ax + b \leq |A| \cdot |x| + b.$$

whence

$$(I_n - |A|)|x| \leq b.$$

Eventually, premultiply by $(I_n - |A|)^{-1} = \sum_{k=0}^{\infty} |A|^k \geq 0$. □

Condition for 2^n solutions

Theorem

If $2|A||b| < b$, then the AVE has 2^n solutions, lying in the interiors of the particular orthants.

Proof.

For each $s \in \{\pm 1\}^n$, we want to show unique solvability of

$$(A + \mathrm{diag}(s))x = b, \quad \mathrm{diag}(s)x > 0.$$

By substitution $y \equiv \mathrm{diag}(s)x$,

$$(A\,\mathrm{diag}(s) + I_n)y = b, \quad y > 0.$$

From $2|A||b| < b$ we have $\rho(A\,\mathrm{diag}(s)) < \frac{1}{2}$, so

$$y = (I_n + A\,\mathrm{diag}(s))^{-1}b = \sum_{k=0}^{\infty}(-A\,\mathrm{diag}(s))^k b \overset{!}{>} 0$$

This happens if

$$b > \sum_{k=1}^{\infty} |A|^k b,$$

$$b > (I_n - |A|)^{-1}|A|b. \qquad \square$$

Unique solvability (1/2)

Interval matrices

- $[A - I_n, A + I_n] = \{B \in \mathbb{R}^{n \times n} : |A - B| \leq I_n\}$
- $[A - I_n, A + I_n]$ is regular if each matrix $B \in [A - I_n, A + I_n]$ is nonsingular

Theorem (Wu & Li, 2018)

The AVE has a unique solution for each $b \in \mathbb{R}^n$ if and only if $[A - I_n, A + I_n]$ is regular.

- Analogous to nonsingularity of A for $Ax = b$
- For LCP the condition is P-matrix property (all principal minors are positive)
- Which is NP-hard

Unique solvability (2/2)

Interval matrices

- $[A - I_n, A + I_n] = \{B \in \mathbb{R}^{n \times n} : |A - B| \leq I_n\}$
- $[A - I_n, A + I_n]$ is regular if each matrix $B \in [A - I_n, A + I_n]$ is nonsingular

Theorem (Wu & Li, 2018)

The AVE has a unique solution for each $b \in \mathbb{R}^n$ if and only if $[A - I_n, A + I_n]$ is regular.

Sufficient conditions:

$$\rho(|A^{-1}|) < 1 \quad \text{or} \quad \sigma_{\min}(A) > 1$$

- AVE is efficiently solvable then
- Open problem: Is AVE efficiently solvable if $[A - I_n, A + I_n]$ is regular?

Method for $\sigma_{\min}(A) > 1$ (Mangasarian & Meyer, 2006)

The iterations

$$x_{k+1} := -A^{-1}|x_k| + A^{-1}b, \quad k = 1, \dots \tag{$\star$}$$

converge and in polynomial time the right orthant is determined.

Proof.

We have $\sigma_{\min}(A) > 1 \;\Leftrightarrow\; \sigma_{\max}(A^{-1}) < 1 \;\Leftrightarrow\; \|A^{-1}\| < 1$.

Function $f(x) = -A^{-1}|x| + A^{-1}b$ given by ($\star$) is a contraction:

$$\begin{aligned} \|f(x) - f(y)\| &= \|A^{-1}(|x| - |y|)\| \\ &\le \|A^{-1}\| \cdot \||x| - |y|\| \\ &\le \|A^{-1}\| \cdot \|x - y\|. \end{aligned}$$

By the fixed-point theorem, there is a unique fixed-point. □

Nonnegative solvability (1/2)

Theorem

The following conditions are equivalent:

1. *The AVE system has a unique nonnegative solution for each $b \geq 0$;*
2. *the AVE system has a nonnegative solution for each $b \geq 0$;*
3. $(A - I_n)^{-1} \geq 0$.

Theorem (Kuttler, 1971)

An interval matrix $[A - I_n, A + I_n]$ is inverse nonnegative if and only if

$$(A - I_n)^{-1} \geq 0 \text{ and } (A + I_n)^{-1} \geq 0.$$

Proposition (One implication)

If $[A - I_n, A + I_n]$ is inverse nonnegative, then for each $b \geq 0$, the AVE system has a unique solution, and this solution is nonnegative.

- Effectively computable by the Newton method in at most n iterations.

Nonnegative solvability (2/2)

Theorem

The following conditions are equivalent:

1. *The AVE system has a unique nonnegative solution for each* $b \geq 0$*;*
2. *the AVE system has a nonnegative solution for each* $b \geq 0$*;*
3. $(A - I_n)^{-1} \geq 0$.

Theorem (Kuttler, 1971)

An interval matrix $[A - I_n, A + I_n]$ *is inverse nonnegative if and only if*

$$(A - I_n)^{-1} \geq 0 \text{ and } (A + I_n)^{-1} \geq 0.$$

Proposition (Second implication)

Let $[A - I_n, A + I_n]$ *be regular. If for each* $b \geq 0$ *the AVE has a unique solution that is nonnegative, then* $[A - I_n, A + I_n]$ *is inverse nonnegative.*

- This implication needs the regularity assumption (can be relaxed?).

Finitely and infinitely many solutions

Proposition

There is no AVE such that each orthant contains infinitely many solutions.

Proof.

The matrix $A + \operatorname{diag}(s)$ cannot be singular for every $s \in \{-1, +1\}^n$. □

Example

Consider the system $x + |x| = 0$. All orthants contain infinitely many solutions, except for the positive orthant.

Proposition

The set $\Sigma(b)$ is finite for each $b \in \mathbb{R}^n$ if and only if $A + \operatorname{diag}(s)$ is nonsingular for each $s \in \{-1, +1\}^n$.

Proposition

Checking whether $A + \operatorname{diag}(s)$ is nonsingular for each $s \in \{-1, +1\}^n$ is co-NP-hard on the class of problems with A having rank one.

Boundedness

Simple observations

- Regularity of $[A - I_n, A + I_n]$ implies boundedness of $\Sigma(b)$ for every b.
- The converse implication is not true in general; simply consider $A = 0$.

Proposition

The set $\Sigma(b)$ is bounded for each $b \in \mathbb{R}^n$ if and only if $Ax + |x| = 0$ has only the trivial solution $x = 0$.

Proposition

Checking whether $Ax + |x| = 0$ has a non-trivial solution is NP-hard on the class of problems with A having rank one.

Simple observations

- Regularity of $[A - I_n, A + I_n]$ implies uniqueness and thus convexity.

Proposition

The set Σ is convex if and only if it is located in one orthant only, i.e., there is $s \in \{-1, +1\}^n$ such that $\operatorname{diag}(s)x \geq 0$ for each $x \in \Sigma$.

Proposition

Checking convexity of Σ is co-NP-hard on the class of problems with $b = 0$ and A having rank one.

Simple observations

- Regularity of $[A - I_n, A + I_n]$ implies uniqueness and thus connectedness.

Proposition

If $b = 0$, then the solution set of AVE is connected.

Proof.

In each orthant, the corresponding solution set is connected and contains the origin, via which is the overall solution set connected. □

- No complete characterization known.

Conclusion

- Characterization of some matrix classes.
- Classification of computational complexity.
- Open problems left.

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