Interval Linear Programming and Its Applications

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Outline

Interval Linear Programming – Introduction

Consider an LP problem

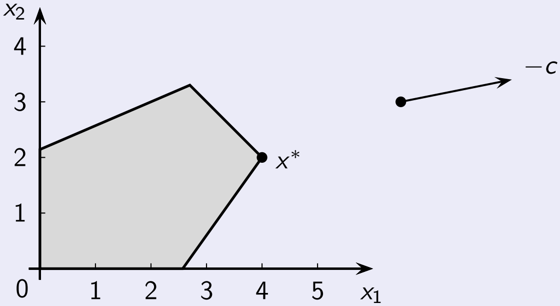
$$\min\ c^T x \ \text{ subject to }\ Ax \le b,\ x \ge 0$$

where

$$A = \begin{pmatrix} -3 & 7 \\ 7 & -5 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 15 \\ 18 \\ 6 \end{pmatrix}, \quad c = \begin{pmatrix} -5 \\ -1 \end{pmatrix}.$$

optimal solution:
$x^* = (4, 2)^T$

optimal value:
$c^T x^* = -22$



Consider an LP problem

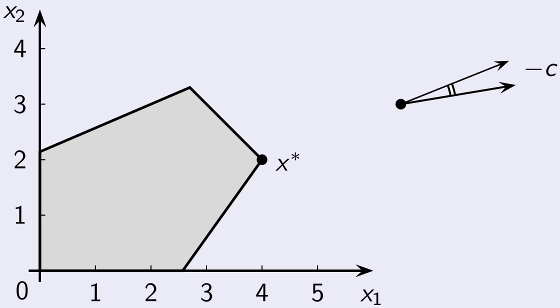
$$\min\ c^T x \ \text{ subject to }\ Ax \le b,\ x \ge 0$$

where

$$A = \begin{pmatrix} -3 & 7 \\ 7 & -5 \\ 1 & 1 \end{pmatrix},\quad b = \begin{pmatrix} 15 \\ 18 \\ 6 \end{pmatrix},\quad c \in \begin{pmatrix} -[5,6] \\ -[1,2] \end{pmatrix}.$$

optimal solution:
$x^* = (4,2)^T$

optimal value:
$c^T x^* \in -[22,28]$



Interval Linear Programming – Introduction

Consider an LP problem

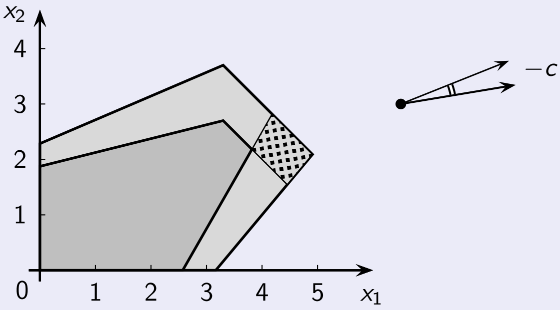
$$\min\ c^T x \ \text{ subject to } \ Ax \le b,\ x \ge 0$$

where

$$A \in \begin{pmatrix} -[2,3] & [7,8] \\ [6,7] & -[4,5] \\ 1 & 1 \end{pmatrix}, \quad b \in \begin{pmatrix} [15,16] \\ [18,19] \\ [6,7] \end{pmatrix}, \quad c \in \begin{pmatrix} -[5,6] \\ -[1,2] \end{pmatrix}.$$

optimal solution:
dotted area

optimal value:
$c^T x^* \in -[21.27, 33.64]$



Interval Linear Programming – Introduction

Linear programming

$$f(A, b, c) \equiv \min \ c^T x \ \text{ subject to } \ Ax \overset{(\leq)}{=} b, \ (x \geq 0)$$

Interval data

Given interval matrix

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = [A_c - A_\Delta, A_c + A_\Delta]$$

and interval vectors $\boldsymbol{b}$ and $\boldsymbol{c}$,

Interval linear programming

Family of linear programs with $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$, in short

$$f(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \equiv \min \ \boldsymbol{c}^T x \ \text{ subject to } \ \boldsymbol{A}x \overset{(\leq)}{=} \boldsymbol{b}, \ (x \geq 0).$$

Main goals

- determine the optimal value range;
- determine a tight enclosure to the optimal solution set.

Interval Linear Programming – Optimal Value Range

Definition

$$\underline{f} := \min\ f(A,b,c)\ \text{ subject to }\ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},$$
$$\overline{f} := \max\ f(A,b,c)\ \text{ subject to }\ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}.$$

Theorem (Vajda, 1961)

We have for type ($\min\ \boldsymbol{c}^T x$ *subject to* $\boldsymbol{A}x \le \boldsymbol{b},\ x \ge 0$)

$$\underline{f} = \min\ \underline{c}^T x\ \text{ subject to }\ \underline{A}x \le \overline{b},\ x \ge 0,$$
$$\overline{f} = \min\ \overline{c}^T x\ \text{ subject to }\ \overline{A}x \le \underline{b},\ x \ge 0.$$

Theorem (Machost, 1970, Rohn, 1984)

We have for type ($\min\ \boldsymbol{c}^T x$ *subject to* $\boldsymbol{A}x = \boldsymbol{b},\ x \ge 0$)

$$\underline{f} = \min\ \underline{c}^T x\ \text{ subject to }\ \underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b},\ x \ge 0,$$
$$\overline{f} = \max_{s \in \{\pm 1\}^m} f(A_c - \operatorname{diag}(s)A_\Delta,\ b_c + \operatorname{diag}(s)b_\Delta,\ \overline{c}).$$

Interval Linear Programming – Complexity

Summary of complexity of the basic problems (general case)

	$\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$	$\boldsymbol{A}x \leq \boldsymbol{b}$	$\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$
optimal value range	$\underline{f}$ polynomial, $\overline{f}$ NP-hard	$\underline{f}$ NP-hard, $\overline{f}$ polynomial	polynomial
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	??	NP-hard	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	NP-hard	NP-hard	NP-hard
basis stability	co-NP-hard	co-NP-hard	co-NP-hard

Interval Linear Programming – Complexity

Summary of complexity of the basic problems (A non-interval)

	$\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$	$\boldsymbol{A}x \leq \boldsymbol{b}$	$\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$
optimal value range	$\underline{f}$ polynomial, $\overline{f}$ NP-hard	$\underline{f}$ NP-hard, $\overline{f}$ polynomial	polynomial
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	polynomial	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	polynomial	NP-hard	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	polynomial	polynomial	polynomial
basis stability	polynomial	polynomial	polynomial

Summary of complexity of the basic problems (A, b non-interval)

	$\boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$	$\boldsymbol{A}x \leq \boldsymbol{b}$	$\boldsymbol{A}x \leq \boldsymbol{b},\ x \geq 0$
optimal value range	$\underline{f}$ polynomial, $\overline{f}$ polynomial	$\underline{f}$ NP-hard, $\overline{f}$ polynomial	polynomial
strong feasibility	polynomial	polynomial	polynomial
weak feasibility	polynomial	polynomial	polynomial
strong unboundedness	polynomial	polynomial	polynomial
weak unboundedness	polynomial	NP-hard	polynomial
strong optimality	polynomial	co-NP-hard	polynomial
weak optimality	polynomial	polynomial	polynomial
basis stability	polynomial	polynomial	polynomial

Next Section

Verification – Motivation

Example (Rump, 1988)

Consider the expression

$$f = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + \frac{a}{2b},$$

with $a = 77617$ and $b = 33096$.

Calculations from 80's:

single precision	$f \approx 1.172603\ldots$
double precision	$f \approx 1.1726039400531\ldots$
extended precision	$f \approx 1.172603940053178\ldots$
the **true** value	$f = -0.827386\ldots$

Ordóñez and Freund, 2003

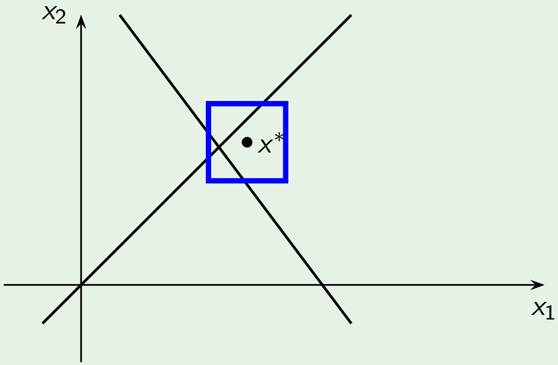
- 72% of real-life LP problems recorded in Netlib repository are ill-conditioned and many commercial solvers failed to solve them.

Verification for Linear Equations

Verification of a system of linear equations

Given a real system $Ax = b$ and x^* approximate solution, find $x^* \in \boldsymbol{x} \in \mathbb{IR}^n$ such that $A^{-1}b \in \boldsymbol{x}$.

Example



Verification for Linear Equations

Example

Let A be the Hilbert matrix of size 10 (i.e., $a_{ij} = \frac{1}{i+j-1}$), and $b := Ae$.

Then $Ax = b$ has the solution $x = e = (1, \ldots, 1)^T$.

Solution by Matlab	Enclosure by ε-inflation method
0.999999999235452	[0.99999973843401, 1.00000026238575]
1.000000065575364	[0.99999843048508, 1.00000149895660]
0.999998607887449	[0.99997745481481, 1.00002404324710]
1.000012638750021	[0.99978166603900, 1.00020478046370]
0.999939734980300	[0.99902374408278, 1.00104070076742]
1.000165704992114	[0.99714060702796, 1.00268292103727]
0.999727989024899	[0.99559932282378, 1.00468935360003]
1.000263042205847	[0.99546972629357, 1.00425202249136]
0.999861803020249	[0.99776781605377, 1.00237789028988]
1.000030414871015	[0.99947719419921, 1.00049082925529]

Overestimation factor about 20; compare $\kappa(A) \approx 1.6 \cdot 10^{13}$.

Verification in Linear Programming

Consider a linear program

$$\min \; c^T x \;\; \text{subject to} \;\; Ax = b, \; x \geq 0.$$

Let B^* be an optimal basis, f^* optimal value and x^* optimal solution. All these are numerically computed.

Verification of the optimal basis (Jansson, 1988)

- confirmation that B^* is (unique) optimal basis,

Verification of the optimal value (Neumaier & Shcherbina, 2004)

- finding $f^* \in \boldsymbol{f} \in \mathbb{IR}$ such that $\boldsymbol{f}$ contains the optimal value,

Verification of the optimal solution

- finding $x^* \in \boldsymbol{x} \in \mathbb{IR}^n$ such that $\boldsymbol{x}$ contains the (unique) optimum.

Relation

basis $\rightarrow$ optimal solution $\rightarrow$ optimal value

Verification of Optimal Basis

Non-interval case

Basis B is optimal iff

- C1 A_B is non-singular;
- C2 $A_B^{-1}b \geq 0$;
- C3 $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$.

Verification of condition C2

- Compute verification interval $\boldsymbol{x}_B$ for $A_B x_B = b$,
- check $\underline{x}_B \geq 0$ (resp. $\underline{x}_B > 0$ for uniqueness)

Verification of condition C3

- Compute verification interval $\boldsymbol{y}$ for $A_B^T y = c_B$,
- check $c_N^T - \boldsymbol{y}^T A_N \geq 0$ (resp. $c_N^T - \boldsymbol{y}^T A_N > 0$ for uniqueness).

Verification Challenges

Verification challenges and obstacles

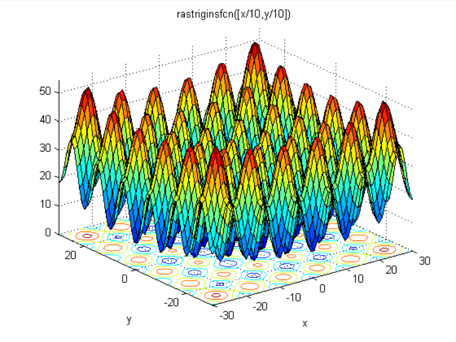
- Verification of degenerate problems
  (in particular verification optimal solutions and basis).
- Handling ill-posed LP problems (e.g., matrix A has not full row rank).
  Many practical problems, e.g. in NETLIB, are mostly ill-posed
  (Keil and Jansson, 2006).

Next Section

Global Optimization

Example (Find the **global** minimum of Rastrigin's function)

$$f(x) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2))$$

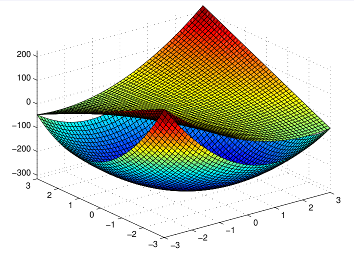


Global optimization ingredients

- branch & bound
- lower and upper bounds (linearizations, convexifications,...)

Lower bounds

- interval arithmetic
- convex underestimating functions (αBB method)



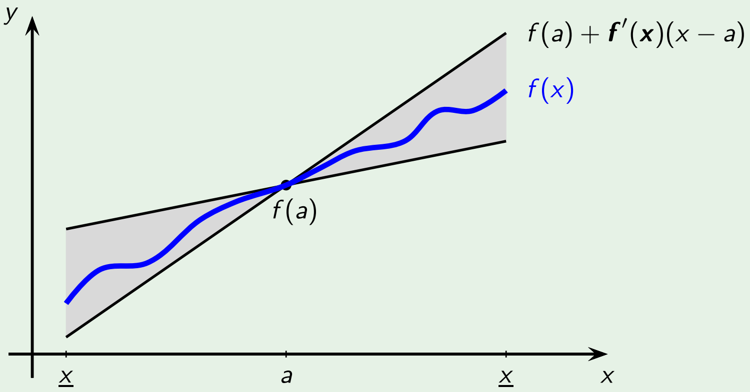
- McCormick envelopes: For every $y \in \boldsymbol{y} \in \mathbb{IR}$ and $z \in \boldsymbol{z} \in \mathbb{IR}$:

$$yz \geq \max\{\underline{y}z + \underline{z}y - \underline{y}\underline{z},\ \overline{y}z + \overline{z}y - \overline{y}\overline{z}\}$$

- Reformulation Linearization Technique (RLT)
- semidefinite programming, ...
- interval linear programming

Interval Linearization

Example (Interval linearization of a nonlinear function)



Theorem (Mean value form)

For $\boldsymbol{x} \in \mathbb{IR}$ and $a \in \boldsymbol{x}$ we have

$$f(x) \subseteq f(a) + \boldsymbol{f}'(\boldsymbol{x})(x - a) \quad \forall x \in \boldsymbol{x}.$$

Interval Linearization in Global Optimization

Global optimization problem

$$\begin{aligned} \min\ & f(x) \\ \text{subject to}\ & h_i(x) = 0, \quad i \in I, \\ & g_j(x) \le 0, \quad j \in J. \end{aligned}$$

Interval linearization on a box $\boldsymbol{x}$ around $a \in \boldsymbol{x}$

We get an interval linear program, rigorous outer approximation

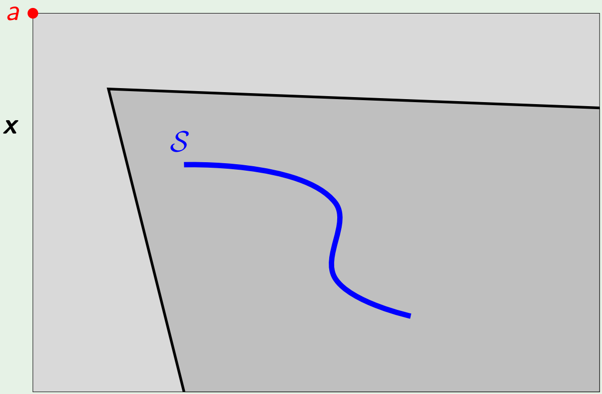
$$\begin{aligned} \min\ & f(a) + \nabla \boldsymbol{f}(\boldsymbol{x})^T(x - a)) \\ \text{subject to}\ & h_i(a) + \nabla \boldsymbol{h}_i(\boldsymbol{x})^T(x - a) = 0, \quad i \in I, \\ & g_j(a) + \nabla \boldsymbol{g}_j(\boldsymbol{x})^T(x - a) \le 0, \quad j \in J. \end{aligned}$$

Questions: Selection of $a \in \boldsymbol{x}$

- Case $a = \underline{x}$ (or any other vertex of $\boldsymbol{x}$): leads to LP
- General case: piecewise linear

Example

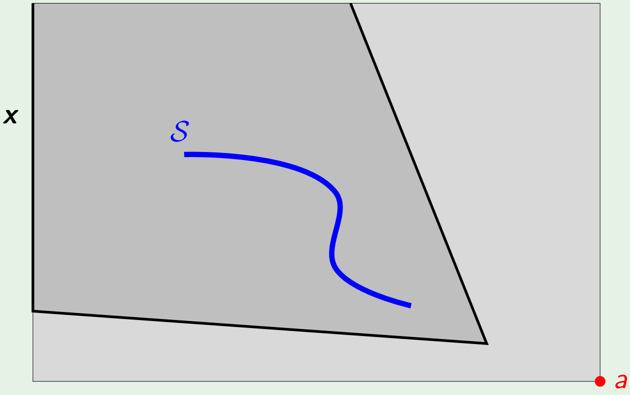
Typical situation when choosing a to be vertex:



Interval Linearization in Global Optimization

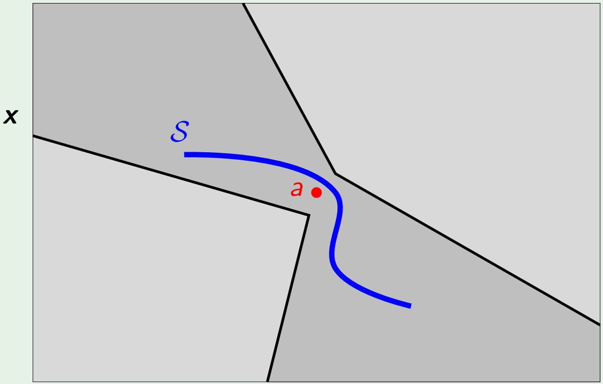
Example

Typical situation when choosing a to be the opposite vertex:



Example

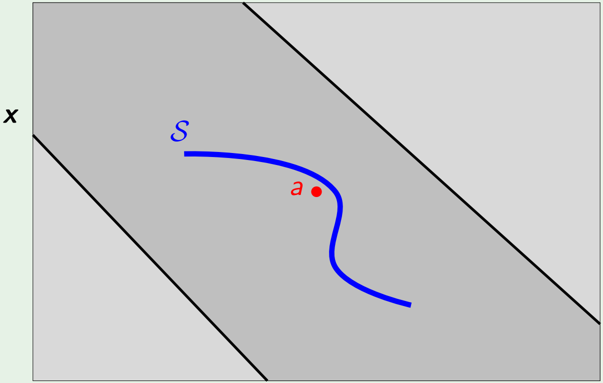
Typical situation when choosing $a = x_c$:



Interval Linearization in Global Optimization

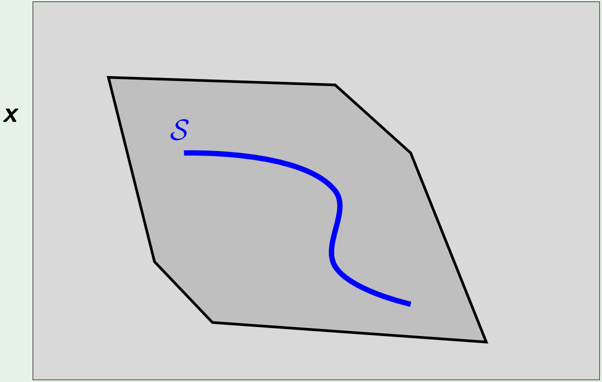
Example

Typical situation when choosing $a = x_c$ (after linearization):



Example

Typical situation when choosing all of them:



Next Section

Sensitivity Measure – Definition

Setup

Consider real LP problem

$$f(A, b, c) = \min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0,$$

and intervals

$$\begin{aligned} \boldsymbol{A}_\alpha &:= [A - \alpha A_\Delta, A + \alpha A_\Delta], \\ \boldsymbol{b}_\alpha &:= [b - \alpha b_\Delta, b + \alpha b_\Delta], \\ \boldsymbol{c}_\alpha &:= [c - \alpha c_\Delta, c + \alpha c_\Delta], \end{aligned}$$

depending on $\alpha \geq 0$.

Sensitivity measure

$$\begin{aligned} d_w &:= \lim_{\alpha \to 0^+} \frac{\overline{f}(\boldsymbol{A}_\alpha, \boldsymbol{b}_\alpha, \boldsymbol{c}_\alpha) - f(A, b, c)}{\alpha}, \\ d_r &:= \frac{1}{\|(A_\Delta, b_\Delta, c_\Delta)\|_F} d_w. \end{aligned}$$

Sensitivity Measure – Computation

Proposition

If the LP problem has the unique nondegenerate optimal solution x^*, and if y^* is a dual optimal solution, then

$$d_w = |y^*|^T A_\Delta x^* + b_\Delta^T |y^*| + c_\Delta^T x^*.$$

Degenerate case assumptions

- Matrix A has full row rank and there is a primal feasible $x^0 > 0$.
- The dual feasible set has nonempty interior.

Proposition

We have

$$d_w = d_w(B) := |y^*(B)|^T A_\Delta x^*(B) + b_\Delta^T |y^*(B)| + c_\Delta^T x^*(B)$$

for certain optimal basis B.

- Corollary: lower and upper bounds on d_w.

Sensitivity Measure – Computation

Computational complexity

- It is NP-hard to check if $d_w \geq 1$.
- It is NP-hard to check $\max_{B \in \mathcal{B}} d_w(B) \geq 1$.

Special cases for the nondegenerate

- If $A_\Delta = 0$, $b_\Delta = 0$ and $c_\Delta = e_j$, then $d_w = x_j^*$.
- If $A_\Delta = 0$, $b_\Delta = e_i$ and $c_\Delta = 0$, then $d_w = |y_i^*|$.
- If $A_\Delta = e_i e_j^T$, $b_\Delta = 0$ and $c_\Delta = 0$, then $d_w = |y_i^* x_j^*|$.

Sensitivity Measure – Examples

Example

Consider

$$\max\ c^T x \ \text{ subject to } \ -e \leq Ax \leq e.$$

with $A_\Delta = |A|$, $b_\Delta = |b|$ and $c_\Delta = |c|$.

n	$A = I_n$		random		Vandermonde		Hilbert	
	d_w	d_r	d_w	d_r	d_w	d_r	d_w	d_r
2	2.545	0.876	3.953	1.338	23.35	8.287	144.3	52.13
3	7.055	1.894	9.747	2.693	56.67	9.801	$1.2 \cdot 10^4$	3468
4	5.392	1.308	26.60	6.295	2034	78.66	$1.4 \cdot 10^7$	$3.7 \cdot 10^6$
5	6.646	1.436	152.3	30.36	$1.2 \cdot 10^5$	641.7	$3.6 \cdot 10^{10}$	$9.0 \cdot 10^9$
6	9.640	1.887	106.0	17.75	$2.9 \cdot 10^6$	1457	$4.8 \cdot 10^{13}$	$1.1 \cdot 10^{13}$
7	13.76	2.468	27.09	3.777	$8.0 \cdot 10^7$	3063	$8.7 \cdot 10^{16}$	$1.8 \cdot 10^{16}$
8	9.683	1.666	92.10	12.20	$5.5 \cdot 10^7$	136.3	–	–
9	14.66	2.342	205.8	23.59	$2.8 \cdot 10^9$	376.2	–	–
10	16.54	2.498	5251	575.2	$1.9 \cdot 10^{10}$	120.5	–	–

Sensitivity Measure – Examples

Example (Netlib data)

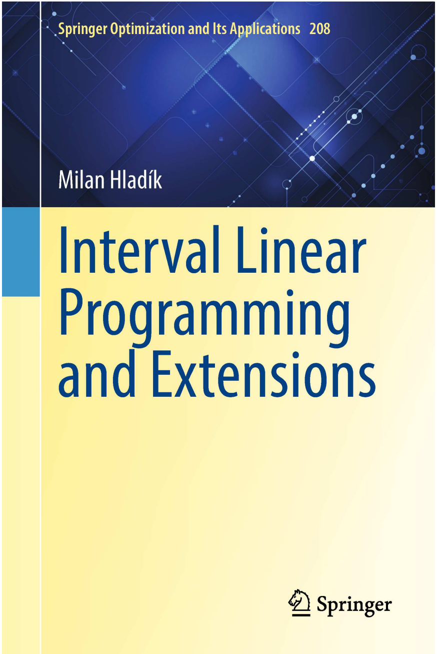
name	vars	constr	$d_w(B)$	$d_r(B)$	$f(A,b,c)$
BANDM	472	305	4686 7584	4.128 6.707	-158.6 -78.44
CAPRI	353	271	$1.5 \cdot 10^5$ $1.5 \cdot 10^5$	20.4 20.39	2690 2690
GREENBEB	5405	2392	$1.7 \cdot 10^7$ $2.4 \cdot 10^7$	12650 17990	$-4.3 \cdot 10^6$ $-4.3 \cdot 10^6$
MAROS	1443	846	$2.4 \cdot 10^6$ $2.4 \cdot 10^6$	15.7 15.77	-58060 -58060
PILOT	3652	1441	13140 13130	1.654 1.652	-557.5 -557.5
SCSD1	760	77	50.42 66	0.6369 0.8337	8.667 8.667
SHIP04L	2118	402	$8.8 \cdot 10^6$ $8.8 \cdot 10^6$	320.5 320.5	$1.8 \cdot 10^6$ $1.8 \cdot 10^6$

Sensitivity Measure – Challenges

Challenges and obstacles – degenerate problems

- Efficient upper bounds
- Computational complexity

References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