An Invitation to Absolute Value Equations: From Theory to Computation

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What are Absolute Value Equations (AVE)?

$$Ax + |x| = b \qquad (A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n)$$

Various extensions: $Ax + B|x| = b, \ldots$

The solution set of AVE

Definition

The solution set of AVE is $\Sigma = \{x \in \mathbb{R}^n;\ Ax + |x| = b\}$.

- May possess up to 2^n isolated points (example: $|x| = e$, where $e = (1, \dots, 1)^T$)
- Any value in $\{1, \dots, 2^n\}$ attained as the number of solutions of certain AVE.

Theorem

The solution set Σ forms a convex polyhedron in each orthant.

Proof.

In the orthant described $\mathrm{diag}(s)x \geq 0,\ s \in \{\pm 1\}^n$ we have

$$|x| = \mathrm{diag}(s)x.$$

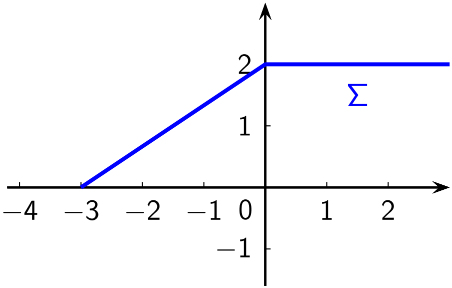
So AVE reads $(A + \mathrm{diag}(s))x = b$. □

The solution set of AVE

Consider the AVE

$$\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} x + |x| = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

Its solution set:

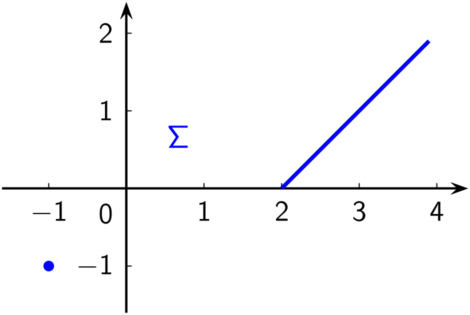


The solution set of AVE

Consider the AVE

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} x + |x| = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Its solution set:



Equivalence to LCP (1/2)

The linear complementarity problem (LCP)

$$y = Mz + q, \;\; y^T z = 0, \;\; y, z \geq 0.$$

Reduction AVE $\to$ LCP.

- Assume $A + I_n$ is nonsingular (reductions avoiding this exist).
- Write x as $x = x^+ - x^-$, where $x^+, x^- \geq 0$, $(x^+)^T x^- = 0$.
- Then $|x| = x^+ + x^-$
- Now, AVE reads $A(x^+ - x^-) + x^+ + x^- = b$, or after rearranging,

$$x^+ = (A + I_n)^{-1}(A - I_n)x^- + (A + I_n)^{-1}b.$$

Equivalence to LCP (2/2)

The linear complementarity problem (LCP)

$$y = Mz + q,\ \ y^T z = 0,\ \ y, z \geq 0.$$

Reduction LCP $\to$ AVE (Mangasarian, 2007).

- Assume $M - I_n$ is nonsingular (obtained by scaling M).
- Observation: $ab = 0,\ a, b \geq 0 \ \Leftrightarrow\ a + b = |a - b|$.
- Thus we can write LCP as

$$z + Mz + q = |z - Mz - q|.$$

- Substituting $x \equiv z - Mz - q$, we have $z = (I_n - M)^{-1}(x + q)$ and system reads

$$(M + I_n)(M - I_n)^{-1}x + |x| = 2(I_n - M)^{-1}q.$$

NP-completeness

Theorem (Mangasarian, 2007)

Checking solvability of AVE is NP-complete.

Proof.

Reduction from Set-Partitioning:

$$\text{Given } a \in \mathbb{Z}^n, \text{ exists } x \in \{\pm 1\}^n : a^T x = 0?$$

Write it as

$$|x| = e, \; a^T x = 0.$$

Equivalently in the canonical form

$$\begin{aligned} |x| &= e, \\ a^T x + |x_{n+1}| &= 0, \\ -a^T x + |x_{n+2}| &= 0. \end{aligned}$$

□

Conditions for unique solvability (1/2)

Interval matrices

- $[A - I_n, A + I_n] = \{B \in \mathbb{R}^{n \times n};\ |A - B| \leq I_n\}$
- $[A - I_n, A + I_n]$ is regular if each matrix $B \in [A - I_n, A + I_n]$ is nonsingular

Theorem (Wu & Li, 2018)

The AVE has a unique solution for each $b \in \mathbb{R}^n$ if and only if $[A - I_n, A + I_n]$ is regular.

- Analogous to nonsingularity of A for $Ax = b$
- For LCP the condition is P-matrix property (all principal minors are positive)
- Which is NP-hard

Conditions for unique solvability (2/2) $Ax + |x| = b$

Interval matrices

- $[A - I_n, A + I_n] = \{B \in \mathbb{R}^{n \times n};\ |A - B| \leq I_n\}$
- $[A - I_n, A + I_n]$ is regular if each matrix $B \in [A - I_n, A + I_n]$ is nonsingular

Theorem (Wu & Li, 2018)

The AVE has a unique solution for each $b \in \mathbb{R}^n$ if and only if $[A - I_n, A + I_n]$ is regular.

Sufficient conditions:

$$\rho(|A^{-1}|) < 1 \quad \text{or} \quad \sigma_{\min}(A) > 1$$

- AVE is efficiently solvable then
- Open: Is AVE efficiently solvable if $[A - I_n, A + I_n]$ is regular?

Method for $\sigma_{\min}(A) > 1$

Method for $\sigma_{\min}(A) > 1$ (Mangasarian & Meyer, 2006)

The iterations

$$x_{k+1} := -A^{-1}|x_k| + A^{-1}b, \quad k = 1, \dots \qquad (\star)$$

converge and in polynomial time the right orthant is determined.

Proof.

We have $\sigma_{\min}(A) > 1 \;\Leftrightarrow\; \sigma_{\max}(A^{-1}) < 1 \;\Leftrightarrow\; \|A^{-1}\| < 1.$

Function $f(x) = -A^{-1}|x| + A^{-1}b$ given by $(\star)$ is a contraction:

$$\begin{aligned}\|f(x) - f(y)\| &= \|A^{-1}(|x| - |y|)\| \\ &\le \|A^{-1}\| \cdot \||x| - |y|\| \\ &\le \|A^{-1}\| \cdot \|x - y\|.\end{aligned}$$

By the fixed-point theorem, there is a unique fixed-point. □

Nonnegative solvability

Theorem
The following conditions are equivalent:

1. *AVE has a unique nonnegative solution for each* $b \geq 0$*;*
2. *AVE has a nonnegative solution for each* $b \geq 0$*;*
3. $(A - I_n)^{-1} \geq 0$.

Theorem (Kuttler, 1971)
$[A - I_n, A + I_n]$ *is inverse nonnegative if and only if*

$$(A - I_n)^{-1} \geq 0 \text{ and } (A + I_n)^{-1} \geq 0.$$

Proposition
If $[A - I_n, A + I_n]$ *is inverse nonnegative, then for each* $b \geq 0$*, the AVE system has a unique solution, and this solution is nonnegative.*

- Computable by the Newton method in at most n iterations.

Methods – general approaches

- Once you know the right orthant $s \in \{\pm 1\}^n$, then simply solve
$$(A + \mathrm{diag}(s))x = b$$
- Newton methods (employing generalized Hessians)
- Picard iterations
  For example, $x^{k+1} = A^{-1}\big(|x^k| - b\big), \quad k = 1, 2, \dots$
- Optimization reformulations (bilinear, concave,...)

Methods – optimization reformulations (1/2) $Ax + |x| = b$

Concave minimization by Mangasarian (2007)

$$\min \ e^T(b - Ax - |x|) \ \text{ subject to } \ Ax + |x| \leq b.$$

- The feasible set is described $Ax + x \leq b$, $Ax - x \leq b$.
- The minimum is 0 iff AVE is solvable.

Theorem (Zamani & H., 2021)

If $[A - I_n, A + I_n]$ is regular, then each local optimum is a global optimum.

Bilinear program by Mangasarian & Meyer (2006)

$$\min \ (b - Ax - x)^T(b - Ax + x) \ \text{ subject to } \ Ax + |x| \leq b.$$

- The minimum is 0 iff AVE is solvable.

Methods – optimization reformulations (2/2)

Linear programming approach I.

$$\min\ c^T x \ \text{ subject to }\ Ax + |x| \leq b.$$

Theorem (Zamani & H., 2021)

If $(A - I_n)^{-1} \geq 0$, $(A + I_n)^{-1} \geq 0$ *and* $c > 0$, *then the optimal solution solves AVE.*

Linear programming approach II.

$$\min\ c^T Ax \ \text{ subject to }\ Ax + |x| \leq b.$$

Theorem (Zamani & H., 2021)

(1) *If* $[A - I_n, A + I_n]$ *is regular, then there exists* $c \geq 0$ *such that the solution of AVE is the optimum of the linear program.*

(2) *If* $\rho(|A^{-1}|) < \frac{1}{2}$ *and* c *is the Perron vector of* $|A^{-T}| + \varepsilon$, *then the linear program yields the solution of AVE.*

Conditions for unsolvability (1/2)

From Reduction AVE $\rightarrow$ LCP:

$$(A + I_n)x^+ + (I_n - A)x^- = b,\ (x^+)^T x^- = 0,\ x^+, x^- \geq 0.$$

$$(A + I_n)x^+ + (I_n - A)x^- = b,\ \cancel{(x^+)^T x^- = 0,}\ x^+, x^- \geq 0.$$

Now, apply the Farkas lemma.

Theorem (Mangasarian & Meyer, 2006)

If

$$-y \leq A^T y \leq y,\ \ b^T y < 0$$

is solvable, then AVE is unsolvable.

Conditions for unsolvability (2/2)

Theorem (H., 2018)

The AVE is unsolvable if

$$\rho(|A|) < 1 \ \text{ and } \ (I_n - |A|)^{-1} b \text{ is not nonnegative.}$$

Lemma

If $\rho(|A|) < 1$, then each solution x of AVE satisfies

$$|x| \leq (I_n - |A|)^{-1} b.$$

Proof.

For each solution

$$|x| = -Ax + b \leq |A| \cdot |x| + b.$$

whence

$$(I_n - |A|)|x| \leq b.$$

Eventually, premultiply by $(I_n - |A|)^{-1} = \sum_{k=0}^{\infty} |A|^k \geq 0$. □

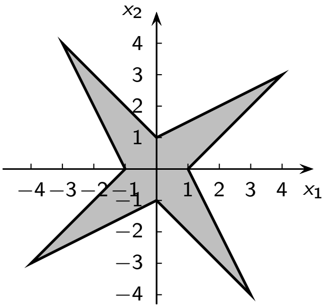
Relation to interval analysis

- Interval matrix $[\underline{A}, \overline{A}] = [A_c - A_\Delta, A_c + A_\Delta]$
- Interval system $[\underline{A}, \overline{A}]x = [\underline{b}, \overline{b}]$. Its solution set

$$\Sigma = \{x;\ Ax = b,\ \underline{A} \le A \le \overline{A},\ \underline{b} \le b \le \overline{b}\}$$

Theorem (Oettli–Prager, 1964)

$$\Sigma = \{x;\ |A_c x - b_c| \le A_\Delta |x| + b_\Delta\}$$



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- Interval matrix $[\underline{A}, \overline{A}] = [A_c - A_\Delta, A_c + A_\Delta]$
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$$\Sigma = \{x;\ Ax = b,\ \underline{A} \le A \le \overline{A},\ \underline{b} \le b \le \overline{b}\}$$

Theorem (Oettli–Prager, 1964)

$$\Sigma = \{x;\ |A_c x - b_c| \le A_\Delta |x| + b_\Delta\}$$

Theorem (Convex hull theorem, Rohn 1989)

If $[\underline{A}, \overline{A}]$ is regular, then

$$\mathit{conv}\,\Sigma = \mathit{conv}\,\{x_s;\ s \in \{\pm 1\}^n\},$$

where x_s is the unique solution of the AVE

$$A_c x - \operatorname{diag}(s) A_\Delta |x| = b_c + \operatorname{diag}(s) b_\Delta.$$

- To solve this AVE, Rohn also proposed a *sign accord algorithm*. ($\le 2^n$ iterations, $\le n$ usually)

Conclusion

Absolute value equations / programming

- Hot and interesting topic
- Many results in theory and methods, open problems

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M. Hladík, H. Moosaei, F. Hashemi, S. Ketabchi, and P. M. Pardalos.
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