Robustness Properties of Absolute Value Linear Programming Problems and Relations to Interval Analysis

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Part I

Absolute Value Equations

What Are Absolute Value Equations?

Definition

Absolute value equations (AVE):

$$Ax + |x| = b \qquad (A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n)$$

Generalized absolute value equations (GAVE):

$$Ax + B|x| = b \qquad (A, B \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n)$$

Properties

- The solution set forms a convex polyhedron in each orthant.
- May possess up to 2^n isolated points.
  (Example: $|x| = e$, where $e = (1, \dots, 1)^T$
  Remark: each value between 1 and 2^n is attained)
- Checking solvability of AVE is NP-complete (Mangasarian, 2007).
- Equivalent to the linear complementarity problem (LCP)

$$y = Mz + q,\ y^T z = 0,\ y, z \geq 0$$

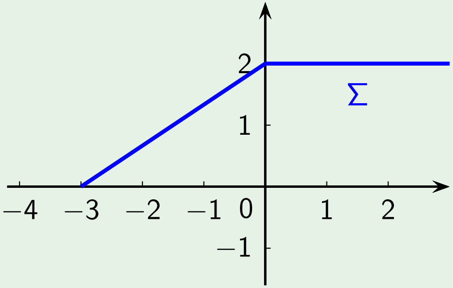
The Solution Set of AVE – Example 1

Example

Consider the absolute value equations

$$\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} x + |x| = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

Its solution set:

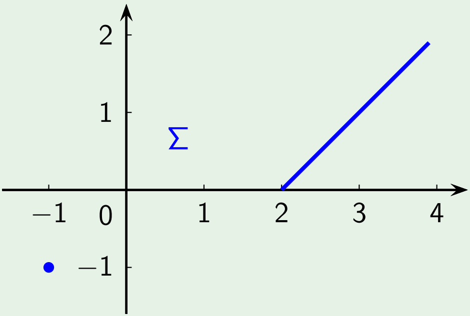


Example

Consider the absolute value equations

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} x + |x| = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Its solution set:



Relation to Interval Analysis: Unique Solvability of AVE

Theorem (Wu & Li, 2018)

The AVE system

$$Ax + |x| = b$$

has a unique solution for each $b \in \mathbb{R}^n$ if and only if the interval matrix

$$[A - I_n, A + I_n]$$

is regular.

Remarks

- Analogous to nonsingularity of A for system $Ax = b$
- NP-hard
- Open problem: Can we find the solution efficiently then?
- Sufficient conditions: $\rho(|A^{-1}|) < 1$, or $\sigma_{\min}(A) > 1$

Relation to Interval Analysis: Unique Solvability of GAVE

Theorem (Wu & Shen, 2021)

The GAVE system

$$Ax + B|x| = b$$

has a unique solution for each $b \in \mathbb{R}^n$ *if and only if the matrix*

$$A + BD$$

is nonsingular for each $D \in [-I_n, I_n]$.

Remarks

- Equivalent characterization: regularity of the interval matrix

$$\begin{pmatrix} A & B \\ [-I_n, I_n] & I_n \end{pmatrix}.$$

- Thus, Rohn's 40 necessary and sufficient conditions apply.

Part II

Absolute Value Linear Programming

What is Absolute Value Linear Programming?

Absolute value linear programming

Linear programming with absolute values

$$\max\ c^T x \ \text{ subject to }\ Ax - D|x| \leq b$$

Assumption: $D \geq 0$

Negative coefficients can be reformulated as linear constraints

- Example: $2x + |x| \leq 3$ rewrite as $2x + y \leq 3$, $-y \leq x \leq y$

Hard and challenging problem: Reduction from integer programming

Consider a 0-1 integer linear program

$$\max\ c^T x \ \text{ subject to }\ Ax \leq b,\ x \in \{0, 1\}^n.$$

The problem equivalently states

$$\max\ c^T x \ \text{ subject to }\ Ax \leq b,\ |2x - e| = e.$$

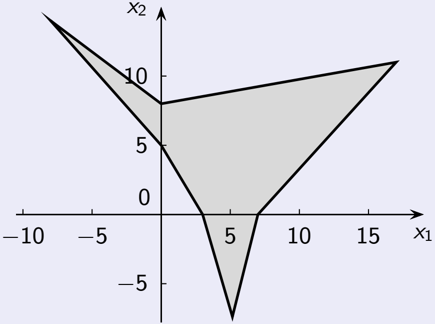
Basic Properties

Our problem max $c^T x$ subject to $Ax - D|x| \leq b$

- nonconvex and nonsmooth optimization problem
- the feasible set can be disconnected: $|x| = e$

Proposition

The feasible set is a convex polyhedral set inside each orthant.



Robust Solvability

Denote the feasible set

$$\mathcal{M} = \mathcal{M}(b) = \{x \in \mathbb{R}^n \colon Ax - D|x| \le b\}.$$

Proposition

The feasible set $\mathcal{M}(b)$ is nonempty for each $b \in \mathbb{R}^n$ if and only if it is nonempty for $b := -e$.

Proposition

It is NP-hard to check $\mathcal{M}(-e) \neq \emptyset$.

Robust Boundedness and Connectedness

Proposition

The feasible set ($Ax - D|x| \le b$) is bounded for each $b \in \mathbb{R}^n$ iff the system

$$Ax - D|x| \le 0$$

has only the trivial solution $x = 0$.

Proposition

It is a co-NP-complete problem to check if the feasible set is bounded.

Proposition

The feasible set is connected if the system is solvable:

$$(A + D)u - (A - D)v \le b, \;\; u, v \ge 0. \qquad (\star)$$

- Let u, v be a solution of $(\star)$.
- Then $x^* := u - v$ solves $\tilde{A}x \le b$ for every $\tilde{A} \in [A - D, A + D]$.
- Thus, every two feasible points are connected via x^*.

Relation to Interval Analysis: The Feasible Set

Our problem

Recall that

$$f^* = \max\ c^T x \ \text{ subject to } \ Ax - D|x| \leq b.$$

The feasible set

The feasible set is the united solution set of an interval linear system

$$[A - D, A + D]x \leq b.$$

That is,

$$\{x : Ax - D|x| \leq b\} = \bigcup_{A' \in [A-D, A+D]} \{x : A'x \leq b\}$$

Relation to Interval Analysis: Optimal Value Interpretation

Our problem

Recall that

$$f^* = \max\ c^T x\ \text{ subject to }\ Ax - D|x| \leq b.$$

Interpretation of f^*

We have that f^* is equal to the best-case optimal value of

$$\max\ c^T x\ \text{ subject to }\ [A - D, A + D]x \leq b,$$

that is,

$$f^* = \max_{\tilde{A} \in [A-D, A+D]}\ \max\ c^T x\ \text{ subject to }\ \tilde{A} \leq b.$$

Relation to Interval Analysis: Duality

Primal problem

Our problem

$$f^* = \max \; c^T x \; \text{ subject to } \; Ax - D|x| \le b.$$

Dual problem

If the linear system

$$(A + D)u - (A - D)v \le b, \; u, v \ge 0$$

is solvable, then f^* is the worst-case optimal value of

$$\min \; b^T y \; \text{ subject to } \; [A - D, A + D]^T y = c, \; y \ge 0.$$

That is,

$$f^* = \max_{\tilde{A} \in [A-D, A+D]} \min \; b^T y \; \text{ subject to } \; \tilde{A}^T y = c, \; y \ge 0.$$

Relation to Interval Analysis: Theorem of the Alternatives

Theorem (Rohn, 2004)

Let $A, D \in \mathbb{R}^{n\times n}$*, where* $D \geq 0$*. Then exactly one of the following alternatives holds:*

1. *the interval system*

$$Ax - [-D, D]|x| = b$$

   is strongly uniquely solvable (each realization has a unique solution);

2. *the inequality*

$$|Ax| \leq D|x|$$

   has a nontrivial solution.

Integrality

Proposition

The vertices of

$$\mathcal{M} = \{x \in \mathbb{R}^n \colon Ax - D|x| \leq b\}.$$

are integral for every $b \in \mathbb{Z}^m$ if and only if matrix $(A - D\,\mathrm{diag}(s))^T$ is unimodular for each $s \in \{\pm 1\}^n$.

Proposition

There is no subset $\mathcal{S} \subseteq \{\pm 1\}^n$ of size at most $2^{n-1} - 1$ such that the condition on unimodularity can be reduced to $s \in \mathcal{S}$.

Open problem

What is the actual complexity of testing integrality?

Proposition

Let $\mathrm{rank}(D) = 1$. Then the condition on unimodularity is satisfied if and only if $(A - D\,\mathrm{diag}(s))^T$ is unimodular for each $s \in \{\pm 1, 0\}^n : \|s\|_0 \leq 2$.

Conclusion

The End