A quantified approach to stability in interval linear programming

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Motivation

Classical Transportation Problem

$$
\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} \quad & \textstyle\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \dots, n, \\
& \textstyle\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \dots, m, \\
& x_{ij} \geq 0, \quad i = 1, \dots, m, \ j = 1, \dots, n,
\end{aligned}
$$

where a_i is the supply, b_j the demand, and c_{ij} the cost.

- Changing supplies or demands changes the optimal solution, but optimal basis B may remain optimal.
- Basis B says which routes will be active and which not.

The Question

- For any admissible variation of demands b_j, exists a feasible adjustment of supplies a_i such that basis B remains optimal?

Interval Data

Interval Vector

$$\boldsymbol{v} = [\underline{v}, \overline{v}] = \{v \in \mathbb{R}^n : \underline{v} \leq v \leq \overline{v}\}$$

We will also use

$$\inf(\boldsymbol{v}) = \underline{v}, \quad \sup(\boldsymbol{v}) = \overline{v}$$

Interval Arithmetic

For two intervals $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\begin{aligned}
\boldsymbol{a} + \boldsymbol{b} &= [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \\
\boldsymbol{a} - \boldsymbol{b} &= [\underline{a} - \overline{b}, \overline{a} - \underline{b}], \\
\boldsymbol{ab} &= [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})], \\
\boldsymbol{a}/\boldsymbol{b} &= [\min(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}), \max(\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b})], \quad 0 \notin \boldsymbol{b}.
\end{aligned}$$

Problem Formulation

Linear Programming Problem

$$\min \ c^T x \ \text{ subject to } \ Ax = b, \ x \geq 0$$

Input

- Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be interval vectors
- Each interval is associated either with $\exists$ or with $\forall$
- Without loss of generality $\boldsymbol{b} = (\boldsymbol{b}^{\exists}; \boldsymbol{b}^{\forall})$ and $\boldsymbol{c} = (\boldsymbol{c}^{\exists}; \boldsymbol{c}^{\forall})$

Problem Formulation

For a given basis B, decide if for each $(b^{\forall}, c^{\forall}) \in (\boldsymbol{b}^{\forall}, \boldsymbol{c}^{\forall})$ there is $(b^{\exists}, c^{\exists}) \in (\boldsymbol{b}^{\exists}, \boldsymbol{c}^{\exists})$ such that basis B is optimal for (b, c).

Basis Optimality Criteria

1. (feasibility) $A_B^{-1} b \geq 0$
2. (optimality) $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$

Basis Feasibility Criterion: $A_B^{-1} b \geq 0$

Case $\boldsymbol{b} = \boldsymbol{b}^{\forall}$

By interval arithmetic, check if $\inf(A_B^{-1}\boldsymbol{b}) \geq 0$

Case $\boldsymbol{b} = \boldsymbol{b}^{\exists}$

Check solvability of the linear system

$$A_B^{-1} b \geq 0, \quad \underline{b} \leq b \leq \overline{b}$$

- This problem is P-complete

Mixed Case $\boldsymbol{b} = (\boldsymbol{b}^{\exists}; \boldsymbol{b}^{\forall})$

- Write the feasibility criterion as $A_B x_B = b, \ x_B \geq 0$
- By splitting the quantifiers write it as

$$A_B^{\forall} x_B = b^{\forall}, \quad A_B^{\exists} x_B = b^{\exists}, \quad x_B \geq 0$$

- For each $b^{\forall}$ such that $b_i^{\forall} \in \{\underline{b}_i^{\forall}, \overline{b}_i^{\forall}\}$, check solvability of

$$A_B^{\forall} x_B = b^{\forall}, \quad \underline{b}^{\exists} \leq A_B^{\exists} x_B \leq \overline{b}^{\exists}, \quad x_B \geq 0$$

Basis Feasibility Criterion: $A_B^{-1} b \geq 0$

Complexity

The condition is exponential in the length of $b^\forall$.

Proposition

Checking basis feasibility is co-NP-hard.

Proposition (Sufficient condition)

Split $A_B^{-1} = (C^\exists \mid C^\forall)$ according to the quantifiers in $\boldsymbol{b}$. If the linear system

$$C^\exists b^\exists + \inf(C^\forall \boldsymbol{b}^\forall) \geq 0, \quad \underline{b}^\exists \leq b^\exists \leq \overline{b}^\exists$$

is feasible, then feasibility criterion holds.

Proof.

The idea is based on reversing the quantifiers. □

Basis Optimality Criterion: $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$

Case $\boldsymbol{c} = \boldsymbol{c}^\forall$

By interval arithmetic, check if $\inf\left(\boldsymbol{c}_N^T - \boldsymbol{c}_B^T (A_B^{-1} A_N)\right) \geq 0^T$.

Case $\boldsymbol{c} = \boldsymbol{c}^\exists$

Check solvability of the linear system

$$c_N^T - c_B^T A_B^{-1} A_N \geq 0^T, \quad \underline{c} \leq c \leq \overline{c}.$$

Mixed Case $\boldsymbol{c} = (\boldsymbol{c}^\exists; \boldsymbol{c}^\forall)$ co-NP-hard

- Write the optimality criterion as $A_B^T y = c_B,\ A_N^T y \leq c_N$,
- By splitting the quantifiers write it as

$$(A_B^\forall)^T y = c_B^\forall,\ (A_B^\exists)^T y = c_B^\exists,\ (A_N^\forall)^T y \leq c_N^\forall,\ (A_N^\exists)^T y \leq c_N^\exists$$

- For each $c_B^\forall$ such that $(c_B^\forall)_i \in \{(\underline{c}_B^\forall)_i, (\overline{c}_B^\forall)_i\}$, check solvability of

$$(A_B^\forall)^T y = (c_B^\forall),\ \underline{c}_B^\exists \leq (A_B^\exists)^T y \leq \overline{c}_B^\exists,\ (A_N^\forall)^T y \leq \underline{c}_N^\forall,\ (A_N^\exists)^T y \leq \overline{c}_N^\exists$$

Example

Example (Transportation Problem from Xie et al. (2017))

$$C = \begin{pmatrix} 15 & 25 & 54 & 5 & 25 \\ 31 & 21 & 87 & 29 & 46 \\ 2 & 15 & 10 & 60 & 30 \end{pmatrix}, \quad a = \begin{pmatrix} 120 \\ 150 \\ 200 \end{pmatrix}, \quad b = \begin{pmatrix} 90 \\ 60 \\ 120 \\ 50 \\ 150 \end{pmatrix}.$$

The optimal solution is

$$X^* = \begin{pmatrix} 0 & 0 & 0 & 50 & 70 \\ 10 & 60 & 0 & 0 & 80 \\ 80 & 0 & 120 & 0 & 0 \end{pmatrix},$$

Assume:

- demands known with 10% uncertainty, $\boldsymbol{b}^{\forall} = [b - 0.1|b|, b + 0.1|b|]$
- supplies can be adjusted within 10%, $\boldsymbol{a}^{\exists} = [a - 0.1|a|, a + 0.1|a|]$

Results:

- optimal basis is stable (sufficient condition applies)
- if uncertainty increased to 20%, basis no more stable

Extensions to Uncertainty in the Matrix

Recall: Basis Optimality Criteria

1. (feasibility) $A_B^{-1} b \geq 0$
2. (optimality) $c_N^T - c_B^T A_B^{-1} A_N \geq 0^T$

Intervals only in $\boldsymbol{A}_N$ and $\forall$-quantified

1. Basis Feasibility: trivial
2. Basis Optimality: rewrite the criterion as

$$A_B^T y = c_B, \quad \overline{A}_N^T y \leq c_N.$$

Then the problem reduces to the case just with interval $\boldsymbol{c}$

$$A_B^T y = c_B, \quad \overline{A}_N^T y^1 - \underline{A}_N^T y^2 \leq c_N, \quad y = y^1 - y^2, \quad y^1, y^2 \geq 0$$

General Case with Intervals in $\boldsymbol{A}$

- a finite reduction exists for $\forall$-quantification
- sufficient conditions exist

Example: Diet Problem

Example (Stigler's Nutrition Model)

Nutritive value of foods (per dollar spent)

	calorie (1000)	protein (g)	calcium (g)	iron (mg)	vitamin-a (1000iu)	vitamin-b1 (mg)	vitamin-b2 (mg)	niacin (mg)	vitamin-c (mg)
wheat	44.7	1411	2.0	365		55.4	33.3	441	
cornmeal	36	897	1.7	99	30.9	17.4	7.9	106	
canned milk	8.4	422	15.1	9	26	3	23.5	11	60
margarine	20.6	17	.6	6	55.8	.2			
cheese	7.4	448	16.4	19	28.1	.8	10.3	4	
peanut-b	15.7	661	1	48		9.6	8.1	471	
lard	41.7				.2		.5	5	
liver	2.2	333	.2	139	169.2	6.4	50.8	316	525
pork roast	4.4	249	.3	37		18.2	3.6	79	
salmon	5.8	705	6.8	45	3.5	1	4.9	209	
green beans	2.4	138	3.7	80	69	4.3	5.8	37	862
cabbage	2.6	125	4	36	7.2	9	4.5	26	5369
onions	5.8	166	3.8	59	16.6	4.7	5.9	21	1184
potatoes	14.3	336	1.8	118	6.7	29.4	7.1	198	2522
spinach	1.1	106		138	918.4	5.7	13.8	33	2755
sweet-pot	9.6	138	2.7	54	290.7	8.4	5.4	83	1912
peaches	8.5	87	1.7	173	86.8	1.2	4.3	55	57
prunes	12.8	99	2.5	154	85.7	3.9	4.3	65	257
lima beans	17.4	1055	3.7	459	5.1	26.9	38.2	93	
navy beans	26.9	1691	11.4	792		38.4	24.6	217	
demand	3	70	.8	12	5	1.8	2.7	18	75

Example: Diet Problem

Example (Stigler's Nutrition Model)

`http://www.gams.com/modlib/libhtml/diet.htm`.

- $n = 20$ different types of food,
- $m = 9$ nutritional demands,
- a_{ij} is the the amount of nutrient j contained in one unit of food i,
- b_i is the required minimal amount of nutrient j,
- c_j is the price per unit of food j,
- minimize the overall cost

The model reads

$$\min c^T x \;\text{ subject to }\; Ax \geq b,\; x \geq 0.$$

The entries a_{ij} are not stable!

Example: Diet Problem

Example (Stigler's Nutrition Model)

We compute

- Optimal basis $B = (1, 8, 12, 15, 20)$
  (wheat, liver, cabbage, spinach and navy beans)

Suppose:

- the entries of A can vary up to δ_1 of their nominal values
- the nutritional demands are usually not hard constraints, the tolerances for b are δ_2

The results on stability:

- For $\delta_1 = 2.5\%$ and $\delta_2 = 2.5\%$, basis is not stable
- For $\delta_1 = 1.5\%$ and $\delta_2 = 2.5\%$, basis is stable
- For $\delta_1 = 0.9\%$ and $\delta_2 = 2.5\%$, basis is stable
  (sufficient condition succeeds)

Conclusion

Novel type of Robustness of a Basis

- Based of forall-exists quantification of interval parameters
- Allows for more complex and flexible variations of data
- In general computationally hard, but sufficient conditions work
- Open problem:
  characterize the case of $\exists$-quantified interval parameters in A

Interval Methods Group

Group on Interval Methods

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