Combinatorial Aspects of Absolute Value Linear Programming

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8th International Symposium on Combinatorial Optimization
ISCO 2024, Tenerife, Spain
May 22 – 24, 2024

Part I

Absolute Value Equations

What are Absolute Value Equations?

Definition

Absolute value equations (AVE):

$$Ax + |x| = b \qquad (A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n)$$

Generalized absolute value equations:

$$Ax + B|x| = b \qquad (A, B \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n)$$

Properties

- The solution set forms a convex polyhedron in each orthant.
- May possess up to 2^n isolated points.
  (Example: $|x| = e$, where $e = (1, \ldots, 1)^T$
  Remark: each value between 1 and 2^n is attained)
- Checking solvability of AVE is NP-complete (Mangasarian, 2007).
- Equivalent to the linear complementarity problem (LCP)

$$y = Mz + q,\ y^T z = 0,\ y, z \geq 0$$

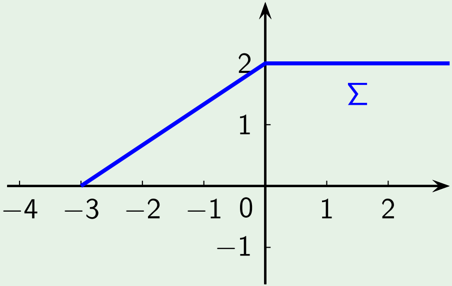
The solution set of AVE – Example 1

Example

Consider the absolute value equations

$$\begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} x + |x| = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

Its solution set:



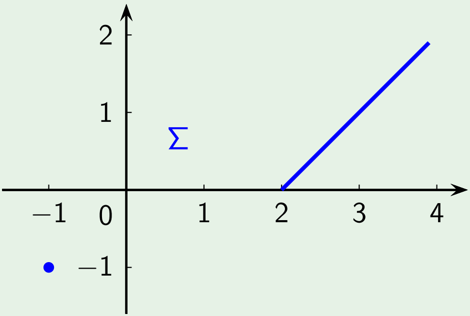
The solution set of AVE – Example 2

Example

Consider the absolute value equations

$$\begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} x + |x| = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Its solution set:



Unique solvability

Interval notation

- $[A \pm I_n] = \{B \in \mathbb{R}^{n\times n} : |A - B| \leq I_n\}$
- $[A \pm I_n]$ is regular if each matrix $B \in [A \pm I_n]$ is nonsingular

Theorem (Wu & Li, 2018)

The AVE system $Ax + |x| = b$ has a unique solution for each $b \in \mathbb{R}^n$ if and only if $[A \pm I_n]$ is regular.

- Analogous to nonsingularity of A for system $Ax = b$
- NP-hard
- Open problem: Can we find the solution efficiently then?

Motivation from LCP

The linear complementarity problem (equivalent to AVE)

$$y = Az + q,\ \ y^T z = 0,\ \ y, z \geq 0.$$

More than 50 matrix classes...

matrix type	definition
P-matrix	unique solution for each q
principally nondegenerate	finitely many solutions (incl. 0) for each q
strictly copositive	at least one solution for each q
semimonotone	unique solution for each $q > 0$
column sufficient	the solution set is convex (or empty)
R_0-matrix	the solution set is bounded
Q-matrix	at least one solution for each q

AVE – topological properties

Proposition

The AVE has a unique nonnegative solution for each $b \geq 0$ if and only if $(A + I_n)^{-1} \geq 0$.

Proposition

There is no AVE such that each orthant contains infinitely many solutions.

- Example. $x + |x| = 0$
  All orthants have infinitely many solutions, except the positive one.

Proposition

The solution set is finite for each $b \in \mathbb{R}^n$ if and only if $A + \mathrm{diag}(s)$ is nonsingular for each $s \in \{\pm 1\}^n$.

Proposition

This property is co-NP-hard to check, even if $\mathrm{rank}(A) = 1$.

Part II

Absolute Value Linear Programming

What is Absolute Value Linear Programming?

Absolute value linear programming

Linear programming with absolute values

$$\max \; c^T x \; \text{ subject to } \; Ax - D|x| \leq b$$

Assumption: $D \geq 0$

Negative coefficients can be reformulated as linear constraints

- Example: $2x + |x| \leq 3$ rewrite as $2x + y \leq 3$, $-y \leq x \leq y$

Hard and challenging problem

- NP-hard to check for feasibility and other issues

Formulation Power

Integer linear programming

Consider a 0-1 integer linear program

$$\max\ c^T x\ \text{ subject to }\ Ax \le b,\ x \in \{0,1\}^n.$$

The problem equivalently states

$$\max\ c^T x\ \text{ subject to }\ Ax \le b,\ |2x - e| = e.$$

Interval linear programming

Our problem

$$\max\ c^T x\ \text{ subject to }\ Ax - D|x| \le b$$

is equivalent to the best case of

$$\max\ c^T x\ \text{ subject to }\ [A - D, A + D]x \le b.$$

Indeed,

$$\{x : Ax - D|x| \le b\} = \bigcup_{A' \in [A-D, A+D]} \{x : A'x \le b\}$$

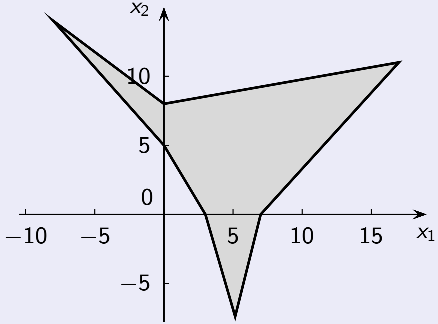
Basic Properties

Our problem max $c^T x$ subject to $Ax - D|x| \le b$

- nonconvex and nonsmooth optimization problem
- the feasible set can be disconnected: $|x| = e$

Proposition

The feasible set is a convex polyhedra set inside each orthant.



Solvability

Denote the feasible set

$$\mathcal{M} = \mathcal{M}(b) = \{x \in \mathbb{R}^n \colon Ax - D|x| \leq b\}.$$

Proposition

The feasible set $\mathcal{M}(b)$ is nonempty for each $b \in \mathbb{R}^n$ if and only if it is nonempty for $b := -e$.

Proposition

It is NP-hard to check $\mathcal{M}(-e) \neq \emptyset$.

Boundedness and Connectedness

Proposition

The feasible set is bounded for each $b \in \mathbb{R}^n$ if and only if the system

$$Ax - D|x| \le 0$$

has only the trivial solution $x = 0$.

Proposition

It is a co-NP-complete problem to check if the feasible set is bounded.

Proposition

The feasible set is connected if the system is solvable:

$$(A + D)u - (A - D)v \le b, \;\; u, v \ge 0. \qquad (\star)$$

- Let u, v be a solution of $(\star)$.
- Then $x^* := u - v$ solves $\tilde{A}x \le b$ for every $\tilde{A} \in [A - D, A + D]$.
- Thus, every two feasible points are connected via x^*.

Duality

Primal problem

Our problem

$$f^* = \max\ c^T x \ \text{ subject to }\ Ax - D|x| \le b.$$

Dual problem

Interval linear program

$$\min\ b^T y \ \text{ subject to }\ [A - D, A + D]^T y = c,\ y \ge 0$$

If $(\star)$ is solvable, then the worst case optimal value is equal to f^*.

Integrality

Proposition

The vertices of

$$\mathcal{M} = \{x \in \mathbb{R}^n \colon Ax - D|x| \leq b\}.$$

are integral for every $b \in \mathbb{Z}^m$ if and only if matrix $(A - D\,\mathrm{diag}(s))^T$ is unimodular for each $s \in \{\pm 1\}^n$.

Proposition

There is no subset $\mathcal{S} \subseteq \{\pm 1\}^n$ of size at most $2^{n-1} - 1$ such that the condition on unimodularity can be reduced to $s \in \mathcal{S}$.

Open problem

What is the actual complexity of testing integrality?

Proposition

Let $\mathrm{rank}(D) = 1$. Then the condition on unimodularity is satisfied if and only if $(A - D\,\mathrm{diag}(s))^T$ is unimodular for each $s \in \{\pm 1, 0\}^n : \|s\|_0 \leq 2$.

Conclusion

- theoretical properties
- numerical methods (TODO)

Interval Methods Group

Group on Interval Methods

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