On transformations in interval linear programming: just be careful!

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Introduction to Interval Analysis

Where interval data do appear

- numerical analysis (handling rounding errors)
  - \( \frac{1}{3} \in [0.33333333333333, 0.33333333333334] \)
  - \( \pi \in [3.1415926535897932384, 3.1415926535897932385] \)
- statistical estimation
  - confidence intervals, prediction intervals (future prices,\ldots)
- measurement errors
  - fuel consumption, stiffness in truss construction, velocity (75 ± 2 km/h)
- discretization
  - time is split in days
  - day range of stock prices – daily min / max
- missing data

Interval notation

A compact interval is \([a, \bar{a}] = [a_c - a\Delta, a_c + a\Delta]\) (for matrices entrywise)

We consider deterministic intervals with no distribution on them.
**Interval Linear Programming**

### Interval linear programming

Family of linear programs

\[
f(A, b, c) = \min_c c^T x \quad \text{subject to} \quad Ax = b, \ x \geq 0
\]

with \( A \in \mathcal{A}, \ b \in \mathcal{b}, \ c \in \mathcal{c}. \) (\( \mathcal{A} = [\underline{A}, \overline{A}] = [A_c - A_\Delta, A_c + A_\Delta] \in \mathbb{IR}^{m \times n} \))

### Optimal solution set \( S \)

- The union of all optimal solutions over all realizations of intervals.
- It is hard to handle \( S \) (characterize, approximate, \ldots).

### Optimal value range \( f = [\underline{f}, \overline{f}] \)

The best case and the worst optimal value

\[
\underline{f} := \min f(A, b, c) \quad \text{subject to} \quad A \in \mathcal{A}, \ b \in \mathcal{b}, \ c \in \mathcal{c}, \\
\overline{f} := \max f(A, b, c) \quad \text{subject to} \quad A \in \mathcal{A}, \ b \in \mathcal{b}, \ c \in \mathcal{c}.
\]

- For some LP forms easy, for some hard to compute.
Example

Consider an interval linear program

$$\max \ (5, 6, 1, 2)^T x \ \text{s.t.} \ \left( \begin{array}{cc} -2 & 7 \\ 6 & 7 \\ 1 \\ -4 & 5 \end{array} \right) x \leq \left( \begin{array}{c} 15 \\ 18 \\ 6 \end{array} \right), \ x \geq 0.$$
The optimal solution set $S$ may be disconnected and nonconvex.

Consider the interval LP problem

$$\max x_2 \quad \text{subject to} \quad [-1, 1]x_1 + x_2 \leq 0, \ x_2 \leq 1.$$
Transformations in Action

Splitting equations to double inequalities

\[ Ax = b \iff Ax \leq b, \ Ax \geq b. \]

Imposing nonnegativity

\[ Ax = b \iff Ax^1 - Ax^2 = b, \ x^1, x^2 \geq 0, \]
\[ Ax \leq b \iff Ax^1 - Ax^2 \leq b, \ x^1, x^2 \geq 0. \]

Substitution

\[ x \equiv Sy, \ \text{where } S \text{ is nonsingular}. \]

Always we have (before and after a transformation):

\[ f \subseteq f', \ S \subseteq S' \]
Splitting Equations to Double Inequalities

\[ Ax = b \implies Ax \leq b, \ Ax \geq b. \]

**Properties**

- It is not equivalent (double occurrences of intervals).
- The same (feasible) solution set.
- The best case optimal value $\underline{f}$ remains the same.
- The worst case optimal value $\overline{f}$ and optimal set $S$ may change. Typically they change enormously!
Splitting Equations to Double Inequalities

Example

\[
\begin{align*}
\min -x_1 & \quad \text{subject to} \quad [1, 2]x_1 - x_2 = 0, \quad 0 \leq x_1, x_2 \leq 10.
\end{align*}
\]

- We have \( f = [-10, -5] \) and \( S = \text{conv} \{(5, 10)^T, (10, 10)^T \} \).
- After transformation: \( f' = [-10, -5] \cup \{0\} \) and \( S' = S \cup \{(0, 0)^T\} \).
Splitting Equations to Double Inequalities

Example

\[ \text{min } -x_1 \text{ subject to } [0, 1]x_1 - x_2 = 0, \ x_1, x_2 \geq 0. \]

After transformation:

\[ \text{min } -x_1 \text{ subject to } [0, 1]x_1 - x_2 \leq 0, \ [0, 1]x_1 - x_2 \geq 0, \ x_1, x_2 \geq 0. \]

- We have \( S = \emptyset \).
- After transformation: \((0, 0)^T \in S'\).

Proposition

The transformation of one interval linear equation

\[ a^T x = b \rightarrow a^T x \leq b, \ a^T x \geq b \]

yields a system with an infeasible realization if

1. \( b_\Delta > 0 \), or
2. \( a_\Delta > 0 \) and \( x = 0 \) is not a commonly feasible to every realization.
Imposing Nonnegativity

\[ \begin{align*}
\mathbf{A} \mathbf{x} &= \mathbf{b} \quad \rightarrow \quad \mathbf{A} \mathbf{x}^1 - \mathbf{A} \mathbf{x}^2 &= \mathbf{b}, \quad x^1, x^2 \geq 0, \\
\mathbf{A} \mathbf{x} &\leq \mathbf{b} \quad \rightarrow \quad \mathbf{A} \mathbf{x}^1 - \mathbf{A} \mathbf{x}^2 \leq \mathbf{b}, \quad x^1, x^2 \geq 0.
\end{align*} \]

Properties

- It is not equivalent (double occurrences of intervals)
- The worst case optimal value $\bar{f}$ remains the same.
- The best case optimal value $\underline{f}$, feasible and optimal set $\mathcal{S}$ may change. Typically they change enormously!
Proposition

Let \( \mathbf{a} \in \mathbb{R}^n \), \( a_\Delta \neq 0 \), and \( \mathbf{b} \in \mathbb{R} \). Then

1. \( \mathbf{a}^T \mathbf{x} = \mathbf{b} \) is transformed to \( \mathbf{a}^T x^1 - \mathbf{a}^T x^2 = \mathbf{b} \), \( x^1, x^2 \geq 0 \), the solution set of which is \( \mathbb{R}^n \);

2. \( \mathbf{a}^T \mathbf{x} \leq \mathbf{b} \) is transformed to \( \mathbf{a}^T x^1 - \mathbf{a}^T x^2 \leq \mathbf{b} \), \( x^1, x^2 \geq 0 \), the solution set of which is \( \mathbb{R}^n \).

Proposition

Suppose the problem is feasible and \( (c_\Delta)_i > 0 \) for some \( i \).

Then the best case optimal value after the transformation \( f' = -\infty \).
Example

\[
\begin{aligned}
\min \ x_2 & \quad \text{subject to} \ x_1 + x_2 \geq 10, \ -[1, 2]x_1 + [0, 1]x_2 \geq 0. \\
\text{After transformation:} & \\
\min \ x_1^1 - x_2^2 & \quad \text{subject to} \ x_1^1 - x_1^2 + x_2^1 - x_2^2 \geq 10, \\
& \quad - [1, 2]x_1^1 + [1, 2]x_1^2 + [0, 1]x_2^1 - [0, 1]x_2^2 \geq 0.
\end{aligned}
\]

- We have \( f = [5, 10] \).
- After transformation: \( f' = (-\infty, 10] \).
  (There is an unbounded realization.)
\[ x \equiv Sy, \quad \text{where } S \text{ is nonsingular.} \]

**Proposition**

Let $S$ be such that it is not a permuted diagonal matrix.

Then there is an interval LP problem for which $f' < f \leq \bar{f} < \bar{f}'$.

The overestimation can be arbitrarily large.
Conclusion

- Even a simple transformation of the constraints may lead to a drastic change of the model properties and outputs.
- We have to maximally utilize the original structure of the data and avoid unprofessional manipulation with the data and the model.
- Be careful!
Group on Interval Methods

https://kam.mff.cuni.cz/gim